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A Massera type criterion for a partial neutral functional differential equation *

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Abstract

We prove the existence of periodic solutions for partial neutral functional differential equations with delay, using a Massera type criterion.

1 Introduction

By using a Massera type criterion, we prove the existence of a periodic solution for the partial neutral functional differential equation

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t), \quad t > 0, \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{D}, \tag{1.2}$$

where A is the infinitesimal generator of a compact analytic semigroup of linear operators, $(T(t))_{t\geq 0}$, on a Banach space X. The history x_t, $x_t(\theta) = x(t+\theta)$, belongs to an appropriate phase space $\mathcal{D}$ and $G, F : \mathbb{R} \times \mathcal{D} \to X$ are continuous functions.

The article by Massera [10] is a pioneer work in the study of the relations between the boundedness of solutions and the existence of periodic solutions. In [10], this relation was explained for a two dimensional periodic ordinary differential equation. Subsequently, several authors considered similar relations, see for example Yoshizawa for a n-dimensional differential equation; Lopes and Hale for n-dimensional ordinary and functional equations with delay and Yong [14], for functional differential equations. Recently Ezzinbi [1], using a Massera type criterion, showed the existence of a periodic solution for the partial functional differential equation

$$\begin{gathered} \dot{x}(t) = Ax(t) + F(t, x_t), \\ x_0 = \varphi \in \mathcal{C} = C([-r, 0] : X), \end{gathered} \tag{1.3}$$

where A is the infinitesimal generator of a compact semigroup of bounded linear operators on a Banach space.

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Our purpose in this paper is to establish similar existence results, as those in [1], for the partial neutral functional differential equation with delay (1.1)-(1.2).

Neutral differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature of neutral functional differential equations with finite delay is the Hale book [6] and the references therein. The work in partial neutral functional differential equations with infinite delay was initiated by Hernández & Henríquez in [8, 7]. In these papers, it is proved the existence of mild, strong and periodic solutions for a neutral equation

$$\begin{gathered} \frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t), \\ x_0 = \varphi \in \mathcal{B}. \end{gathered} \tag{1.4}$$

where A is the infinitesimal generator of an analytic semigroup of linear operators on a Banach space and $\mathcal{B}$ is a phase space defined axiomatically. In general, the results were obtained using the semigroup theory and the Sadovskii fixed point Theorem.

For the rest of this paper, X will denote a Banach space with norm $\|\cdot\|$. $A : D(A) \to X$ will denote the infinitesimal generator of a compact analytic semigroup, $(T(t))_{t\geq 0}$, of linear operators on X. For the theory of C_0 semigroups, we refer the reader to Pazy [11]. However, we will review some notation and properties that will be used in this work.

It is well known that there exist $\tilde{M} \geq 1$ and $\rho \in \mathbb{R}$ such that $\|T(t)\| \leq \tilde{M}e^{\rho t}$, for every $t \geq 0$. If $(T(t))_{t\geq 0}$ is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $\alpha \in (0, 1]$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \quad x \in D(-A)^\alpha,$$

defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|\cdot\|_\alpha$, then the following properties are well known ([11], pp. 74):

Lemma 1.1 *If the above conditions hold, then*

1. *If* $0 < \alpha \leq 1$, *then* X_α *is a Banach space.*
2. *If* $0 < \beta < \alpha \leq 1$ *then* $X_\alpha \hookrightarrow X_\beta$ *and the imbedding is compact whenever the resolvent operator of* A *is compact.*
3. *For every* $0 < \alpha \leq 1$ *there exists* $C_\alpha > 0$ *such that*

$$\|(-A)^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad t > 0.$$

In what follows, to avoid unnecessary notation, we suppose that $0 \in \rho(A)$ and that for $0 < \vartheta \leq 1$

$$\|T(t)\| \leq \tilde{M}, \quad t \geq 0, \quad \text{and} \quad \|(-A)^\vartheta T(t)\| \leq \frac{C_\vartheta}{t^\vartheta}, \quad t > 0, \tag{1.5}$$

for some positive constant C_θ.

In this paper, $0 < \beta \leq 1$ and $\omega > 0$ are fixed numbers, $G, F : \mathbb{R} \times \mathcal{D} \to X$ are continuous and we use the following conditions

$\mathbf{H_1}$ The function G is X_β-valued and $(-A)^\beta G$ is continuous.

$\mathbf{H_2}$ $G(t,\psi) = V(t,\psi) + h(t)$ where V, h are X_β-valued; $(-A)^\beta V$, $(-A)^\beta h$ are continuous; $(-A)^\beta V(\cdot,\psi)$, $(-A)^\beta h$ are ω-periodic and $(-A)^\beta V(t,\cdot)$ is linear.

$\mathbf{H_3}$ $G(t,\psi) = V(t,\psi) + G_1(t,\psi)$ where the functions V, G_1 are X_β-valued; $(-A)^\beta V$, $(-A)^\beta G_1$ are continuous; $(-A)^\beta V(\cdot,\psi)$, $(-A)^\beta G_1(\cdot,\psi)$ are ω-periodic and $(-A)^\beta V(t,\cdot)$ is linear.

$\mathbf{H_4}$ $F(t,\psi) = L(t,\psi) + f(t)$ where L, f are continuous; $L(\cdot,\psi)$, f are ω-periodic and $L(t,\cdot)$ is linear.

$\mathbf{H_5}$ $F(t,\psi) = L(t,\psi) + F_1(t,\psi)$ where L, F_1 are continuous; $L(\cdot,\psi)$, $F_1(\cdot,\psi)$ are ω-periodic and $L(t,\cdot)$ is linear.

$\mathbf{H_6}$ For every $R > 0$ and all $T > 0$, the set of functions

$$\{s \to G(s,x_s) : x \in C([-r,T] : X), \sup_{\theta\in[-r,T]} \|x(\theta)\| \leq R\}$$

is equicontinuous on $[0,T]$.

$\mathbf{H_7}$ For every $R > 0$ and all $T > 0$, the set of functions

$$\{s \to G(s,x_s) : x \in C_b((-\infty,T] : X), \sup_{\theta\in[-\infty,T]} \|x(\theta)\| \leq R\}$$

is equicontinuous on $[0,T]$.

