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# Uniqueness for radial Ginzburg-Landau type minimizers *

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**Abstract**

We prove the uniqueness of radial minimizers of a Ginzburg-Landau type functional. We present also an analysis of the the location of the zeros of the radial minimizer.

## 1 Introduction

For $n \geq 2$, let $B = \{x \in \mathbb{R}^n; |x| < 1\}$ and $\partial B$ its boundary. On this domain, we find minimizers to the Ginzburg-Landau-type functional

$$E_\varepsilon(u, B) = \frac{1}{p}\int_B |\nabla u|^p + \frac{1}{4\varepsilon^p}\int_B (1 - |u|^2)^2, \quad (p \geq n)$$

on the class of functions

$$W = \{u(x) = f(r)\frac{x}{|x|} \in W^{1,p}(B, \mathbb{R}^n); f(1) = 1, r = |x|\}.$$

Such minimizer is denoted by $u_\varepsilon$ and is called a *radial minimizer.*

Many authors have studied the existence, uniqueness and asymptotic behaviour of $u_\varepsilon$ as $\varepsilon \to 0$. For $p = n = 2$, studies of asymptotic behaviour can be found in [1, 11], studies of uniqueness in [7], and other related topics in [2, 3, 5, 10]. For $p = n > 2$ and $p > n = 2$, the asymptotic behaviour was studied in [6] and [9], respectively. However, uniqueness was not mentioned there. In this paper, we prove the following results for $p \geq n$.

**Theorem 1.1** *Assume $u_\varepsilon$ is a radial minimizer of $E_\varepsilon(u, B)$. Then for any given $\eta \in (0, 1/2)$ there exists a positive constant $h = h(\eta)$ such that*

$$Z_\varepsilon = \{x \in B; |u_\varepsilon(x)| < 1 - \eta\} \subset B(0, h\varepsilon) = \{x \in \mathbb{R}^n; |x| < h\varepsilon\}.$$

**Theorem 1.2** *The radial minimizer of $E_\varepsilon(u, B)$ on $W$ is unique.*

This article is organized as follows: In §2, we present some basic properties of minimizers. In §3, we prove Theorem 1.1 which implies, in particular, that the zeros of $u_\varepsilon$ are contained in $B(0, h\varepsilon)$. We conclude with the proof of Theorem 1.2 in §4.

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## 2 Preliminaries

For a function $u(x) = f(r)\frac{x}{|x|}$, written in polar coordinates, we have

$$|\nabla u| = (f_r^2 + (n-1)r^{-2}f^2)^{1/2}, \quad \int_B |u|^p = |S^{n-1}| \int_0^1 r^{n-1}|f|^p \, dr,$$

$$\int_B |\nabla u|^p = |S^{n-1}| \int_0^1 r^{n-1}(f_r^2 + (n-1)r^{-2}f^2)^{p/2} \, dr.$$

It is easily seen that $f(r)\frac{x}{|x|} \in W^{1,p}(B, \mathbb{R}^n)$ implies $f(r)r^{\frac{n-1}{p}-1}$ and $f_r(r)r^{\frac{n-1}{p}}$ are in $L^p(0,1)$. Conversely, if $f(r)$ is in $W^{1,p}_{\rm loc}(0,1]$, and $f(r)r^{\frac{n-1}{p}-1}$, $f_r(r)r^{\frac{n-1}{p}}$ are in $L^p(0,1)$, then $f(r)\frac{x}{|x|} \in W^{1,p}(B, \mathbb{R}^n)$. Thus if we denote

$$V = \{f \in W^{1,p}_{\rm loc}(0,1]; r^{\frac{n-1}{p}} f_r \in L^p(0,1), r^{(n-1-p)/p} f \in L^p(0,1), f(1) = 1\},$$

then $V = \{f(r); u(x) = f(r)\frac{x}{|x|} \in W\}$.

**Proposition 2.1** *The set $V$ defined above is a subset of $\{f \in C[0,1]; f(0) = 0\}$.*

**Proof.** Let $f \in V$. If $p > n$, let $h(r) = f(r^{\frac{p-1}{p-n}})$. Then

$$\begin{aligned}
\int_0^1 |h'(r)|^p \, dr &= (\frac{p-1}{p-n})^p \int_0^1 |f'(r^{\frac{p-1}{p-n}})|^p r^{\frac{p(n-1)}{p-n}} \, dr \\
&= (\frac{p-1}{p-n})^{p-1} \int_0^1 s^{n-1}|f'(s)|^p \, ds < \infty
\end{aligned}$$

by noting $f_s(s)s^{(n-1)/p} \in L^p(0,1)$.

