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# Positive solutions of nonlinear elliptic equations in a half space in $\mathbb{R}^2$ *

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**Abstract**

We study the existence and the asymptotic behaviour of positive solutions of the nonlinear equation $\Delta u + f(., u) = 0$, in the domain $D = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$, with $u = 0$ on the boundary. The aim is to prove some existence results for the above equation in a general setting by using a fixed-point argument.

## 1 Introduction

In [12], Zeddini considered the nonlinear elliptic problem

$$\begin{aligned} \Delta u \ + \ f(., u) = 0 \quad & \text{in } D \\ u > 0 \quad & \text{in } D \\ u = 0 \quad & \text{on } \partial D, \end{aligned} \tag{1.1}$$

in the sense of distributions, where $D$ is the outside of the unit disk in $\mathbb{R}^2$ and $f$ is a nonnegative function in $D \times (0, \infty)$ non-increasing with respect to the second variable. Then, when $f$ is in a certain Kato class, he proved the existence of infinitely many positive continuous solutions on $\overline{D}$. More precisely, he showed that for each $b > 0$, there exists a positive continuous solution $u$ satisfying

$$\lim_{|x| \to \infty} \frac{u(x)}{Log|x|} = b.$$

Note that the existence results of problem (1.1) have been extensively studied for the special nonlinearity $f(x, t) = p(x)q(t)$, for both bounded and unbounded domain $D$ in $\mathbb{R}^n (n \geq 1)$, with smooth compact boundary (see for example [3, 4, 5, 6] and the references therein). On the other hand, in [7, 9, 10, 11], the authors considered the problem

$$\begin{aligned} \Delta u \ + \ g(., u) = 0 \quad & \text{in } D \\ u > 0 \quad & \text{in } D \\ u = 0 \quad & \text{on } \partial D, \end{aligned} \tag{1.2}$$

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where there is no restriction on the sign of $g$, and $D$ is an unbounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a compact Lipschitz boundary. Then they proved the existence of infinitely many solutions provided that $g$ is in a certain Kato class. Namely, they showed that there exists a number $b_0 > 0$ such that for each $b \in (0, b_0]$, there exists a positive continuous solution $u$ in $\overline{D}$ satisfying

$$\lim_{|x|\to\infty} \frac{u(x)}{h(x)} = b,$$

where $h$ is a positive solution of the homogeneous Dirichlet problem $\triangle u = 0$ in $D$, $u = 0$ on $\partial D$.
In this paper, we consider the domain

$$D = \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\},$$

which has a non-compact boundary. The purpose of this paper is two-folded. One is to introduce a new Kato class $K$ of functions on $D$ and to study the properties of this class. The other is to investigate the existence of positive continuous solutions on (1.1) and (1.2). Indeed, we shall establish some existence theorems for problems (1.1) and (1.2), when $f$ and $g$ are required to satisfy suitable assumptions related to the class $K$. Note that solutions of these problems are understood as distributional solutions in $D$.

The outline of the paper is as follows. In section 2, we prove some inequalities on the Green's function $G(x,y) = \frac{1}{4\pi} Log(1 + \frac{4x_2y_2}{|x-y|^2})$ of the Laplacian in $D$. In particular, we establish the fundamental inequality

$$\frac{G(x,y)G(y,z)}{G(x,z)} \leq C_0 \big[\frac{y_2}{x_2} G(x,y) + \frac{y_2}{z_2} G(y,z)\big]$$

which is called the 3G-Theorem. This enable us to define and study, in section 3, a new Kato class $K$ on $D$.

**Definition** A Borel measurable function $\varphi$ in $D$ belongs to the class $K$ if $\varphi$ satisfies

$$\lim_{\alpha\to 0} \sup_{x\in D} \int_{(|x-y|\leq\alpha)\cap D} \frac{y_2}{x_2} G(x,y)|\varphi(y)|dy = 0\,, \tag{1.3}$$

$$\lim_{M\to\infty} \sup_{x\in D} \int_{(|y|\geq M)\cap D} \frac{y_2}{x_2} G(x,y)|\varphi(y)|\,dy = 0\,. \tag{1.4}$$

To study Problem (1.1) in section 4, we assume that $f$ satisfies:

(H1) $f : D \times (0,\infty) \to [0,\infty)$ is measurable, continuous and non-increasing with respect to the second variable.

(H2) For all $c > 0$, $f(.,c) \in K$.

(H3) For all $c > 0$, $V(f(.,c)) > 0$, where $V = (-\Delta)^{-1}$ is the potential kernel associated to $\Delta$.

As usual, we denote by $\mathcal{B}(D)$ the set of Borel measurable functions in $D$ and $\mathcal{B}^+(D)$ the set of nonnegative functions. $C(D)$ will denote the set of continuous functions in $D$ and

$$C_0(D) = \{v \in C(D) : \lim_{x\to\partial D} v(x) = \lim_{|x|\to\infty} v(x) = 0\}.$$

Throughout this paper, the letter $C$ will denote a generic positive constant which may vary from line to line.

**Theorem 1.1** *Assume (H1)-(H3). Then for each $b > 0$, the problem (1.1) has at least one positive solution $u$ continuous on $\overline{D}$ and satisfying*

$$\lim_{x_2\to\infty} \frac{u(x)}{x_2} = b.$$

*Moreover, we have for $x$ in $D$,*

$$bx_2 \le u(x) \le bx_2 + \min\big(\delta, \int_D G(x,y)f(y, by_2)dy\big),$$

*where $\delta = \inf_{\alpha>0}(\alpha + \|Vf(.,\alpha)\|_\infty)$.*

**Theorem 1.2** *Assume (H1)-(H3). Then the problem (1.1) has a unique solution $u \in C_0(D)$, satisfying*

$$\frac{x_2}{C(|x|+1)^2} \le u(x) \le \min(\delta, \int_D G(x,y)f(y, \frac{y_2}{C(|y|+1)^2})dy), \quad \forall x \in D.$$

We point out, that for some functions $f$ of the type $f(x,t) = p(x)t^{-\sigma}$, with $\sigma \ge 0$, we get better estimates on the solution. Namely for each $x \in D$, we have

$$u(x) \le C\frac{x_2^{\frac{1}{1+\sigma}}}{(|x|+1)^{\frac{2}{1+\sigma}}},$$

for some positive constant $C$.

In section 5, we consider Problem (1.2) under the following hypotheses:

(A1) The function $g$ is measurable on $D \times (0,\infty)$, continuous with respect to the second variable and satisfies

$$|g(x,t| \le t\psi(x,t) \quad \text{for } (x,t) \in D \times (0,\infty),$$

where $\psi$ is a nonnegative measurable function on $D \times (0,\infty)$ such that the function $t \to \psi(x,t)$ is nondecreasing on $(0,\infty)$ and $\lim_{t\to 0}\psi(x,t) = 0$.

(A2) The function defined as $x \to \psi(x, x_2)$ on $D$ belongs to the class $K$.

**Theorem 1.3** *Assume (A1)-(A2). Then (1.2) has infinitely many solutions. More precisely, there exists $b_0 > 0$ such that for each $b \in (0, b_0]$, there exists a solution $u$ of (1.2) continuous on $D$ and satisfying*

$$\frac{b}{2}x_2 \le u(x) \le \frac{3b}{2}x_2 \quad \text{and} \quad \lim_{x_2\to\infty}\frac{u(x)}{x_2} = b.$$

## 2 Properties of Green's function

**Lemma 2.1** *For $x$ and $y$ in $D$, we have the following properties:*

- *(i) If $x_2y_2 \leq |x-y|^2$, then $\max(x_2, y_2) \leq \frac{\sqrt{5}+1}{2}|x-y|$.*
- *(ii) If $|x-y|^2 \leq x_2y_2$, then $\frac{3-\sqrt{5}}{2}x_2 \leq y_2 \leq \frac{3+\sqrt{5}}{2}x_2$.*

**Proof** (i) If $x_2y_2 \leq |x-y|^2$ then $|y-\widetilde{x}| \geq \frac{\sqrt{5}}{2}x_2$, where $\widetilde{x} = (x_1, \frac{3}{2}x_2)$. It follows that

$$|y-x| \geq |y-\widetilde{x}| - |x-\widetilde{x}| \geq \frac{\sqrt{5}-1}{2}x_2.$$

i.e., $x_2 \leq \frac{\sqrt{5}+1}{2}|x-y|$. Thus, interchange the role of $x$ and $y$, we obtain (i).
(ii) If $|x-y|^2 \leq x_2y_2$ then $|x_2-y_2|^2 \leq x_2y_2$. Hence

$$\big[y_2 - \frac{3+\sqrt{5}}{2}x_2\big]\big[y_2 - \frac{3-\sqrt{5}}{2}x_2\big] \leq 0.$$