This paper has four sections. In section 2, we discuss the existence of a periodic solution for a partial functional neutral differential equation defined on $\mathbb{R}\times C([-r,0] : X)$. In section 3, by employing the results in section 2, we consider the existence of a periodic solution for a neutral equation with unbounded delay modeled on $\mathbb{R}\times\mathcal{B}$, where $\mathcal{B}$ is a phase space defined axiomatically as in Hale and Kato [3]. The section 4 is reserved for examples. Our results are based on the properties of analytic semigroups and the ideas and techniques in Hernández & Henríquez [8, 7] and Ezzinbi [1].

Throughout this paper, $x(\cdot,\varphi)$ denotes a solution of (1.1)- (1.2). In addition, $B_r(x : Z)$, ($B_r[x : Z]$) will be the open (the closed) ball in a metric space Z with center at x and radius r. For a bounded function $\xi : [a,b] \to [0,\infty)$ and $a \leq t \leq b$ we will employ the notation $\xi_{a,t}$ for

$$\xi_{a,t} = \sup\{\xi(s) : s \in [a,t]\}. \tag{1.6}$$

If $\mathcal{D}$ is a Banach phase space, the norm in $\mathcal{D}$ will be denoted by $\|\cdot\|_{\mathcal{D}}$.

We remark that for the proofs of our results we will use the following results.

Theorem 1.2 (**[3]**) *Let Y be a Banach space and $\Gamma := \Gamma_1 + y$ where $\Gamma_1 : Y \to Y$ is a bounded linear operator and $y \in Y$. If there exist $x_0 \in Y$ such that the set $\{\Gamma^n(x_0) : n \in \mathbb{N}\}$ is relatively compact in Y, then Γ has a fixed point in Y.*

Theorem 1.3 (**[2]**) *Let X be a Banach space and M be a nonempty convex subset of X. If $\Gamma : M \to 2^X$ is a multivalued map such that*

(i) *For every $x \in M$, the set $\Gamma(x)$ is nonempty, convex and closed,*

(ii) *The set $\Gamma(M) = \bigcup_{x \in M} \Gamma x$ is relatively compact,*

(iii) *Γ is upper semi-continuous,*

then Γ has a fixed point in M.

2 A periodic solution for a partial neutral differential equation with bounded delay

In this section, we prove the existence of a periodic solution of the initial value problem

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t), \tag{2.1}$$

$$x_0 = \varphi \in \mathcal{C} = C([-r, 0] : X). \tag{2.2}$$

Definition A function $x : [-r, T] \to X$ is a mild solution of the abstract Cauchy problem (2.1)-(2.2) if: $x_0 = \varphi$; the restriction of $x(\cdot)$ to the interval $[0, T]$ is continuous; for each $0 \le t < T$ the function $AT(t-s)G(s, x_s)$, $s \in [0, t)$, is integrable and

$$\begin{aligned} x(t) &= T(t)(\varphi(0) + G(0, \varphi)) - G(t, x_t) - \int_0^t AT(t-s)G(s, x_s)ds \\ &\quad + \int_0^t T(t-s)F(s, x_s)ds, \quad t \in [0, T]. \end{aligned} \tag{2.3}$$

The existence of mild solutions for the abstract Cauchy problem (2.1)-(2.2) follows from [7, theorems 2.1, 2.2], for this reason, we choose to omit the proof of the next two results.

Theorem 2.1 *Let $\varphi \in \mathcal{C}$, $T > 0$ and assume that the following conditions hold:*

(a) *There exist constants $\beta \in (0, 1)$ and $L \ge 0$ such that the function G is X_β-valued, $L\|(-A)^{-\beta}\| < 1$ and*

$$\|(-A)^\beta G(t, \psi_1) - (-A)^\beta G(s, \psi_2)\| \le L(|t - s| + \|\psi_1 - \psi_2\|_{\mathcal{C}}), \tag{2.4}$$

for every $0 \le s, t \le T$ and $\psi_1, \psi_2 \in \mathcal{C}$.

(b) *The function F is continuous and takes bounded sets into the bounded sets.*

Then there exists a mild solution $x(\cdot,\varphi)$ of the abstract Cauchy problem (2.1)-(2.2) defined on $[-r,b]$, for some $0<b\le T$.

Theorem 2.2 *Let $\varphi\in\mathcal{C}$ and $T>0$. Assume that condition (**a**) of the previous Theorem holds and that there exists $N>0$ such that*

$$\|F(t,\varphi)-F(t,\psi)\|\le N\|\varphi-\psi\|_{\mathcal{C}}, \tag{2.5}$$

for all $0\le t\le T$ and every $\varphi,\psi\in\mathcal{C}$. Then there exists a unique mild solution $x(\cdot,\varphi)$ of (2.1)-(2.2) defined on $[-r,b]$ for some $0<b\le T$. Moreover, b can be chosen as $\min\{T,b_0\}$, where b_0 is a positive constant independent of φ.

To prove the main result of this section, it is fundamental the next result.

Theorem 2.3 *Let $T>r$ and assume that assumption $\mathbf{H_1},\mathbf{H_6}$ hold. Suppose, furthermore, that the following conditions hold.*

(a) *For every $\varphi\in\mathcal{C}$ the set*

$$\mathrm{X}(\varphi)=\{\,x\in C([-r,T]:X):x \text{ is solution of (2.1)-(2.2)}\}$$

is nonempty.

(b) *For every $R>0$, the set*

$$\{(-A)^{\beta}G(s,x_s),\ F(s,x_s)\ :s\in[0,T],\, x\in\mathrm{X}(\varphi) \text{ and } \|\varphi\|_{\mathcal{C}}\le R\},$$

is bounded.

Then the multivalued map $\Upsilon:\mathcal{C}\to 2^{\mathcal{C}}$; $\varphi\to\mathrm{X}_T(\varphi)=\{x_T:x\in\mathrm{X}(\varphi)\}$ is compact, that is, for every $R>0$ the set $\mathcal{U}_{R,T}=\bigcup_{\|\varphi\|_{\mathcal{C}}\le R}\mathrm{X}_T(\varphi)$ is relatively compact in $\mathcal{C}$.