If $p = n$, let $h(r) = f(r^x)$ with $x > 1$ to be determined later. Then for any $y \in (1,p)$,

$$\begin{aligned}
\int_0^1 |h'(r)|^y dr &= x^y \int_0^1 |f'(r^x)|^y r^{(x-1)y} dr \\
&= x^{y-1} \int_0^1 |f'(s)|^y s^{(x-1)(y-1)/x} ds,
\end{aligned}$$

where $s = r^x$. Choose $x, y$ such that $(1-\frac{1}{x})(1-\frac{1}{y}) = \frac{n-1}{n}$. Hence

$$\begin{aligned}
\int_0^1 |h'(r)|^y dr &= x^{y-1} \int_0^1 |f'(s)|^y s^{y(n-1)/n} ds \\
&\le x^{y-1} (\int_0^1 |f'(s)|^n s^{n-1} ds)^{y/n} < \infty.
\end{aligned}$$

Using an interpolation inequality and Young inequality, $\|h\|_{W^{1,y}((0,1),R)} < \infty$ which implies that $h(r) \in C[0,1]$ and hence $f(r) \in C[0,1]$.

Suppose $f(0) > 0$, then $f(r) \geq s > 0$ for $r \in [0, t)$ with $t > 0$ small enough since $f \in C[0, 1]$. We have

$$\int_0^1 r^{n-1-p} f^p \, dr \geq s^p \int_0^t r^{n-1-p} \, dr = \infty,$$

which contradicts $r^{(n-1)/p-1} f \in L^p(0, 1)$. Therefore $f(0) = 0$ and the proof is complete.

**Proposition 2.2** *The functional $E_\varepsilon(u, B)$ achieves its minimum on $W$ by a function $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$.*

**Proof.** Note that $W^{1,p}(B, \mathbb{R}^n)$ is a reflexive Banach space and $E_\varepsilon(u, B)$ is weakly lower-semicontinuous. To prove the existence of minimizers of $E_\varepsilon(u, B)$ in W, it suffices to verify that W is a weakly closed subset of $W^{1,p}(B, \mathbb{R}^n)$. Clearly W is a convex subset of $W^{1,p}(B, \mathbb{R}^n)$. Now we prove that $W$ is a closed subset of $W^{1,p}(B, \mathbb{R}^n)$. Let $u_k = f_k(r)\frac{x}{|x|} \in W$ and

$$\lim_{k\to\infty} u_k = u, \quad \text{in } W^{1,p}(B, \mathbb{R}^n).$$

By the embedding theorem there exists a subsequence $u_k = f_k(r)\frac{x}{|x|}$ such that

$$\lim_{k\to\infty} f_k = f, \quad \text{in } C(0, 1]$$

and $u = f(r)\frac{x}{|x|}$. Combining this with $f_k(1) = 1$, we see that $f(1) = 1$. Thus $u \in W$.

**Proposition 2.3** *The minimizer $u_\varepsilon$ is a weak radial solution of*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{1}{\varepsilon^p} u(1 - |u|^2), \quad \textit{on } B, \tag{2.1}$$

$$u|_{\partial B} = x. \tag{2.2}$$

**Proof.** Denote $u_\varepsilon$ by $u$. For any $t \in [0, 1)$and $\phi = f(r)\frac{x}{|x|} \in C_0^\infty(B, \mathbb{R}^n)$, we have $u + t\phi \in W$ as long as $t$ is small sufficiently. Since $u$ is a minimizer we obtain

$$\frac{dE_\varepsilon(u + t\phi, B)}{dt}|_{t=0} = 0,$$

namely,

$$\begin{aligned} 0 =& \frac{d}{dt}|_{t=0} \int_B (\frac{1}{p}|\nabla(u + t\phi)|^p + \frac{1}{4\varepsilon^p}(1 - |u|^2)^2) dx \\ =& \int_B |\nabla u|^{p-2}\nabla u \nabla\phi dx - \frac{1}{\varepsilon^p}\int_B u\phi(1 - |u|^2) dx. \end{aligned}$$

By a limit process we see that the test function $\phi$ can be any member of $\{\phi = f(r)\frac{x}{|x|} \in W^{1,p}(B, \mathbb{R}^n); \phi|_{\partial B} = 0\}$.

As in [6, Lemma 2.2], we also have the following statement.

**Proposition 2.4** *Let $u_\varepsilon$ be a weak radial solution of (2.1)-(2.2). Then $|u_\varepsilon| \leq 1$ on $\overline{B}$.*

**Proof.** Taking $\phi = u - \frac{u}{|u|}\min(1,|u|)$. Let $B_+ = \{x \in B; |u| > 1, a.e. \text{ on } B\}$. Noting

$$\nabla\phi = 0, a.e. \quad on \quad B \setminus B_+; \quad \nabla\phi = \nabla u(1 - \frac{1}{|u|}) + \frac{u(u\nabla u)}{|u|^3}, a.e. \quad on \quad B_+,$$

we have

$$\int_{B_+} |\nabla u|^p(1 - \frac{1}{|u|}) + \int_{B_+} |\nabla u|^{p-2}\frac{(u\nabla u)^2}{|u|^3} + \frac{1}{\varepsilon^p}\int_{B_+} |u|(|u|^2 - 1)(|u| - 1) = 0.$$

This implies that $|B_+| = 0$. Thus $|u_\varepsilon| \le 1$.