**Proposition 2.2** *There exists $C>0$ such that, for all $x$ and $y$ in $D$*

$$\frac{x_2y_2}{C(|x|+1)^2(|y|+1)^2} \leq G(x,y) \leq \frac{1}{\pi}\frac{x_2y_2}{|x-y|^2}. \tag{2.1}$$

$$\frac{1}{\pi}\frac{y_2^2}{|x-y|^2+4x_2y_2} \leq \frac{y_2}{x_2}G(x,y) \leq C(1+G(x,y)). \tag{2.2}$$

**Proof** Recall that the Green's function $G$ of $\Delta$ in $D$ is

$$G(x,y) = \frac{1}{4\pi}Log(1+\frac{4x_2y_2}{|x-y|^2}). \tag{2.3}$$

To prove (2.1) and the first inequality in (2.2), we use that

$$\frac{t}{1+t} \leq Log(1+t) \leq t, \forall t \geq 0, \quad \text{and} \quad |x-y| \leq (|x|+1)(|y|+1), \quad \forall x,y \in D.$$

The second inequality in (2.2) follows from Lemma 2.1. Indeed, if $x_2y_2 \leq |x-y|^2$ then

$$\frac{y_2}{x_2}G(x,y) \leq C\frac{y_2^2}{|x-y|^2} \leq C$$

and if $|x-y|^2 \leq x_2y_2$ then

$$\frac{y_2}{x_2}G(x,y) \leq CG(x,y). \quad \diamondsuit$$

**Theorem 2.3 (3G-Theorem)** *There exists a constant $C_0>0$ such that for all $x$, $y$ and $z$ in $D$, we have*

$$\frac{G(x,z)\,G(z,y)}{G(x,y)} \leq C_0\big[\frac{z_2}{x_2}G(x,z) + \frac{z_2}{y_2}G(y,z)\big]. \tag{2.4}$$

**Proof.** Let $N(x,y)=\frac{x_2y_2}{G(x,y)}$, for $x$ and $y$ in $D$. Then (2.4) is equivalent to

$$N(x,y)\le C_0(N(y,z)+N(z,x)). \tag{2.5}$$

Using the inequalities $\frac{t}{1+t}\le Log(1+t)\le t, \forall t\ge 0$, we deduce by (2.1) and (2.2) that for all $x$ and $y$ in $D$,

$$\pi|x-y|^2\le N(x,y)\le \pi(|x-y|^2+4x_2y_2). \tag{2.6}$$

Then to prove (2.5), we need to consider two cases:
Case i: $x$ and $y$ in $D$ with $x_2y_2\le|x-y|^2$. Then by (2.6), for all $z$ in $D$,

$$N(x,y)\le 5\pi|x-y|^2\le 10\pi(|x-z|^2+|z-y|^2)\le 10(N(x,z)+N(z,y)).$$

Case ii: $x$ and $y$ in $D$ with $|x-y|^2\le x_2y_2$. Then by Lemma 2.1,

$$\frac{3-\sqrt{5}}{2}x_2\le y_2\le\frac{3+\sqrt{5}}{2}x_2.$$

If $|x-z|^2\le x_2z_2$ or $|y-z|^2\le y_2z_2$, then by Lemma 2.1

$$\frac{3-\sqrt{5}}{2}x_2\le z_2\le\frac{3+\sqrt{5}}{2}x_2,\quad\text{or}\quad \frac{3-\sqrt{5}}{2}y_2\le z_2\le\frac{3+\sqrt{5}}{2}y_2.$$

Recall that for all $a$ and $b$ in $(0,\infty)$,

$$\frac{ab}{a+b}\le\min(a,b)\le 2\frac{ab}{a+b},$$

and for all $x,y$ and $z$ in $D$, $|x-y|^2\le 4\max(|x-z|^2,|z-y|^2)$, then in this case we have

$$Log(1+\frac{4x_2y_2}{|x-y|^2})\ge C\min\big[Log(1+\frac{4z_2y_2}{|z-y|^2}),Log(1+\frac{4x_2z_2}{|x-z|^2})\big].$$

Which is equivalent to (2.5).
If $|x-z|^2\ge x_2z_2$ and $|y-z|^2\ge y_2z_2$, then using (2.6) and Lemma 2.1, we obtain

$$\begin{aligned}N(x,y)&\le 5\pi x_2y_2\le C|x-z||y-z|\\&\le C(|x-z|^2+|y-z|^2)\le C(N(x,z)+N(y,z)).\quad\diamondsuit\end{aligned}$$

Now we are ready to study the properties of the functional class $K$.

# 3 The class K.

**Proposition 3.1** *Let $\varphi$ be a function in $K$. Then the function $y\to y_2^2\varphi(y)$ is in $L^1_{\text{loc}}(\overline{D})$.*

**Proof** Since $\varphi \in K$, then by (1.3) there exists $\alpha > 0$ such that

$$\sup_{x\in D}\int_{(|x-y|\le\alpha)\cap D}\frac{y_2}{x_2}G(x,y)|\varphi(y)|\,dy \le 1.$$

Let $R > 0$ and $a_1, \dots, a_n$ in $B(0,R)\cap D$ with $B(0,R)\cap D \subset \cup_{1\le i\le n} B(a_i,\alpha)$. Then by (2.2), there exists $C > 0$ such that for all $i \in \{1,\dots,n\}$ and $y \in B(a_i,\alpha)\cap D$

$$y_2^2 \le C\frac{y_2}{(a_i)_2}G(a_i,y).$$

Hence, we have

$$\begin{aligned}\int_{B(0,R)\cap D} y_2^2|\varphi(y)|dy &\le C\sum_{1\le i\le n}\int_{(|x_i-y|\le\alpha)\cap D}\frac{y_2}{(a_i)_2}G(a_i,y)|\varphi(y)|dy\\ &\le Cn\sup_{x\in D}\int_{(|x-y|\le\alpha)\cap D}\frac{y_2}{x_2}G(x,y)|\varphi(y)|dy\\ &\le Cn<\infty \end{aligned}$$

♢

In the sequel, we use the notation

$$\|\varphi\| = \sup_{x\in D}\int_D \frac{y_2}{x_2}G(x,y)|\varphi(y)|\,dy\,. \tag{3.1}$$

**Proposition 3.2** *If $\varphi \in K$, then $\|\varphi\| < +\infty$.*

**Proof** Let $\alpha > 0$ and $M > 0$. Then we have

$$\begin{aligned}\int_D \frac{y_2}{x_2}G(x,y)\,|\varphi(y)|\,dy &\le \int_{(|x-y|\le\alpha)\cap D}\frac{y_2}{x_2}G(x,y)|\varphi(y)|dy\\ &\quad+\int_{(|y|\ge M)\cap D}\frac{y_2}{x_2}G(x,y)|\varphi(y)|dy\\ &\quad+\int_{(|x-y|\ge\alpha)\cap(|y|\le M)\cap D}\frac{y_2}{x_2}G(x,y)|\varphi(y)|dy.\end{aligned}$$

By (2.3), we have

$$\int_{(|x-y|\ge\alpha)\cap(|y|\le M)\cap D}\frac{y_2}{x_2}G(x,y)|\varphi(y)|dy \le C\int_{B(0,M)\cap D} y_2^2|\varphi(y)|dy.$$

Thus the result follows immediately from (1.3)), (1.4) and Proposition 3.1. ♢

**Proposition 3.3** *Let $\varphi$ be a function in $K$ and $h$ be a positive superharmonic function in $D$.*
*a) For $x_0 \in \overline{D}$,*

$$\lim_{r\to 0}\sup_{x\in D}\frac{1}{h(x)}\int_{B(x_0,r)\cap D} G(x,y)\,h(y)\,|\varphi(y)|\,dy \;=0 \tag{3.2}$$

$$\lim_{M\to+\infty}\sup_{x\in D}\frac{1}{h(x)}\int_{D\cap(|y|\geq M)}G(x,y)\,h(y)\,|\varphi(y)|\,dy=0. \tag{3.3}$$