Proof: Let $R>0$ and $\mathcal{U}_R=\bigcup_{\|\varphi\|_{\mathcal{C}}\le R}\mathrm{X}(\varphi)$. From **(b)**, we fix $N>0$ such that $\|(-A)^{\beta}G(s,x_s)\|\le N$ and $\|F(s,x_s)\|\le N$ for every $x\in\mathcal{U}_R$ and every $s\in[0,T]$. In order to use the Ascoli Theorem, we divide the proof in two steps.
Step 1 The set $\mathcal{U}_R(t)=\{x(t):x\in\mathcal{U}_R\}$ is relatively compact for $t\in(0,T]$. Let $0<\epsilon<t\le T$. Since $(T(t))_{t\ge 0}$ is analytic, the operator function $s\to AT(s)$ is continuous in the uniform operator topology on $(0,T]$, which by the estimate

$$\|(-A)^{1-\beta}T(t-s)(-A)^{\beta}G(s,x_s)\|\le\frac{C_{1-\beta}N}{(t-s)^{1-\beta}},\quad s\in[0,t),\, x\in\mathcal{U}_R \tag{2.6}$$

implies that the function $s \to \|AT(t-s)G(s,x_s)\|$ is integrable on $[0,t)$ for every $x \in \mathcal{U}_R$. Under the previous conditions, for $x \in \mathcal{U}_R$ we get

$$\begin{aligned}
x(t) &= T(\epsilon)T(t-\epsilon)(\varphi(0)+G(0,\varphi)) + (-A)^{-\beta}(-A)^{\beta}G(t,x_t) \\
&\quad + T(\epsilon)\int_0^{t-\epsilon}(-A)^{1-\beta}T(t-s-\epsilon)(-A)^{\beta}G(s,x_s)ds \\
&\quad + \int_{t-\epsilon}^{t}(-A)^{1-\beta}T(t-s)(-A)^{\beta}G(s,x_s)ds \\
&\quad + T(\epsilon)\int_0^{t-\epsilon}T(t-s-\epsilon)F(s,x_s)ds + \int_{t-\epsilon}^{t}T(t-s)F(s,x_s)ds,
\end{aligned}$$

and hence

$$x(t) \in T(\epsilon)\tilde{M}B_{R+N}[0,X] + (-A)^{-\beta}B_N[0,X] + W_t^1 + C_\epsilon^1 + W_t^2 + C_\epsilon^2,$$

where each W_t^i is compact, $\operatorname{diam}(C_\epsilon^1) \le 2C_{1-\beta}N\frac{\epsilon^\beta}{\beta}$ and $\operatorname{diam}(C_\epsilon^2) \le 2\tilde{M}N\epsilon$. Since $(-A)^{-\beta}$ is compact, these remarks imply that $\mathcal{U}_R(t)$ is totally bounded and consequently relatively compact in X.

Step 2 $\mathcal{U}_R$ is equicontinuous on $(0,T]$. Let $0 < \epsilon < t_0 < t \le T$. The strong continuity of $(T(t))_{t\ge 0}$ implies that the set of functions $\{s \to T(s)x : x \in T(\epsilon)B_{R+N}(0,X)\}$ is equicontinuous on $[0,T]$. Let $0 < \delta < \epsilon$ be such that

$$\begin{gathered}
\|T(s)x - T(s')x\| < \epsilon, \quad x \in T(\epsilon)B_{R+N}(0,X), \\
\|G(t,u_t) - G(t_0,u_{t_0})\| < \epsilon, \quad u \in C([-r,T]:X),\ \|u\|_{-r,T} \le R+N,
\end{gathered}$$

when $|s-s'| < \delta$, $0 \le s,s' \le T$ and $0 \le t - t_0 < \delta$.

Under the above conditions, for $x \in \mathcal{U}_R$ and $0 \le t - t_0 < \delta$ we get

$$\begin{aligned}
&\|x(t_0) - x(t)\| \\
&\le \|(T(t_0-\epsilon) - T(t-\epsilon))T(\epsilon)(\varphi(0)+G(0,\varphi))\| \\
&\quad + \|G(t_0,x_{t_0}) - G(t,x_t)\| \\
&\quad + \int_0^{t_0-\epsilon}\|(-A)^{1-\beta}T(t_0-s-\epsilon)(I-T(t-t_0))T(\epsilon)(-A)^{\beta}G(s,x_s)\|ds \\
&\quad + \int_{t_0-\epsilon}^{t_0}\|I-T(t-t_0)\|\|(-A)^{1-\beta}T(t_0-s)(-A)^{\beta}G(s,x_s)\|ds \\
&\quad + \int_{t_0}^{t}\|(-A)^{1-\beta}T(t-s)(-A)^{\beta}G(s,x_s)\|ds \\
&\quad + \int_0^{t_0-\epsilon}\|T(t_0-s-\epsilon)(I-T(t-t_0))T(\epsilon)F(s,x_s)\|ds \\
&\quad + \int_{t_0-\epsilon}^{t_0}\|(T(t_0-s)-T(t-s))F(s,x_s)\|ds + \int_{t_0}^{t}\|T(t-s)F(s,x_s)\|ds \\
&\le 2\epsilon + \epsilon C_{1-\beta}\frac{(t_0-\epsilon)^\beta}{\beta} + 2\tilde{M}C_{1-\beta}N\frac{\epsilon^\beta}{\beta} + NC_{1-\beta}\frac{(t-t_0)^\beta}{\beta} + \epsilon\tilde{M}(t_0-\epsilon) \\
&\quad + 2\tilde{M}N\epsilon + \tilde{M}N(t-t_0),
\end{aligned}$$

and hence

$$\|x(t_0) - x(t)\| \le c_1\epsilon + c_2\epsilon^\beta + c_3\delta^\beta + c_4\delta,$$

where the constants c_i are independent of $x(\cdot)$. Thus, $\mathcal{U}_R$ is equicontinuous from the right side at $t_0 > 0$. The equicontinuity in $t_0 > 0$ is proved in similar form, we omit details. Thus, $\mathcal{U}_R$ is equicontinuous on $(0, T]$.