*b) For all $x\in D$ and $C_0$ as in Theorem 2.3,*

$$\int_D G(x,y)\,h(y)\,|\varphi(y)|\,dy\leq 2C_0\|\varphi\|\,h(x). \tag{3.4}$$

**Proof** Let $h$ be a positive superharmonic function in $D$. Then by [8;Theorem 2.1, p.164], there exists a sequence $(f_n)_n$ of positive measurable functions in $D$ such that

$$h(y)=\sup_n\int_D G(y,z)f_n(z)\,dz\,.$$

Hence, we need only to verify (3.2), (3.3) and (3.4) for $h(y)=G(y,z)$, uniformly for $z\in D$.
a) Let $r>0$. By using Theorem 2.3, we obtain

$$\begin{aligned}&\frac{1}{G(x,z)}\int_{B(x_0,r)\cap D}G(x,y)\,G(y,z)\,|\varphi(y)|\,dy\\ &\leq\ 2C_0\sup_{\xi\in D}\int_{B(x_0,r)\cap D}\frac{y_2}{\xi_2}G(\xi,y)\,|\varphi(y)|\,dy\,.\end{aligned}$$

Let $\alpha>0$ and $M>0$. Then by (2.1), we have

$$\begin{aligned}\int_{B(x_0,r)\cap D}\frac{y_2}{x_2}G(x,y)\,|\varphi(y)|\,dy &\leq\int_{B(x_0,r)\cap D\cap(|x-y|\leq\alpha)}\frac{y_2}{x_2}G(x,y)\,|\varphi(y)|\,dy\\ &\quad+C\int_{B(x_0,r)\cap D\cap(|x-y|\geq\alpha)\cap(|y|\leq M)}y_2^2|\varphi(y)|\,dy\\ &\quad+\int_{B(x_0,r)\cap D\cap(|y|\geq M)}\frac{y_2}{x_2}G(x,y)\,|\varphi(y)|\,dy.\end{aligned}$$

Then (3.2) follows from (1.3), (1.4) and Proposition 3.1. On the other hand, we have

$$\begin{aligned}&\frac{1}{G(x,z)}\int_{(|y|\geq M)\cap D}G(x,y)\,G(y,z)\,|\varphi(y)|\,dy\\ &\leq\ 2C_0\sup_{\xi\in D}\int_{(|y|\geq M)\cap D}\frac{y_2}{\xi_2}G(\xi,y)\,|\varphi(y)|\,dy\end{aligned}$$

which converges to zero as $M\to\infty$. This gives (3.3).
b) By using Theorem 2.3, we obtain

$$\frac{1}{G(x,z)}\int_D G(x,y)\,G(y,z)\,|\varphi(y)|\,dy\leq 2C_0\|\varphi\|.$$

**Corollary 3.4** *Let $\varphi$ be a function in $K$. Then we have*

$$\sup_{x\in D}\int_D G(x,y)\,|\varphi(y)|\,dy<\infty, \tag{3.5}$$

$$\int_D \frac{y_2}{(|y|+1)^2}|\varphi(y)|<\infty, \tag{3.6}$$

$$\int_{D\cap(|y|\le M)} y_2|\varphi(y)|dy<\infty,\quad \forall M>0. \tag{3.7}$$

**Proof** Inequality (3.5) follows from (3.4) with $h=1$ in $D$ and Proposition 3.2. Let $x_0\in D$. Then by (2.1) and (3.5), we have

$$\int_D \frac{y_2}{(|y|+1)^2}|\varphi(y)|dy\le C\frac{(|x_0|+1)^2}{|x_0|}(\sup_{x\in D}\int_D G(x,y)|\varphi(y)|dy)<\infty,$$

which gives (3.6). Inequality (3.7) follows immediately from (3.6).

**Proposition 3.5** *Let $\varphi\in K$. Then the function*

$$V\varphi(x)=\int_D G(x,y)\varphi(y)dy$$

*is defined in $D$ and is in $C_0(D)$.*

**Proof** Let $x_0\in D$ and $r>0$. Let $x,x'\in B(x_0,\frac{r}{2})\cap D$. Then for $M>0$

$$\begin{aligned}
&|V\varphi(x)-V\varphi(x')|\\
&\le \int_D |G(x,y)-G(x',y)||\varphi(y)|dy\\
&\le 2\sup_{\xi\in D}\int_{B(x_0,r)\cap D} G(\xi,y)|\varphi(y)|dy+2\sup_{\xi\in D}\int_{(|y|\ge M)\cap D} G(\xi,y)|\varphi(y)|dy\\
&\quad+\int_{D\cap(|y-x_0|\ge r)\cap(|y|\le M)} |G(x,y)-G(x',y)||\varphi(y)|dy.
\end{aligned}$$

By (2.1), there exists $C>0$ such that for all $x\in B(x_0,\frac{r}{2})\cap D$, for all $y\in B(0,M)\cap(D\backslash B(x_0,r))$,

$$G(x,y)\le Cy_2.$$

Moreover, $G(x,y)$ is continuous on $(x,y)\in(B(x_0,\frac{r}{2})\cap D)\times(D\backslash B(x_0,r))$. Then by (3.7) and Lebesgue's theorem, we have that

$$\int_{D\cap(|y-x_0|\ge r)\cap(|y|\le M)} |G(x,y)-G(x',y)||\varphi(y)|dy\to 0\quad\text{as } |x-x'|\to 0.$$

Hence, we obtain by (3.2) and (3.3) with $h=1$ that $V\varphi$ is continuous in $D$. Now, we will show that

$$\lim_{x\to\partial D}V\varphi(x)=\lim_{|x|\to+\infty}V\varphi(x)=0.$$

Let $x_0 \in \partial D$ and $r > 0$. Let $x \in B(x_0, \frac{r}{2}) \cap D$. Then for $M > 0$,

$$\begin{aligned} |V\varphi(x)| &\leq \int_D G(x,y)\,|\varphi(y)|\,dy \\ &\leq \sup_{\xi \in D} \int_{B(x_0,r)\cap D} G(\xi,y)|\varphi(y)|dy + \sup_{\xi\in D}\int_{(|y|\geq M)\cap D} G(\xi,y)|\varphi(y)|dy \\ &\quad + \int_{D\cap(|y-x_0|\geq r)\cap(|y|\leq M)} G(x,y)|\varphi(y)|dy. \end{aligned}$$

Since

$$\int_{D\cap(|y-x_0|\geq r)\cap(|y|\leq M)} G(x,y)|\varphi(y)|dy \leq Cx_2 \int_{D\cap(|y|\leq M)} y_2|\varphi(y)|dy,$$

then we obtain by (3.7), (3.2) and (3.3) with $h = 1$ that

$$\lim_{x\to\partial D} V\varphi(x) = 0.$$

Let $M > 0$ and $x$ in $D$ such that $|x| \geq M + 1$, then we have

$$\begin{aligned} |V\varphi(x)| &\leq \int_D G(x,y)|\varphi(y)|\,dy \\ &\leq \int_{(|y|\leq M)\cap D} G(x,y)\,|\varphi(y)|\,dy + \int_{(|y|\geq M)\cap D} G(x,y)|\varphi(y)|\,dy\,. \end{aligned}$$

Since $G(x,y) \leq C\frac{x_2 y_2}{(|x|-M)^2}$, for $|y| \leq M$, then from (3.7) and (3.3) with $h = 1$, we deduce that

$$\lim_{|x|\to+\infty} V\varphi(x) = 0$$

**Proposition 3.6** *Let $\lambda$, $\mu$ be in $\mathbb{R}$ and $\theta$ be the function defined on $D$ by*

$$\theta(y) = \frac{1}{(|y|+1)^{\mu-\lambda} y_2^{\lambda}}.$$

*Then $\theta \in K$ if and only if $\lambda < 2 < \mu$.*

**Proof** Let $\lambda < 2 < \mu$ and $\alpha > 0$. Then we have

$$\begin{aligned} I &= \int_{(|x-y|\leq\alpha)\cap D} \frac{y_2}{x_2} Log(1+\frac{4x_2y_2}{|x-y|^2}) \frac{1}{(|y|+1)^{\mu-\lambda}y_2^{\lambda}} dy \\ &\leq \int_{(|x-y|\leq\alpha)\cap D_1} \frac{y_2}{x_2} Log(1+\frac{4x_2y_2}{|x-y|^2}) \frac{1}{(|y|+1)^{\mu-\lambda}y_2^{\lambda}} dy \\ &\quad + \int_{(|x-y|\leq\alpha)\cap D_2} \frac{y_2}{x_2} Log(1+\frac{4x_2y_2}{|x-y|^2}) \frac{1}{(|y|+1)^{\mu-\lambda}y_2^{\lambda}} dy \\ &= I_1 + I_2, \end{aligned}$$

where

$$D_1 = \{y \in D : x_2 y_2 \le |x-y|^2\} \quad \text{and} \quad D_2 = \{y \in D : |x-y|^2 \le x_2 y_2\}.$$