From the steps 1 and 2, it follow that $\{x|_{[\mu,T]} : x \in \mathcal{U}_R\}$ is relatively compact in $C([\mu, T] : X)$ for every $\mu > 0$, which in turn implies that $\mathcal{U}_{R,T}$ is relatively compact in $C([-r, 0] : X)$. This completes the proof.

Corollary 2.4 *Assume that the hypothesis in Theorems 2.2 and 2.3 are fulfilled. Then the map $\varphi \to x_T(\cdot, \varphi)$ is a completely continuous function.*

Proof: The assertion follows from (2.6), the Lebesgue dominated convergence Theorem and Theorem 2.3. We omit details.

Definition A function $x : \mathbb{R} \to X$ is an ω-periodic solution of equation (2.1) if: $x(\cdot)$ is a mild solution of (2.1) and $x(t + \omega) = x(t)$ for every $t \in \mathbb{R}$.

Using the ideas and techniques in [8], it is possible to establish sufficient conditions for the existence of global solutions of (2.1). In what follows, we always assume that the mild solutions are defined on $[0, \infty)$.

Theorem 2.5 *Let conditions $\mathbf{H_2}$, $\mathbf{H_4}$ and $\mathbf{H_6}$ be satisfied. If the equation (2.1) has a bounded mild solution, then there exists an ω-periodic solution of (2.1).*

Proof: For a mild solution $x(\cdot) = x(\cdot, \varphi)$, we introduce the decomposition $x(\cdot) = v(\cdot) + z(\cdot)$ where $v(\cdot)$ is the mild solution of

$$\frac{d}{dt}(u(t) + V(t, u_t)) = Au(t) + L(t, u_t),$$
$$u_0 = \varphi,$$

and $z(\cdot)$ is the mild solution of

$$\frac{d}{dt}(u(t) + V(t, u_t) + h(t)) = Au(t) + L(t, u_t) + f(t),$$
$$u_0 = 0.$$

Let $y : [-r, \infty) \to X$ be a bounded mild solution of (2.1) and $\Gamma : \mathcal{C} \to \mathcal{C}$ be the map $\Gamma(\varphi) := \Gamma_1(\varphi) + z_\omega := v_\omega + z_\omega$. Since Γ_1 is a bounded linear operator and $\bigcup_{n\ge1} \Gamma^n(y_0) = \{y_{n\omega} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}$, see Theorem 2.3, it follows from Theorem 1.2 that Γ has a fixed point in $\mathcal{C}$. This fixed point give a periodic solution. The proof is complete.

In what follows CP is the space $CP = \{u : \mathbb{R} \to X : u \text{ is } \omega\text{-periodic}\}$ endowed with the uniform convergence topology.

Now we prove the main result of this work.

Theorem 2.6 *Let conditions $\mathbf{H_3}, \mathbf{H_5}$ and $\mathbf{H_6}$ be satisfied and assume that the following conditions are fulfilled.*

(**a**) *The functions $(-A)^\beta V$, $(-A)^\beta G_1$, L and F_1 takes bounded sets into the bounded sets.*

(**b**) *There is $\rho > 0$ such that for every $v \in B_\rho[0, CP]$ the neutral equation*
$$\frac{d}{dt}(x(t) + V(t, x_t) + G_1(t, v_t)) = Ax(t) + L(t, x_t) + F_1(t, v_t),$$
has an ω-periodic solution $u(\cdot, v) \in B_\rho[0, CP]$.

Then the equation (2.1) has an ω-periodic solution.

Proof: On $B_\rho = B_\rho[0, CP]$, we define the multivalued map $\Gamma : B_\rho \to 2^{B_\rho}$ by: $x \in \Gamma(v)$ if, and only if,
$$\begin{aligned} x(t) &= T(t-s)(x(s) + V(s, x_s) + G_1(s, v_s)) - V(t, x_t) - G_1(t, v_t) \\ &\quad - \int_s^t AT(t-\tau)(V(\tau, x_\tau) + G_1(\tau, v_\tau))d\tau \\ &\quad + \int_s^t T(t-\tau)(L(\tau, x_\tau) + F_1(\tau, v_\tau))d\tau, \quad t > s. \end{aligned}$$
Next we prove that Γ verifies the conditions (**i**)-(**iii**) of Theorem 1.3. Clearly assumption (**i**) holds. The condition (**ii**) follows using the steps in the proof of Theorem 2.3. In relation to (**iii**), we observe that from (**ii**) is sufficient to show that Γ is closed. If $(v^n)_{n\in\mathbb{N}}$ and $(x^n)_{n\in\mathbb{N}}$ are convergent sequences in CP to points v, x then
$$\begin{aligned} (-A)^\beta(V(\tau, x^n_\tau) + G_1(\tau, v^n_\tau)) &\to (-A)^\beta(V(\tau, x_\tau) + G_1(\tau, v_\tau)), \\ T(t)(L(\tau, x^n_\tau) + F_1(\tau, v^n_\tau)) &\to T(t)(L(\tau, x_\tau) + F_1(\tau, v_\tau)), \end{aligned}$$
for $\tau \in \mathbb{R}$ and $t \geq s$. From the Lebesgue dominated convergence Theorem, assumption (**a**) and the estimate
$$\|AT(t-s)(V(\tau, x^n_\tau) + G_1(\tau, v^n_\tau))\| \leq C_{1-\beta}\frac{\|(-A)^\beta(V(\tau, x^n_\tau) + G_1(\tau, v^n_\tau))\|}{(t-s)^{1-\beta}},$$
we conclude that
$$\begin{aligned} x(t) &= T(t-s)(x(s) + V(s, x_s) + G_1(s, v_s)) - V(t, x_t) - G_1(t, v_t) \\ &\quad - \int_s^t AT(t-\tau)(V(\tau, x_\tau) + G_1(\tau, v_\tau))d\tau \\ &\quad + \int_s^t T(t-\tau)(L(\tau, x_\tau) + F_1(\tau, v_\tau))d\tau, \quad t > s, \end{aligned}$$
which proves that $x \in \Gamma v$. Thus, Γ is closed and consequently upper semi-continuous.

From Theorem 1.2 the operator Γ has a fixed point. This fixed point is an ω-periodic solution of (2.1). The proof is finished.

3 A periodic solution for a partial neutral differential equation with unbounded delay

In this section, we discuss the existence of an ω-periodic solution for a partial functional neutral differential equation with unbounded delay modeled in the form

$$\frac{d}{dt}(x(t)+G(t,x_t)) = Ax(t)+F(t,x_t), \tag{3.1}$$

$$x_\sigma = \varphi \in \mathcal{B}, \tag{3.2}$$

where the history $x_t : (-\infty,0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically and $F, G : \mathbb{R}\times\mathcal{B} \to X$ are appropriate continuous functions.

For the rest of this paper, $\mathcal{B}$ will be an abstract phase space defined axiomatically as in Hale and Kato [4]. To establish the axioms of the space $\mathcal{B}$, we follow the terminology used in [9], and thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty,0]$ into X, endowed with a semi-norm $\|\cdot\|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-\infty,\sigma+a) \to X$, $a>0$, is continuous on $[\sigma,\sigma+a)$ and $x_\sigma \in \mathcal{B}$, then for every $t\in[\sigma,\sigma+a)$, the following conditions hold:

i) x_t is in $\mathcal{B}$.

ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$.

iii) $\|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma\le s\le t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,

where $H>0$ is a constant; $K, M : [0,\infty)\to[0,\infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in **(A)**, x_t is a $\mathcal{B}$-valued continuous function on $[\sigma,\sigma+a)$.

(B) The space $\mathcal{B}$ is complete.

(C2) If a uniformly bounded sequence $(\varphi^n)_n$ in C_{00} converges to a function φ in the compact-open topology, then $\varphi\in\mathcal{B}$ and $\|\varphi^n-\varphi\|_{\mathcal{B}} \to 0$ as $n\to\infty$.

Example 3.1 We consider the phase space $\mathcal{B} := C_r \times L^p(g;X)$, $r\ge 0$, $1\le p<\infty$, see [9], which consists of all classes of functions $\varphi : (-\infty,0]\to X$ such that φ is continuous on $[-r,0]$, Lebesgue-measurable and $g|\varphi(\cdot)|^p$ is Lebesgue integrable on $(-\infty,-r)$, where $g : (-\infty,-r)\to\mathbb{R}$ is a positive Lebesgue integrable function. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} := \sup\{\|\varphi(\theta)\| : -r\le\theta\le 0\} + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta\right)^{1/p}.$$

We will assume that g satisfies conditions (g-6) and (g-7) in the terminology of [9]. This means that g is integrable on $(-\infty, -r)$ and that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that

$$g(\xi + \theta) \leq \gamma(\xi)\, g(\theta),$$

for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. In this case, $\mathcal{B}$ is a phase space which satisfies axioms (**A**), (**A1**), (**B**) and (**C2**) (see [9], Theorem 1.3.8).

Definition A function $x : (-\infty, T] \to X$ is a mild solution of the abstract Cauchy problem (3.1)-(3.2) if: $x_0 = \varphi$; the restriction of $x(\cdot)$ to the interval $[0, T]$ is continuous; for each $0 \leq t < T$ the function $AT(t-s)G(s, x_s)$, $s \in [0, t)$, is integrable and

$$\begin{aligned} x(t) =& T(t)(\varphi(0) + G(0, \varphi)) - G(t, x_t) - \int_0^t AT(t-s)G(s, x_s)ds \\ &+ \int_0^t T(t-s)F(s, x_s)ds, \quad t \in [0, T]. \end{aligned} \tag{3.3}$$

Lemma 3.2 *Let Assumptions* $\mathbf{H_1}, \mathbf{H_7}$ *be satisfied. If* $x : (-\infty, T] \to X$ *is a bounded mild solution of (3.1), then the set* $U = \{x_{n\omega} : n \in \mathbb{N}\}$ *is relatively compact in* $\mathcal{B}$.

Proof: Let $(x_{n_k\omega})_{k\in\mathbb{N}}$ be a sequence in U. For $n > 0$, we define the set $U(n, r) = \{x_{n_k\omega}|_{[-r,0]} : n_k \geq n\}$. Using the same arguments in the proof of Theorem 2.3, it follows that $U(n, r)$ is relatively compact in $C([-r, , 0]; X)$ when $n\omega > r$. Now we can choose a subsequence of $(x_{n_k\omega})_{k\in\mathbb{N}}$; which is indicated by the same index, that converges uniformly on compact subsets of $(-\infty, 0]$ to some function $x \in C_b((-\infty, 0] : X)$. Since $\mathcal{B}$ verifies axiom $\mathbf{C\,2}$, it follow that $x \in \mathcal{B}$ and that $x_{n_k\omega} \to x$ in $\mathcal{B}$. Thus, U is relatively compact in $\mathcal{B}$.

Definition A function $x : \mathbb{R} \to X$ is an ω-periodic solution of equation (3.1) if: $x(\cdot)$ is a mild solution of (3.1) and $x(t + \omega) = x(t)$ for every $t \in \mathbb{R}$.

The proofs of the following results are similar to the proofs of Theorems 2.5 and 2.6. We only remark that the continuity of $\varphi \to x_\omega(\cdot, \varphi)$ is discussed in [8].

Theorem 3.3 *Let conditions* $\mathbf{H_2}$, $\mathbf{H_4}$ *and* $\mathbf{H_7}$ *be satisfied. If the equation (3.1) has a bounded mild solution then there exists an* ω*-periodic solution of (3.1).*

Theorem 3.4 *Let conditions* $\mathbf{H_3}, \mathbf{H_5}$ *and* $\mathbf{H_7}$ *be satisfied and assume that the following conditions are fulfilled.*

(**a**) *The functions* $(-A)^\beta V$, $(-A)^\beta G_1$, L *and* F_1 *takes bounded sets into the bounded sets.*

(**b**) *There is $\rho > 0$ such that for every $v \in B_\rho[0, CP]$ the neutral equation*

$$\frac{d}{dt}(x(t) + V(t, x_t) + G_1(t, v_t)) = Ax(t) + L(t, x_t) + F_1(t, v_t),$$

has an ω-periodic solution $u(\cdot, v) \in B_\rho[0, CP]$.