So, using $Log(1+t) \le t$, for $t > 0$ and Lemma 2.1, we obtain

$$\begin{aligned} I_1 &\le \int_{(|x-y|\le\alpha)\cap D_1} \frac{y_2^{2-\lambda}}{|x-y|^2} dy \\ &\le C \int_{(|x-y|\le\alpha)\cap D_1} \frac{1}{|x-y|^\lambda} dy \le C \int_0^\alpha t^{1-\lambda} dt, \end{aligned}$$

which converges to zero as $\alpha \to 0$.
On the other hand, we have from Lemma 2.1, that there is $C > 0$ such that if $y \in D_2$,

$$\frac{1}{C}(|x|+1) \le |y| + 1 \le C(|x|+1).$$

Hence

$$I_2 \le C \frac{1}{x_2^\lambda (|x|+1)^{\mu-\lambda}} \int_{(|x-y|\le\alpha)\cap D_2} Log(1 + \frac{(cx_2)^2}{|x-y|^2}) dy,$$

where $c = 1+\sqrt{5}$. Let $\gamma \in ]\max(0,\lambda), 2[$. Since $Log(1+t^2) \le Ct^\gamma, \forall t \ge 0$, then

$$I_2 \le C \frac{x_2^{\gamma-\lambda}}{(|x|+1)^{\mu-\lambda}} \int_0^{\inf(\alpha, cx_2)} t^{1-\gamma} dt \le C \max(\alpha^{2-\lambda}, \alpha^{2-\gamma}),$$

which converges to zero as $\alpha \to 0$. Now, we will show that

$$\lim_{M\to\infty} \Big( \sup_{x\in D} \int_{(|y|\ge M)} \frac{y_2}{x_2} Log(1 + \frac{4x_2y_2}{|x-y|^2}) \frac{1}{(|y|+1)^{\mu-\lambda} y_2^\lambda} dy \Big) = 0.$$

By the above argument, for $\varepsilon > 0$, there exists $\alpha > 0$ such that

$$\sup_{x\in D} \int_{(|y|\ge M)\cap D\cap(|x-y|\le\alpha)} \frac{y_2}{x_2} Log(1 + \frac{4x_2y_2}{|x-y|^2}) \frac{1}{(|y|+1)^{\mu-\lambda} y_2^\lambda} dy \le \varepsilon.$$

Fixing this $\alpha$ and letting $M > 1$, we have

$$\begin{aligned} &\sup_{x\in D} \int_{(|y|\ge M)\cap D\cap(|x-y|\ge\alpha)} \frac{y_2}{x_2} Log(1 + \frac{4x_2y_2}{|x-y|^2}) \frac{1}{(|y|+1)^{\mu-\lambda} y_2^\lambda} dy \\ &\le \sup_{x\in D} \int_{(|y|\ge M)\cap D\cap(|x-y|\ge\alpha)} \frac{y_2^{2-\lambda}}{|x-y|^2 |y|^{\mu-\lambda}} dy \\ &\le \sup_{|x|\le M/2} \int_{(|y|\ge M)\cap D} \frac{dy}{|x-y|^2 |y|^{\mu-2}} \\ &\quad + \sup_{|x|\ge M/2} \Big[ \int_{(M\vee\frac{|x|}{2}\le|y|\le 2|x|)\cap D\cap(|x-y|\ge\alpha)} \frac{dy}{|x-y|^2|y|^{\mu-2}} \\ &\quad + \int_{(|y|\ge 2|x|)\cap D} \frac{dy}{|x-y|^2|y|^{\mu-2}} \Big] + \sup_{|x|\ge 2M} \int_{(M\le|y|\le\frac{|x|}{2})\cap D} \frac{dy}{|x-y|^2|y|^{\mu-2}} \end{aligned}$$

$$\begin{aligned}
&\le C\Big(\int_{(|y|\ge M)\cap D}\frac{1}{|y|^{\mu}}dy+\sup_{|x|\ge M/2}\frac{Log\frac{3|x|}{\alpha}}{|x|^{\mu-2}}\Big)\\
&\le C\Big(\frac{1}{M^{\mu-2}}+\sup_{|x|\ge M/2}\frac{Log\frac{3|x|}{\alpha}}{|x|^{\mu-2}}\Big),
\end{aligned}$$

which converges to zero as $M\to\infty$. Conversely, if $\theta\in K$ then we have by Proposition 3.5 that

$$\lim_{x_2\to 0}V\theta(x)=\lim_{x_2\to+\infty}V\theta(x)=0,\quad \text{for } x=(0,x_2).$$

On the other hand, it follows from Lemma 2.1 that

$$\begin{aligned}
V\theta(x) &= \frac{1}{4\pi}\int_D Log\Big(1+\frac{4x_2y_2}{|x-y|^2}\Big)\frac{1}{(|y|+1)^{\mu-\lambda}y_2^{\lambda}}dy\\
&\ge C\int_{D\cap(|x-y|^2\le x_2y_2)}\frac{1}{(|y|+1)^{\mu-\lambda}y_2^{\lambda}}dy\\
&\ge C\frac{1}{x_2^{\lambda}(|x|+1)^{\mu-\lambda}}\int_{|\widetilde{x}-y|\le\frac{\sqrt{5}}{2}x_2}dy\ge C\frac{x_2^{2-\lambda}}{(x_2+1)^{\mu-\lambda}},
\end{aligned}$$

where $\widetilde{x}=(0,\frac{3}{2}x_2)$. Hence, it is necessary that $\lambda<2<\mu$. $\diamondsuit$
Moreover, we have the following estimates.

**Proposition 3.7** *There exists $C>0$ such that for all $x$ in $D$, we have*

$$V\theta(x)\le C\frac{x_2^{\mu-2}}{(|x|+1)^{2\mu-4}},\quad \text{if } 2<\mu<\min(3,4-\lambda) \tag{3.8}$$

$$V\theta(x)\le C\frac{x_2}{(|x|+1)^{2}},\quad \text{if } \lambda<1 \text{ and } \mu>3 \tag{3.9}$$

$$V\theta(x)\le C\frac{x_2}{(|x|+1)^{2}}Log\Big(\frac{(|x|+1)^2}{x_2}\Big),\quad \text{if } \begin{cases}\lambda<1\\ \mu=3\end{cases} \text{ or } \begin{cases}\lambda=1\\ \mu\ge 3\end{cases} \tag{3.10}$$

$$V\theta(x)\le C\frac{x_2^{2-\lambda}}{(|x|+1)^{4-2\lambda}},\quad \text{if } 1<\lambda<2 \text{ and } \mu\ge 4-\lambda. \tag{3.11}$$

For the proof, we need the following lemma.

**Lemma 3.8** *Let $\lambda<2$, $B:=\{x\in\mathbb{R}^2,|x|<1\}$, and*

$$w(x)=\int_B G_B(x,y)\frac{1}{(1-|y|)^{\lambda}}dy,\quad \text{for } x\in B,$$

*where $G_B$ is the Green's function of $\Delta$ in $B$. Then for each $x\in B$,*

1) $w(x)\le C(1-|x|)$, *if* $\lambda<1$
2) $w(x)\le C(1-|x|)Log(\frac{2}{1-|x|})$, *if* $\lambda=1$
3) $w(x)\le C(1-|x|)^{2-\lambda}$, *if* $1<\lambda<2$.

**Proof** Since

$$G_B(x,y) = \frac{1}{4\pi} Log(1 + \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}),$$

and the function $w$ is radial, then by elementary calculus we have

$$w(x) = C \int_0^1 Log(\frac{1}{r \vee |x|}) \frac{r}{(1-r)^\lambda} dr,$$

where $r \vee |x| = \max(r, |x|)$. Since $tLog(\frac{1}{t}) \leq 1-t, \forall t \in [0,1]$, then we have

$$w(x) \leq C \int_0^1 \frac{1-(r \vee |x|)}{(1-r)^\lambda} dr.$$

Hence, if $|x| \leq \frac{1}{2}$ then

$$w(x) \leq C \int_0^1 (1-r)^{1-\lambda} dr < \infty,$$

and if $|x| \geq \frac{1}{2}$ then

$$\begin{aligned} w(x) &\leq C[(1-|x|)(\int_0^{\frac{1}{2}} \frac{1}{(1-r)^\lambda} dr + \int_{\frac{1}{2}}^{|x|} \frac{1}{(1-r)^\lambda} dr) + \int_{|x|}^1 (1-r)^{1-\lambda} dr] \\ &\leq C[(1-|x|) + (1-|x|) \int_{\frac{1}{2}}^{|x|} \frac{1}{(1-r)^\lambda} dr + (1-|x|)^{2-\lambda}]. \end{aligned}$$