Then the neutral equation (3.1) has an ω-periodic solution.

4 Applications

In this section, we illustrate some of the results in this work. Let $X = L^2([0, \pi])$ and A and $Ax = x''$ with domain

$$D(A) := \{f(\cdot) \in L^2([0, \pi]) : f''(\cdot) \in L^2([0, \pi]),\ f(0) = f(\pi) = 0\}.$$

It's well known that A is the infinitesimal generator of a C_0 semigroup, $(T(t))_{t\geq 0}$, on X, which is compact, analytic and self-adjoint. Moreover, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenvectors $z_n(\xi) := (2/\pi)^{1/2} \sin(n\xi)$ and the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

(b) If $f \in D(A)$ then $A(f) = -\sum_{n=1}^{\infty} n^2 \langle f, z_n\rangle z_n$.

(c) For $f \in X$, $(-A)^{-\frac{1}{2}} f = \sum_{n=1}^{\infty} \frac{1}{n} \langle f, z_n\rangle z_n$. In particular, $\|(-A)^{-1/2}\| = 1$.

(d) The operator $(-A)^{1/2}$ is given as $(-A)^{1/2} f = \sum_{n=1}^{\infty} n\langle f, z_n\rangle z_n$ on the space $D((-A)^{1/2}) = \{f \in X : \sum_{n=1}^{\infty} n\langle f, z_n\rangle z_n \in X\}$.

(e) For every $f \in X$, $T(t)f = \sum_{n=1}^{\infty} e^{-n^2 t}\langle f, z_n\rangle z_n$. Moreover, it follows from this expression that $\|T(t)\| \leq e^{-t}$, $t \geq 0$, and that $\|(-A)^{1/2}T(t)\| \leq \frac{1}{\sqrt{2}} e^{-t/2} t^{-1/2}$, for $t > 0$.

A neutral equation with bounded Delay

Considering the example in Ezzimby [1], in this section we study the neutral equation

$$\begin{aligned} \frac{d}{dt}[u(t,\xi) &+ \int_{-r}^{0}\int_0^{\pi} a_0(t) b(s,\eta,\xi) u(t+s,\eta) d\eta ds] \\ &= \frac{\partial^2}{\partial \xi^2} u(t,\xi) + a_1(t) x(t-r,\xi) + a_2(t) p(t, x(t-r,\xi)) + q(t,\xi), \end{aligned} \tag{4.1}$$

$$u(t,0) = u(t,\pi) = 0, \quad t \geq 0, \tag{4.2}$$

$$u(\tau,\xi) = \varphi(\tau,\xi), \quad \tau \in [-r,0],\ 0 \leq \xi \leq \pi, \tag{4.3}$$

where

(i) The function $b(\cdot)$ is measurable and
$$\sup_{t\in[-r,\infty)}\int_0^\pi\int_0^\pi b^2(t,\eta,\xi)d\eta d\xi<\infty.$$

(ii) The function $\frac{\partial}{\partial\zeta}b(\tau,\eta,\zeta)$ is measurable; $b(\tau,\eta,\pi)=0$; $b(\tau,\eta,0)=0$ and $N_1r<1$ where
$$N_1:=\int_0^\pi\int_{-r}^0\int_0^\pi\big(\frac{\partial}{\partial\zeta}b(\tau,\eta,\zeta)\big)^2d\eta d\tau d\zeta.$$

(iii) The functions $p,q:\mathbb{R}^2\to\mathbb{R}$ are continuous and ω-periodic in the first variable.

(iv) The substitution operators $g:\mathbb{R}\times X\to X$, $f:\mathbb{R}\to X$ defined by $g(t,x)(\xi)=p(t,x(\xi))$ and $f(t)(\xi)=q(t,\xi)$ are ω-periodic, continuous and there exists $k>0$ such that $\|g(t,x)\|\le k\|x\|$, $\quad (t,x)\in\mathbb{R}\times X$.

(v) The functions $a_1,a_2:\mathbb{R}\to\mathbb{R}$ are continuous, ω-periodic and there exists a constant l such that $-1+|a_1(t)|+|a_2(t)|k\le -l$, $t\ge 0$.

(vi) The function $a_0:\mathbb{R}\to\mathbb{R}$ is continuous, nondecreasing, ω-periodic and $0\le a_0(t)\le(1-e^{-t})$, for $t\ge 0$.

On the space $\mathbb{R}\times\mathcal{C}$, we define the maps
$$\begin{gathered}G(t,\psi)(\xi):=a_0(t)V(t,\psi),\\ V(t,\psi)(\xi):=\int_{-r}^0\int_0^\pi b(s,\eta,\xi)\psi(s,\eta)d\eta ds,\\ L(t,\psi)(\xi):=a_1(t)\psi(-r)(\xi),\\ F_1(t,\psi)(\xi):=a_2(t)g(t,\psi(-r))(\xi)+f(t)(\xi).\end{gathered}$$

With the previous notation, the initial-boundary value problem (4.1)-(4.3) can be written as the abstract Cauchy problem
$$\frac{d}{dt}(x(t)+G(t,x_t))=Ax(t)+L(t,x_t)+F_1(t,x_t),\quad t\ge 0 \tag{4.4}$$
$$x_0=\varphi,\quad \varphi\in C([-r,0]:X)=:\mathcal{C}. \tag{4.5}$$

A straightforward estimation using **(i)-(iv)** shows that the functions G, L and F_1 are continuous. Moreover, from **(d)** and **(ii)**, it follow that G is a bounded linear operator with values in $X_{1/2}$ and that $\|(-A)^{1/2}G(t,\cdot)\|\le a_0(t)(N_1r)^{1/2}$ for every $t\in\mathbb{R}$.