Which implies the result. ♢

**Proof of Proposition 3.7** Let $\gamma : D \to B$ be the Möbius transformation defined by $\gamma(x) = x^* = e - \frac{2(x+e)}{|x+e|^2}$, where $e = (0,1)$. Then for $x, y \in D$,

$$G(x,y) = G_B(x^*, y^*).$$

On the other hand, it is easy to see that

$$\frac{1}{\sqrt{2}}(|x|+1) \leq |x+e| \leq (|x|+1), \forall x \in D. \tag{3.12}$$

Since for $x \in D$, we have $1 - |x^*|^2 = \frac{4x_2}{|x+e|^2}$, then by (3.12) we obtain that

$$\frac{2x_2}{(|x|+1)^2} \leq \delta_B(x^*) = 1 - |x^*| \leq \frac{8x_2}{(|x|+1)^2}. \tag{3.13}$$

It follows that

$$V\theta(x) \leq C \int_D G_B(x^*, y^*) \frac{1}{(|y|+1)^{\mu+\lambda}(\delta_B(y^*))^\lambda} dy$$

$$\leq \quad C \int_B G_B(x^*, \xi) \frac{1}{|\xi - e|^{4-\mu-\lambda}} \frac{1}{(\delta_B(\xi))^\lambda} d\xi.$$

Since $1 - |\xi| \leq |\xi - e| \leq 2, \forall \xi \in B$, we have

$$V\theta(x) \leq C \int_B G_B(x^*, \xi) \frac{1}{(\delta_B(\xi))} d\xi, \text{ if } 4 - \mu - \lambda \leq 0$$

and

$$V\theta(x) \leq C \int_B G_B(x^*, \xi) \frac{1}{(\delta_B(\xi))^{4-\mu}} d\xi, \text{ if } 4 - \mu - \lambda > 0.$$

Thus the required inequalities follow from Lemma 3.8 and (3.13).

# 4 Proofs of Theorems 1.1 and 1.2

For this section, we need some preliminary results. Recall that the potential kernel $V$ is defined on $B^+(D)$ by

$$V\phi(x) = \int_D G(x, y)\phi(y)dy, \, x \in D.$$

Hence, for $\phi \in B^+(D)$ such that $\phi \in L^1_{\rm loc}(D)$ and $V\phi \in L^1_{\rm loc}(D)$, we have in the distributional sense that $\Delta(V\phi) = -\phi$, in $D$. We point out if $V\phi \neq \infty$, we have $V\phi \in L^1_{\rm loc}(D)$, (see [1], p.51). Let us recall that $V$ satisfies the complete maximum principle, i.e for each $\phi \in \mathcal{B}^+(D)$ and $v$ a nonnegative superharmonic function on $D$ such that $V\phi \leq v$ in $\{\phi > 0\}$ we have $V\phi \leq v$ in $D$, (cf. [8], Theorem 3.6, p.175]).

**Lemma 4.1** *Let $h \in \mathcal{B}^+(D)$ and $v$ be a nonnegative superharmonic function on $D$. Then for all $w \in B(D)$ such that $V(h|w|) < \infty$ and $w + V(hw) = v$, we have $0 \leq w \leq v$.*

**Proof** We denote by $w^+ = \max(w, 0)$ and $w^- = \max(-w, 0)$. Since V(h$|w|$) $<$ $\infty$, then we have

$$w^+ + V(hw^+) = v + w^- + V(hw^-).$$

Hence

$$V(hw^+) \leq v + V(hw^-) \quad \text{in } \{w^+ > 0\}.$$

Since $v + V(hw^-)$ is a nonnegative superharmonic function in $D$, then we have as consequence of the complete maximum principle that

$$V(hw^+) \leq v + V(hw^-) \quad \text{in } D,$$

that is $V(hw) \leq v = w + V(hw)$. This implies that $0 \leq w \leq v$.

**Theorem 4.2** *Assume (H1)-(H3). Let $\alpha > 0$ and $b > 0$. Then the problem*

$$(P_\alpha) \qquad \begin{aligned} \Delta u + f(., u) = 0 \quad &\text{in } D \\ u > 0 \quad &\text{in } D \\ u = \alpha \quad &\text{on } \partial D \end{aligned}$$

*has at least one positive solution $u_\alpha \in C(\overline{D})$ satisfying*

$$\lim_{x_2 \to \infty} \frac{u_\alpha(x)}{x_2} = b.$$

**Proof** Let $\alpha > 0$. It follows from (H2) and Proposition 3.5 that $V(f(., \alpha)) \in C_0(D)$. So, in the sequel, we denote

$$\beta = \alpha + \|V(f(., \alpha))\|_\infty.$$

To apply a fixed-point argument, we consider the convex set

$$F = \{w \in C(\overline{D} \cup \{\infty\}) : \alpha \leq w(x) \leq \beta, \ \forall x \in D\}.$$

and on this set we define the integral operator

$$Tw(x) = \alpha + \frac{\alpha}{\alpha + bx_2} \int_D G(x, y) f(y, \frac{(\alpha + by_2)}{\alpha} w(y)) dy, \quad x \in D.$$

By (H1), we have

$$f(y, \frac{(\alpha + by_2)}{\alpha} w(y)) \leq f(y, \alpha), \forall w \in F. \tag{4.1}$$

Then for $w \in F$

$$\alpha \leq Tw(x) \leq \beta \quad \forall x \in D.$$

As in the proof of Proposition 3.5 we show that the family $TF$ is equicontinuous in $\overline{D} \cup \{\infty\}$. In particular, for all $v \in F$, $Tw \in C(\overline{D} \cup \{\infty\})$ and so $TF \subset F$. Moreover, the family $\{Tw(x), w \in F\}$ is uniformly bounded in $\overline{D} \cup \{\infty\}$. It follows by Ascoli's theorem that $TF$ is relatively compact in $C(\overline{D} \cup \{\infty\})$. Next, we prove the continuity of $T$ in $Y$. We consider a sequence $(w_n)$ in $F$ which converges uniformly to a function $w$ in $F$. Then we have

$$\begin{aligned} &|Tw_n(x) - Tw(x)| \\ &\leq \frac{\alpha}{\alpha + bx_2} \int_D G(x, y) |f(y, \frac{(\alpha + by_2)}{\alpha} w_n(y)) - f(y, \frac{(\alpha + by_2)}{\alpha} w(y))| dy. \end{aligned}$$

Since $f$ is continuous with respect to the second variable, we deduce by (4.1), (H2), (3.5) and the Lebesgue's theorem that for each $x \in \overline{D} \cup \{\infty\}$

$$Tw_n(x) \to Tw(x) \quad \text{as } n \to \infty.$$

Since $TY$ is a relatively compact family in $C(\overline{D} \cup \{\infty\})$, we have the uniform convergence, namely

$$\|Tw_n - Tw\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Thus we have proved that $T$ is a compact mapping from $F$ to itself. Hence, by the Schauder's fixed point-theorem, there exists $w_\alpha \in F$ such that

$$w_\alpha(x) = \alpha + \frac{\alpha}{\alpha + bx_2} \int_D G(x,y) f(y, \frac{(\alpha + by_2)}{\alpha} w_\alpha(y)) dy, \forall x \in D.$$

Put $u_\alpha(x) = \frac{(\alpha+bx_2)}{\alpha} w_\alpha(x)$, for $x \in D$. Then we have

$$u_\alpha(x) = \alpha + bx_2 + \int_D G(x,y) f(y, u_\alpha(y)) dy, \forall x \in D. \tag{4.2}$$

By (H1), we have for each $y \in D$,

$$f(y, u_\alpha(y)) \le f(y, \alpha). \tag{4.3}$$

Then we deduce by (H2) and Proposition 3.1 that the map $y \to f(y, u_\alpha(y)) \in L^1_{\text{loc}}(D)$, and by Proposition 3.5, that $V(f(., u_\alpha)) \in C_0(D) \subset L^1_{\text{loc}}(D)$. Apply $\Delta$ on both sides of equality (4.2), we obtain that

$$\Delta u_\alpha + f(., u_\alpha) = 0 \quad \text{in } D \text{ (in the sense of distributions)}.$$

Furthermore, it follows from (4.2) that

$$\alpha + bx_2 \le u_\alpha(x) \le \beta + bx_2, \ \forall x \in D. \tag{4.4}$$

Hence

$$\lim_{x_2 \to \infty} \frac{u_\alpha(x)}{x_2} = b.$$

Now, using (4.3), (H2), Proposition 3.5 and (4.2), we obtain $\lim_{x \to \partial D} u_\alpha(x) = \alpha$. Then, $u_\alpha$ is a positive continuous solution of the problem $(P_\alpha)$.

**Proposition 4.3** *Let $f : D \times (0, \infty) \to [0, \infty)$ be a measurable function satisfying (H1) and $\alpha_1, \alpha_2, b_1, b_2$ be real numbers such that $0 \le \alpha_1 \le \alpha_2$ and $0 \le b_1 \le b_2$. If $u_1$ and $u_2$ are two positive functions continuous on $D$ satisfying for each $x$ in $D$*

$$u_1(x) = \alpha_1 + b_1 x_2 + V(f(., u_1))(x),$$
$$u_2(x) = \alpha_2 + b_2 x_2 + V(f(., u_2))(x).$$

*Then*

$$0 \le u_2(x) - u_1(x) \le \alpha_2 - \alpha_1 + (b_2 - b_1) x_2, \quad \forall x \in D.$$

**Proof** Let $h$ be the function defined on $D$ as

$$h(x) = \begin{cases} \frac{f(x,u_1(x))-f(x,u_2(x))}{u_2(x)-u_1(x)} & \text{if } u_1(x) \neq u_2(x) \\ 0 & \text{if } u_1(x) = u_2(x). \end{cases}$$

Then $h \in B^+(D)$ and

$$u_2(x) - u_1(x) + V(h(u_2 - u_1))(x) = \alpha_2 - \alpha_1 + (b_2 - b_1)x_2.$$

Now, since

$$V(h|u_2 - u_1|) \leq V(f(.,u_1)) + V(f(.,u_2)) \leq u_1 + u_2 < \infty,$$

we deduce the result from Lemma 4.1.

**Proof of Theorem 1.1** Let $(\alpha_n)$ be a sequence of positive real numbers, non-increasing to zero. For each $n \in \mathbb{N}$, we denote by $u_n$ the continuous solution of the problem given by the integral equation (4.2) with $\alpha = \alpha_n$. Then, by Proposition 4.3, the sequence $(u_n)$ decreases to a function $u$. Since

$$u_n(x) - \alpha_n = bx_2 + \int_D G(x,y)f(y,u_n(y))dy \geq bx_2 > 0. \tag{4.5}$$

Then the sequence $(u_n - \alpha_n)$ increases to $u$ and so $u > 0$ in $D$. Hence,

$$u = \inf_n u_n = \sup_n (u_n - \alpha_n)$$

is a positive continuous function in $D$. Using (H1) and applying the monotone convergence theorem, we get

$$u(x) = bx_2 + \int_D G(x,y)f(y,u(y))dy\,, \quad \forall x \in D. \tag{4.6}$$

Then, it follows from (4.6) that $V(f(.,u)) \in L^1_{\rm loc}(D)$. On the other hand, since $u$ is positive in $D$, then by (H2) and Proposition 3.1, the function $y \to f(y,u(y)) \in L^1_{\rm loc}(D)$. Applying $\Delta$ on both sides of equality (4.6), we conclude that $u$ satisfies

$$\Delta u + f(.,u) = 0 \quad \text{in } D.$$

Since for $x$ in $D$ and $n$ in $\mathbb{N}$,

$$0 \leq u_n(x) - \alpha_n \leq u(x) \leq u_n(x) \quad \text{and} \quad \lim_{x_2 \to \infty} \frac{u_n(x)}{x_2} = b,$$

we deduce that

$$\lim_{x \to \partial D} u(x) = 0 \quad \text{and} \quad \lim_{x_2 \to \infty} \frac{u(x)}{x_2} = b.$$

Thus, $u \in C(\overline{D})$ and u is a positive solution of the problem(1.1). Now, let

$$\delta = \inf_{\alpha > 0} (\alpha + \|Vf(.,\alpha\|_\infty).$$

Then by (H3) and (H1), $\delta > 0$. By (4.4) we have that

$$bx_2 \le u(x) \le bx_2 + \delta.$$

By (H1) and (4.6),

$$bx_2 \le u(x) \le bx_2 + \int_D G(x,y)f(y,by_2)dy.$$

Which implies that

$$bx_2 \le u(x) \le bx_2 + \min(\delta, \int_D G(x,y)f(y,by_2)dy).$$

**Corollary 4.4** *Let $0 < b_1 \le b_2$ and $f_1$ and $f_2$ be two nonnegative measurable functions in $D \times (0,\infty)$, satisfying the hypotheses (H1)-(H3), such that $0 \le f_1 \le f_2$. If we denote by $u_j \in C(D)$ the positive solution of the problem (1.1) with $f = f_j$ and $b = b_j$, $j \in \{1,2\}$, given by* (4.6)*, then we have*

$$0 \le u_2 - u_1 \le (b_2 - b_1)x_2 + V(f_2(.,u_2) - f_1(.,u_2)) \quad \textit{in } D.$$

**Proof** It follows from (4.6) that

$$u_1 = b_1 x_2 + V(f_1(.,u_1)) \quad \text{and} \quad u_2 = b_2 x_2 + V(f_2(.,u_2)).$$

Let $h$ be the nonnegative measurable function defined on $D$ by

$$h(x) = \begin{cases} \frac{f_1(x,u_2(x)) - f_1(x,u_1(x))}{u_1(x) - u_2(x)} & \text{if } u_1(x) \ne u_2(x) \\ 0 & \text{if } u_1(x) = u_2(x). \end{cases}$$

Then $h \in B^+(D)$ and we have

$$u_2 - u_1 + V(h(u_2 - u_1)) = (b_2 - b_1)x_2 + V(f_2(.,u_2) - f_1(.,u_2)).$$

Now, since

$$\begin{aligned} V(h|u_2 - u_1|) &\le V(f_1(.,u_2)) + V(f_1(.,u_1)) \\ &\le V(f_2(.,u_2)) + V(f_1(.,u_1)) \\ &= u_2 + u_1 < \infty \end{aligned}$$

and $(b_2 - b_1)x_2 + V(f_2(.,u_2) - f_1(.,u_2))$ is a nonnegative superharmonic function on $D$, we deduce the result from Lemma 4.1. ♢

**Example** Let $\sigma > 0$, $\lambda < 1 - \sigma$ and $\mu > \max(2, 3 - \sigma)$. Suppose that the function $f$ satisfies (H1), (H3) and such that

$$f(y,t) \le \frac{1}{(|y|+1)^{\mu-\lambda} y_2^{\lambda} t^{\sigma}}.$$

Then for each $b > 0$, there exists $C > 0$ such that the problem

$$\begin{gathered}\Delta u + f(.,u) = 0 \quad \text{in } D \\ u > 0 \quad \text{in } D \\ u = 0 \quad \text{on } \partial D \\ \lim_{x_2 \to \infty} \frac{u(x)}{x_2} = b,\end{gathered}$$

has a continuous solution $u$ in $D$ satisfying

$$bx_2 \le u(x) \le Cx_2\,, \quad \forall x \in D.$$

**Proof of Theorem 1.2** Let $\alpha > 0$ and $(b_n)$ be a sequence of positive real numbers, non-increasing to zero. If $u_{\alpha,n}$ denotes the positive continuous solution of the problem $(P_\alpha)$ given by (4.2) for $b = b_n$, then for each $x$ in $D$

$$u_{\alpha,n}(x) = \alpha + b_n x_2 + \int_D G(x,y) f(y, u_{\alpha,n}(y))dy, \tag{4.7}$$

and if $u_n$ denotes the positive continuous solution of the problem (1.1) given by (4.6) for $b = b_n$, then

$$u_n(x) = b_n x_2 + \int_D G(x,y) f(y, u_n(y))dy, \forall x \in D. \tag{4.8}$$

By Proposition 4.3, the sequence $(u_n)$ decreases to a function $u$ and by (H1) the sequence $(u_n - b_n x_2)$ increases to $u$. Then $u$ is a positive continuous function in $D$. Using the monotone convergence theorem, we deduce that $u$ satisfies

$$u(x) = \int_D G(x,y) f(y, u(y))dy, \quad \forall x \in D. \tag{4.9}$$

Moreover, from Proposition 4.3 and (4.3), we have

$$u(x) \le u_{\alpha,n}(x) \le \alpha + Vf(.,\alpha)(x), \quad \forall x \in D. \tag{4.10}$$

Then it follows from Proposition 3.5 that

$$\lim_{x \to \partial D} u(x) = \lim_{|x| \to \infty} u(x) = 0\,.$$

Now, by (4.10) and (H1), we have

$$\int_D G(x,y) f(y,\delta)dy \le u(x) \le \delta \quad \forall x \in D,$$

where $\delta = \inf_{\alpha > 0}(\alpha + \|Vf(.,\alpha\|_\infty)$. Then, we get from (2.1) that

$$\frac{x_2}{C(|x|+1)^2} \int_D \frac{y_2}{(|y|+1)^2} f(y,\delta)dy \le u(x)\,, \quad \forall x \in D.$$