Next we prove that G satisfies $\mathbf{H_6}$. Considering the condition (**vi**), we only proof that V verifies $\mathbf{H_6}$. Let $R>0$ and $x\in C([-r,T];X)$ such that $\|x\|_{-r,T}\le$

R. For $t > 0$ we get

$$\begin{aligned}
&V(t+h, x_{t+h})(\xi) - V(t, x_t)(\xi)\\
&= \int_{-r+t+h}^{-r+t} \int_0^\pi b(\theta - t - h, \eta, \xi) x(\theta, \eta) d\eta d\theta\\
&\quad + \int_{-r+t}^{t} \int_0^\pi (b(\theta - t - h, \eta, \xi) - b(\theta - t, \eta, \xi)) x(\theta, \eta) d\eta d\theta\\
&\quad + \int_{t}^{t+h} \int_0^\pi b(\theta - t - h, \eta, \xi) x(\theta, \eta) d\eta d\theta
\end{aligned}$$

and hence

$$\begin{aligned}
|\frac{d}{dt} V(t, x_t)|_X^2 \;\le\;& 4\|x(-r+t)\|^2 \int_0^\pi \int_0^\pi b^2(-r, \eta, \xi)^2 d\eta d\xi\\
&+ 4\|x\|_\infty^2 r \int_0^\pi \int_{-r+t}^{t} \int_0^\pi (\frac{\partial b}{\partial \theta}(\theta - t, \eta, \xi))^2 d\eta d\theta d\xi\\
&+ 4\|x\|_\infty^2 \int_0^\pi \int_0^\pi b^2(0, \eta, \xi) d\eta d\xi.
\end{aligned}$$

Thus,

$$\|\frac{d}{dt} V(t, x_t)\|^2 \le C(\frac{\partial b}{\partial \theta}, b), \tag{4.6}$$

where $C(\frac{\partial b}{\partial \theta}, b) > 0$ is independent of $t > 0$ and x with $\|x\|_{-r,T} \le R$. This implies that the set

$$\mathcal{U} = \{s \to V(s, x_s) : x \in C([-r, T]; X), \sup_{\theta \in [-r,T]} \|x(\theta)\| \le R\}$$

is equicontinuous from the right side at $t > 0$. The equicontinuity of $\mathcal{U}$ on $\mathbb{R}$ is proved in similar form. Thus, V verifies condition $\mathbf{H_6}$.

Proposition 4.1 *Assume that the above conditions hold and that*

$$l > (N_1 r)^{1/2} \rho (1 + \frac{1}{\sqrt{2}} (2e^{\frac{-1}{2}} + 2)), \tag{4.7}$$

where $\rho = 1 + \|f\|_\infty / l$. Then there exists an ω-periodic solution of (4.1)-(4.3).

Proof: Let $v \in B_\rho(0, CP)$ and $\varphi \in B_\rho(0, \mathcal{C})$. From Theorem 2.1 and the condition $\|(-A)^{1/2} G(t, \cdot)\| \le (N_1 r)^{1/2} < 1$, we know that there exist a local mild solution, $x(\cdot, \varphi)$, of

$$\begin{gathered}
\frac{d}{dt}(y(t) + G(t, y_t)) = Ay(t) + L(t, y_t) + F_1(t, v_t), \quad t \ge 0,\\
y_0 = \varphi.
\end{gathered} \tag{4.8}$$

We claim that $x(\cdot, \varphi)$ is bounded by ρ on $[0, a_\varphi)$, where $[0, a_\varphi)$ is the maximal interval of definition of $x(\cdot, \varphi)$. Assume that the claim is false and let $t_0 =$

$\inf\{t > 0 : \|x(t)\| > \rho\}$. Clearly, $x(t_0) = \rho$. If $t_0 > 1$, by employing the estimates in Ezzinbi [1], pp. 227, we have that

$$\begin{aligned}
\|x(t_0)\| &\leq \rho - l(1-e^{-t_0}) + a_0(t_0)\|V(t_0, x_{t_0})\| \\
&\quad + a_0(t_0)\int_0^{t_0} \|AT(t_0-s)V(s,x_s)\|ds \\
&\leq \rho - l(1-e^{-t_0}) + a_0(t_0)(N_1 r)^{1/2}\rho \\
&\quad + a_0(t_0)(\frac{N_1 r}{2})^{1/2}\int_0^{t_0-1} e^{-\frac{(t_0-s)}{2}}\|x_s\|_{\mathcal{C}}ds \\
&\quad + a_0(t_0)\,(\frac{N_1 r}{2})^{1/2}\int_{t_0-1}^{t_0} \frac{e^{-\frac{(t_0-s)}{2}}}{(t_0-s)^{1/2}}\|x_s\|_{\mathcal{C}}ds \\
&\leq \rho - l(1-e^{-t_0}) + a_0(t_0)(N_1 r)^{1/2}\rho + a_0(t_0)(\frac{N_1 r}{2})^{1/2}\rho(2e^{-\frac{1}{2}}+2),
\end{aligned}$$

thus,

$$\|x(t_0)\| \leq \rho - \big(l - (N_1 r)^{1/2}\rho(1+\frac{2}{\sqrt{2}}(e^{-\frac{1}{2}}+1)\big)(1-e^{-t_0}). \tag{4.9}$$

Similarly, if $t_0 \in (0,1]$

$$\|x(t_0)\| \leq \rho - \big(l - (N_1 r)^{1/2}\rho(1+\frac{2}{\sqrt{2}})\big)(1-e^{-t_0}). \tag{4.10}$$

From (4.7), (4.9) and (4.10), it follows that $\|x(t_0)\| < \rho$, which is a contradiction.