Hence we deduce from (H2) and (3.6) that

$$\frac{x_2}{C(|x|+1)^2} \leq u(x). \tag{4.11}$$

Since f is non-increasing with respect to the second variable, then we have

$$u(x) \leq \min(\delta, \int_D G(x,y) f(y, \frac{y_2}{C(|y|+1)^2}) dy).$$

Finally, we intend to show the uniqueness of the solution. Let u and v be two solutions of (1.1) in $C_0(D)$. Suppose that there exists $x_0 \in D$ such that $u(x_0) < v(x_0)$. Put $w = v - u$. Then $w \in C_0(D)$ and satisfies

$$\Delta w + f(.,v) - f(.,u) = 0, \quad \text{in } D.$$

Let $\Omega = \{x \in D, w(x) > 0\}$. Then $\Omega$ is an open nonempty set in $D$ and by (H3) we deduce that $\Delta w \geq 0$, in $\Omega$ with $w = 0$ on $\partial\Omega$. Hence, by the maximum principle ([2], p.465-466), we get $w \leq 0$ in $\Omega$. Which is in contradiction with the definition of $\Omega$. ♢

We close this section by giving another comparison result for the solutions $u$ of the problem (1.1), in the case of the special nonlinearity $f(x,t) = p(x)q(t)$. The following hypotheses on $p$ and $q$ are adopted.

i) The function $p$ is nontrivial nonnegative and is in $K \cap C^{\gamma}_{\rm loc}(D)$, $0 < \gamma < 1$.

ii) The function $q : (0,\infty) \to (0,\infty)$ is a continuously differentiable and non-increasing.

In the sequel, we define the function $Q$ in $[0,\infty)$ by

$$Q(t) = \int_0^t \frac{1}{q(s)} ds.$$

From the hypothesis adopted on $q$, we note that the function $Q$ is a bijection from $[0,\infty)$ to itself. Then we have the following theorem.

**Theorem 4.5** *Let $u$ be the positive solution of*

$$\Delta u(x) + p(x) q(u(x)) = 0 \quad x \in D,\ u \in C_0(D). \tag{4.12}$$

*Then $q(\delta)Vp \leq u \leq Q^{-1}(Vp)$ in $D$.*

**Proof** Since $u \leq \delta$ in $D$ and $q$ is non-increasing, we deduce from (4.9) that

$$q(\delta)Vp(x) \leq u(x) = \int_D G(x,y)p(y)q(u(y))dy, \quad \forall x \in D.$$

To show the upper estimate, we consider the function $v$ defined in $D$ by

$$v = Q(u) - Vp.$$

Then $v \in C^2(D)$ and

$$\Delta v = \frac{1}{q(u)}\Delta u + p - \frac{q'(u)}{q^2(u)}|\nabla u|^2 \geq 0.$$

In addition, since $Vp \in C_0(D)$, we deduce that $v \in C_0(D)$. Thus, the maximum principle implies that $v \leq 0$.

**Corollary 4.6** *Let $\lambda < 2 < \mu$. Suppose further that the function $p$ satisfies*

$$p(y) \leq \theta(y) \quad \forall y \in D,$$

*where $\theta(y) = 1/((|y|+1)^{\mu-\lambda} y_2^{\lambda})$. Let $u$ be the positive solution of (4.12). Then there exists $C > 0$ such that for each $x \in D$,*

$$\frac{1}{C}\frac{x_2}{(|x|+1)^2} \leq u(x) \leq Q^{-1}(r_{\lambda,\mu}(x)), \forall x \in D,$$

*where $r_{\lambda,\mu}$ is the right hand function in the inequalities of Proposition 3.7.*

**Proof** The lower estimate is obtained from (4.11). Using Theorem 4.5, the upper estimate follows from the monotonicity of $Q^{-1}$ and Proposition 3.7.

**Example** Let $\lambda < 2$, $\mu \geq 4 - \lambda$ and $\sigma \geq 0$. Suppose further that the function $p$ satisfies

$$p(y) \leq \frac{1}{(|y|+1)^{\mu-\lambda} y_2^{\lambda}}, \quad \text{for } y \in D.$$

Then the equation

$$\Delta u + pu^{-\sigma} = 0 \quad \text{in D}, \; u \in C_0(D)$$

has a unique positive solution $u \in C^{2+\gamma}(D)$ which for each $x \in D$ it satisfies:

i) $\frac{1}{C}\frac{x_2}{(|x|+1)^2} \leq u(x) \leq C\frac{x_2^{\frac{2-\lambda}{1+\sigma}}}{(|x|+1)^{\frac{4-2\lambda}{1+\sigma}}}$, if $1 < \lambda < 2$.

ii) $\frac{1}{C}\frac{x_2}{(|x|+1)^2} \leq u(x) \leq C\frac{x_2^{\frac{1}{1+\sigma}}}{(|x|+1)^{\frac{2}{1+\sigma}}}\left[Log(\frac{2(|x|+1)^2}{x_2})\right]^{\frac{1}{1+\sigma}}$, if $\lambda = 1$.

iii) $\frac{1}{C}\frac{x_2}{(|x|+1)^2} \leq u(x) \leq C\frac{x_2^{\frac{1}{1+\sigma}}}{(|x|+1)^{\frac{2}{1+\sigma}}}$, if $\lambda < 1$.

# 5 Proof of Theorem 1.3

Let

$$C_0(\overline{D}) := \{w \in C(\overline{D}) : \lim_{|x|\to\infty} w(x) = 0\}.$$

Then $C_0(\overline{D})$ is a Banach space with the uniform norm $\|w\|_\infty = \sup_{x\in D}|w(x)|$. Let $\varphi_0$ be a positive function belonging to K and let

$$F_0 := \{\varphi \in K : \; |\varphi(x)| \leq \varphi_0(x), \; \forall x \in D\}.$$

**Lemma 5.1** *The family of the functions*

$$\Big\{\int_D \frac{y_2}{x_2} G(.,y)\varphi(y)dy, \varphi \in F_0\Big\}$$

*is uniformly bounded and equicontinuous on* $\overline{D} \cup \{\infty\}$. *Consequently it is relatively compact in* $C_0(\overline{D})$.

**Proof** Let $T$ be the operator defined on $F_0$ as

$$T\varphi(x) = \int_D \frac{y_2}{x_2} G(x,y)\varphi(y)\,dy\,.$$

Then for all $\varphi \in F_0$,

$$|T\varphi(x)| \leq \int_D \frac{y_2}{x_2} G(x,y)\varphi_0(y)\,dy\,.$$

Since $\varphi_0 \in K$, from Proposition 3.2, $\|T\varphi\|_\infty \leq \|\varphi_0\|$ for all $\varphi \in F_0$. Thus the family $T(F_0) = \{T\varphi, \varphi \in F_0\}$ is uniformly bounded.

Now, we prove the equicontinuity of $T(F)$ on $\overline{D} \cup \{\infty\}$. Let $x_0 \in \overline{D}$ and $r > 0$. Let $x, x' \in B(x_0, \frac{r}{2}) \cap D$ and $\varphi \in F_0$, then for $M > 0$,

$$\begin{aligned} |T\varphi(x) - T\varphi(x')| \;\leq\;& 2 \sup_{x\in D} \int_{B(x_0,r)\cap D} \frac{y_2}{x_2} G(x,y)\varphi_0(y)\,dy \\ &+2 \sup_{x\in D} \int_{(|y|\geq M)\cap D} \frac{y_2}{x_2} G(x,y)\varphi_0(y)dy \\ &+\int_{(|x_0-y|\geq r)\cap(|y|\leq M)\cap D} \Big|\frac{G(x,y)}{x_2} - \frac{G(x',y)}{x'_2}\Big| y_2\varphi_0(y)dy. \end{aligned}$$