Now we prove that $a_\varphi = \infty$. Assume that $a_\varphi < \infty$ and let N_2 be the number $N_2 = \rho(|a_1|_\infty + |a_2|_\infty k) + \|f\|_\infty$. For $\epsilon > 0$, we fix $0 < \delta < \frac{a_\varphi}{2}$ such that

$$\|T(s)x - T(s')x\| < \epsilon, \quad x \in T(\epsilon)B_{2\rho+N_2}[0,X],$$

when $|s-s'| < \delta$ and $s, s' \in [0, a_\varphi]$. Let $\frac{a_\varphi}{2} < t < t_0 < a_\varphi$. Using that $a_0(t) \leq 1$ and the estimate in step 2 of the proof of Theorem 2.3, we have

$$\begin{aligned}
\|x(t_0) - x(t)\| &\leq \epsilon + \|G(t_0, x_{t_0}) - G(t, x_t)\| + 2\epsilon\frac{1}{\sqrt{2}}(t_0-\epsilon)^{1/2} \\
&\quad + 4(\frac{N_1 r}{\sqrt{2}})^{1/2}\rho\epsilon^{1/2} + 2(\frac{N_1 r}{\sqrt{2}})^{1/2}\rho(t-t_0)^{1/2} + \epsilon(t_0-\epsilon) \\
&\quad + 2N_2\epsilon + N_2(t-t_0)
\end{aligned}$$

which from (4.6), allows us to conclude that $x(\cdot)$ is uniformly continuous on $[a_\varphi/2, a_\varphi)$. Let $\tilde{x} : [-r, a_\varphi] \to X$ be the unique continuous extension of $x(\cdot)$. From Theorem 2.2, there exists a mild solution, $y(\cdot)$, of (4.4) with initial condition $y_{a_\varphi} = \tilde{x}_{a_\varphi}$. This solution give an extension of $x(\cdot, \varphi)$, which is a contradiction. Thus $x(\cdot, \varphi)$ is defined on $\mathbb{R}$.

From Theorem 2.5, we infer that for every $v \in B_\rho[0, CP]$ there exists an ω-periodic, $u(\cdot, v)$, of (4.8) and that $u(\cdot, v) \in B_\rho[0, CP]$. Finally, the existence of an ω-periodic solution of (4.4) follows from Theorem 2.6. The proof is complete.

A neutral equation with unbounded delay

Next we consider the boundary-value problem

$$\begin{aligned} \frac{\partial}{\partial t}&[u(t,\xi)+\int_{-\infty}^{t}\int_{0}^{\pi} b(s-t,\eta,\xi)u(s,\eta)d\eta ds] \\ &= \frac{\partial^2}{\partial \xi^2}u(t,\xi)+a_0(\xi)u(t,\xi)+\int_{-\infty}^{t} a(s-t)u(s,\xi)\,ds + a_1(t,\xi), \end{aligned} \tag{4.11}$$

$$u(t,0)=u(t,\pi)=0, \quad t\geq 0, \tag{4.12}$$

$$u(\tau,\xi)=\varphi(\tau,\xi), \quad \tau\leq 0, \quad 0\leq\xi\leq\pi, \tag{4.13}$$

which is studied in [8]. Let $\mathcal{B} := C_r \times L^2(g;X)$, $r=0$, be the phase space studied in Example 3.1. In this case, $H=1$; $M(t)=\gamma(-t)^{1/2}$ and $K(t)=1+(\int_{-t}^{0} g(\tau)d\tau)^{1/2}$ for all $t\geq 0$. Assuming the conditions (**i**)-(**iii**) of Example 3.1 in [8], this problem can be written as

$$\begin{gathered} \frac{d}{dt}(x(t)+G(t,x_t)) = Ax(t)+F(t,x_t)+f(t), \\ x_0=\varphi\in\mathcal{B}, \end{gathered}$$

where

$$\begin{gathered} G(t,\psi)(\xi) := \int_{-\infty}^{0}\int_{0}^{\pi} b(s,\,\eta,\,\xi)\psi(s,\,\eta)d\eta ds, \\ F(t,\psi)(\xi) := a_0(\xi)\psi(0,\xi)+\int_{-\infty}^{0} a(s)\psi(s,\xi)ds, \\ f(t) := a_1(t,\cdot). \end{gathered}$$

Moreover, $F(t,\cdot)$ and $G(t,\cdot)$ are bounded linear operators, the range of G is contained in $X_{\frac{1}{2}}$, $\|(-A)^{1/2}G(t,\cdot)\|\leq N_1^{1/2}$ and $\|F(t,\cdot)\|\leq N_2$ where

$$\begin{gathered} N_1 := \int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \frac{1}{g(s)}\big(\frac{\partial}{\partial\zeta}b(s,\eta,\zeta)\big)^2 d\eta ds d\zeta, \\ N_2 := \max\{\,\|a_0\|_\infty, \big(\int_{-\infty}^{0}\frac{a^2(\theta)}{g(\theta)}d\theta\big)^{1/2}\}. \end{gathered}$$

Next we assume that the function $g(\cdot)$ verifies the conditions:

($\mathbf{g_i}$) $\ln(g)$ is uniformly continuous,

($\mathbf{g_{ii}}$) $k_1=\int_{-\infty}^{0} g(\theta)d\theta<\infty$,

($\mathbf{g_{iii}}$) the function $\gamma(\cdot)$ is bounded on $(-\infty,0]$.

Under these conditions, the functions $K(\cdot), M(\cdot)$ are bounded. In the following result we use the symbol K for $\sup_{s\geq 0} K(s)$.

Theorem 4.2 *Assume that the above conditions hold and that*

$$K\Big[N_1 + \frac{2N_1}{\sqrt{2}}e^{-\frac{1}{2}} + \frac{N_1}{\sqrt{2}}\int_0^1 e^{-\frac{s}{2}}s^{-\frac{1}{2}}ds + N_2\Big] < 1. \tag{4.14}$$

If f is continuous and ω-periodic, then there exists an ω-periodic solution of (4.11).

Proof: Using similar estimates that those in the section 4, it follows that condition $\mathbf{H_7}$ holds. From Lemmas 3.1, 3.2 and Proposition 3.4 in [8] we know that each mild solution of (4.11) is bounded on $[0,\infty)$. The existence of an ω-periodic solution for (4.11)-(4.12)-(4.13) is now consequence of Theorem 3.3. The proof is complete.

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