By (2.1), there exists $C > 0$ such that for all $x \in B(x_0, \frac{r}{2}) \cap D$, for all $y \in B(0,M) \cap (D \backslash B(x_0,r))$,

$$\frac{y_2}{x_2} G(x,y)\varphi_0(y) \leq C y_2^2 \varphi_0(y).$$

Moreover, $\frac{G(x,y)}{x_2}$ is continuous on $(x,y) \in (B(x_0, \frac{r}{2}) \cap D) \times (D \backslash B(x_0,r))$. Then by Proposition 3.1 and Lebesgue's theorem, we have

$$\int_{(|x_0-y|\geq r)\cap(|y|\leq M)\cap D} \Big|\frac{G(x,y)}{x_2} - \frac{G(x',y)}{x'_2}\Big| y_2\varphi_0(y)\,dy \to 0,$$

as $|x - x'| \to 0$. Then it follows from (3.2) that

$$|T\varphi(x) - T\varphi(x')| \to 0 \quad \text{as } |x - x'| \to 0$$

uniformly for all $\varphi \in F_0$. On the other hand, to establish compactness we need to show that

$$\lim_{|x|\to+\infty} T\varphi(x) = 0, \quad \text{uniformly for } \varphi \in F_0.$$

Let $M>0$ and $x$ in $D$ such that $|x|\geq M+1$, then

$$\begin{aligned} |T\varphi(x)| &\leq \int_D \frac{y_2}{x_2} G(x,y)\varphi_0(y)\,dy \\ &\leq \int_{(|y|\leq M)\cap D} \frac{y_2}{x_2} G(x,y)\varphi_0(y)\,dy + \int_{(|y|\geq M)\cap D} \frac{y_2}{x_2} G(x,y)\varphi_0(y)\,dy\,. \end{aligned}$$

Since $\lim_{|x|\to+\infty} y_2 \frac{G(x,y)}{x_2} = 0$ uniformly for $|y|\leq M$, and $\frac{y_2}{x_2}G(x,y)\leq \frac{1}{\pi}y_2^2$, for $|x-y|\geq 1$, then from Proposition 3.1, Lebesgue's theorem and (3.3) with $h=1$, we deduce that

$$\lim_{|x|\to+\infty} T\varphi(x) = 0$$

uniformly for all $\varphi\in F_0$. Finally, by Ascoli's theorem, the family $T(F_0)$ is relatively compact in $C_0(\overline{D})$.

**Proof of Theorem 1.3** Let $\beta\in(0,1)$. Then, by (A1),(A2) and Lemma 5.1, the function

$$T_\beta(x) = \int_D \frac{y_2}{x_2} G(x,y)\ \psi(y,\beta y_2)\,dy$$

is continuous on $\overline{D}$ satisfying

$$\lim_{|x|\to+\infty} T_\beta(x) = 0 \quad \text{and} \quad \lim_{\beta\to 0} T_\beta(x) = 0\ \forall x\in\overline{D}.$$

Moreover, the function $\beta\to T_\beta(x)$ is nondecreasing on $(0,1)$. Then, by Dini Lemma, we have

$$\lim_{\beta\to 0}\sup_{x\in D}\int_D \frac{y_2}{x_2} G(x,y)\psi(y,\beta y_2)\,dy = 0.$$

Thus, there exists $\beta\in(0,1)$ such that for each $x\in D$,

$$\int_D \frac{y_2}{x_2} G(x,y)\psi(y,\beta y_2)\,dy \leq \frac{1}{3}.$$

Let $b_0 = \frac{2}{3}\beta$ and $b\in(0,\,b_0]$. In order to apply a fixed-point argument, set

$$S = \big\{w\in C(\overline{D}\cup\{\infty\}) : \frac{b}{2}\leq w(x)\leq \frac{3b}{2},\, x\in D\big\}.$$

Then, $S$ is a nonempty closed bounded and convex set in $C(\overline{D}\cup\{\infty\})$. Define the operator $\Gamma$ on $S$ as

$$\Gamma w(x) = b + \frac{1}{x_2}\int_D G(x,y) g(\,y, y_2 w(y))\,dy\,, \quad x\in D.$$

First, we shall prove that the operator $\Gamma$ maps $S$ into itself. Let $v\in S$, then for any $x\in D$, we have by (A1) that

$$|\Gamma w(x) - b| \leq \frac{3b}{2}\int_D \frac{y_2}{x_2} G(x,y)\,\psi(y,\beta y_2)\,dy \leq \frac{b}{2}.$$

It follows that $\frac{b}{2} \leq \Gamma w \leq \frac{3b}{2}$ and by Lemma 5.1, $\Gamma(S)$ is included in $C(\overline{D}\cup\{\infty\})$. So $\Gamma S \subset S$.

Next, we shall prove the continuity of $\Gamma$ in the supremum norm. Let $(w_k)_k$ be a sequence in $S$ which converges uniformly to $w \in S$. It follows from (A1) and Lebesgue's theorem that

$$\forall\, x \in D, \quad \Gamma w_k(x) \to \Gamma w(x) \quad \text{as } k \to +\infty.$$

Since $\Gamma(S)$ is a relatively compact family in $C(\overline{D} \cup \{\infty\})$, then the pointwise convergence implies the uniform convergence. Thus we have proved that $\Gamma$ is a compact mapping from $S$ to itself. Now the Schauder fixed-point theorem implies the existence of $w \in S$ such that $\Gamma w = w$. For $x \in D$, put $u(x) = x_2 w(x)$. Therefore we have

$$u(x) = bx_2 + \int_D G_D(x,y) g(y, u(y))\, dy.$$

Since $g(y,u(y)) \leq y_2 \psi(y, y_2)$, then we have by $(A_2)$ and (3.7) that $y \to g(y, u(y)$ is in $\mathrm{L}^1_{\rm loc}(D)$. Applying $\Delta$ in both sides of the above equation, we get

$$\Delta u + g(.,u) = 0, \quad \text{in} D.$$

It is clear that $u$ is a solution of (1.2), continuous on $D$,

$$\frac{b}{2} x_2 \leq u(x) \leq \frac{3b}{2} x_2 \quad \text{and} \quad \lim_{x_2 \to +\infty} \frac{u(x)}{x_2} = b.$$

**Example** Let $\sigma > 0$ and $\lambda < 2 < \mu$. Let $p$ be a measurable function in $D$ such that

$$|p(x)| \leq \frac{C}{(|x|+1)^{\mu-\lambda} x_2^{\lambda+\sigma}}, \quad \forall x \in D.$$

Then there exists $b_0 > 0$ such that for each $b \in (0, b_0]$, the problem

$$\begin{gathered} \Delta u(x) + p(x) u^{\sigma+1}(x) = 0, \quad x \in D \\ u(x) > 0, \quad x \in D \\ u\big|_{\partial D} = 0 \end{gathered}$$

has a solution $u$ continuous on $D$ and satisfying

$$\frac{b}{2} x_2 \leq u(x) \leq \frac{3b}{2} x_2 \quad \text{and} \quad \lim_{x_2 \to \infty} \frac{u(x)}{x_2} = b.$$

## References

[1] K. L. Chung, Z. Zhao: From Brownian Motion to Schrödinger's Equation. Springer,Berlin (1995)

[2] R. Dautray, J. L. Lions, et al: Analyse mathématique et calcul numérique pour les sciences et les techniques, Coll. C.E.A Vol 2, L'opérateur de Laplace, Masson (1987)

[3] A. L. Edelson: Entire solutions of singular elliptic equations, J. Math. Anal. Appl. 139, 523-532 (1989)

[4] A. V. Lair, A. W. Shaker: Entire solutions of a singular semilinear elliptic problem, J. Math. Anal. Appl. 200, 498-505 (1996)

[5] A. V. Lair, A. W. Shaker: Classical and weak solutions of a singular semilinear elliptic problem, J. Math. Anal. Appl. 211, 371-385 (1997)

[6] A. C. Lazer, P. J. Mckenna: On a singular nonlinear elliptic boundary-value problem, Proc. Amer. Mat. Soc. 111, 721-730 (1991)

[7] L. Mâatoug: Positive solutions of a nonlinear elliptic equation in $\{x \in \mathbb{R}^2 : |x| > 1\}$. Potential Analysis. 16(2), 193-203 (2002).

[8] S. Port, C. Stone: Brownian motion and classical Potential theory. Probab. Math, Statist, Academic Press, New York, 1978.

[9] U. Ufuktepe, Z. Zhao: Positive solutions of nonlinear elliptic equations in the Euclidien plane. Proc.Amer. Math. Soc. Vol. 126, 3681-3692 (1998)

[10] Z. Zhao: On the existence of positive solutions of nonlinear elliptic equations-a Probabilistic Potential theory approach. Duke. Math. J. Vol 69, 247-258 (1993)

[11] Z. Zhao: Positive solutions of nonlinear second order ordinary differential equations. Proc. Amer. Math. Soc. Vol. 121 (2), 465-469 (1994).

[12] N. Zeddini: Positive solutions of singular elliptic equations outside the unit disk, Electron. J. Diff. Eqns. No. 53, 1-20 (2001).

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