**Proposition 2.5** *Assume $u_\varepsilon$ is a weak radial solution of (2.1)-(refe2.2). Then there exist positive constants $C_1, \rho$ which are independent of $\varepsilon$, such that*

$$\|\nabla u_\varepsilon(x)\|_{L(B(x,\rho\varepsilon/8))} \le C_1\varepsilon^{-1}, \quad if \quad x \in B(0, 1 - \rho\varepsilon), \tag{2.3}$$

$$|u_\varepsilon(x)| \ge \frac{10}{11}, \quad if \quad x \in \overline{B} \setminus B(0, 1 - 2\rho\varepsilon). \tag{2.4}$$

**Proof.** Let $y = x\varepsilon^{-1}$ in (2.1) and denote $v(y) = u(x)$, $B_\varepsilon = B(0, \varepsilon^{-1})$. Then

$$\int_{B_\varepsilon} |\nabla v|^{p-2}\nabla v\nabla\phi = \int_{B_\varepsilon} v(1 - |v|^2)\phi, \quad \phi \in W_0^{1,p}(B_\varepsilon, \mathbb{R}^n). \tag{2.5}$$

This implies that $v(y)$ is a weak solution of (2.5). By using the standard discuss of the Holder continuity of weak solution of (2.5) on the boundary (for example see Theorem 1.1 and Line 19-21 of Page 104 in [4]) we can see that for any $y_0 \in \partial B_\varepsilon$ and $y \in B(y_0, \rho_0)$ (where $\rho_0 > 0$ is a constant independent of $\varepsilon$), there exist positive constants $C = C(\rho_0)$ and $\alpha \in (0, 1)$, both independent of $\varepsilon$, such that

$$|v(y) - v(y_0)| \le C(\rho_0)|y - y_0|^\alpha.$$

Choose $\rho > 0$ sufficiently small such that

$$y \in B(y_0, 2\rho) \subset B(y_0, \rho_0), \quad \text{and} \quad C(\rho_0)|y - y_0|^\alpha \le \frac{1}{11}, \tag{2.6}$$

then

$$|v(y)| \ge |v(y_0)| - C(\rho_0)|y - y_0|^\alpha = 1 - C(\rho_0)|y - y_0|^\alpha \ge \frac{10}{11}.$$

Let $x = y\varepsilon$. Thus $|u_\varepsilon(x)| \ge 10/11$, if $x \in B(x_0, 2\rho\varepsilon)$, where $x_0 \in \partial B$. This implies (2.4). Taking $\phi = v\zeta^p, \zeta \in C_0^\infty(B_\varepsilon, R)$ in (2.5), we obtain

$$\int_{B_\varepsilon} |\nabla v|^p\zeta^p \le p\int_{B_\varepsilon} |\nabla v|^{p-1}\zeta^{p-1}|\nabla\zeta||v| + \int_{B_\varepsilon} |v|^2(1 - |v|^2)\zeta^p.$$

For $\rho$ as in (2.6), setting $y \in B(0, \varepsilon^{-1} - \rho)$, $B(y, \rho/2) \subset B_\varepsilon$,

$$\zeta = \begin{cases} 1 & \text{in } B(y, \rho/4), \\ 0 & \text{in } B_\varepsilon \setminus B(y, \rho/2) \end{cases}$$

and $|\nabla\zeta| \le C(\rho)$, we have

$$\int_{B(y,\rho/2)} |\nabla v|^p \zeta^p \le C(\rho)\int_{B(y,\rho/2)} |\nabla v|^{p-1}\zeta^{p-1} + C(\rho).$$

Using Holder's inequality, we can derive $\int_{B(y,\rho/4)} |\nabla v|^p \le C(\rho)$. Combining this with the Tolksdroff' theorem in [12] yields

$$\|\nabla v\|^p_{L^\infty(B(y,\rho/8))} \le C(\rho)\int_{B(y,\rho/4)} (1+|\nabla v|)^p \le C(\rho)$$

which implies

$$\|\nabla u\|_{L^\infty(B(x,\varepsilon\rho/8))} \le C(\rho)\varepsilon^{-1}.$$

**Proposition 2.6** *Let $u_\varepsilon$ be a radial minimizer of $E_\varepsilon(u,B)$. Then*

$$E_\varepsilon(u_\varepsilon,B) \le C\varepsilon^{n-p} + C, \quad \text{when} \quad p>n, \tag{2.7}$$

$$E_\varepsilon(u_\varepsilon,B) \le \frac{1}{n}(n-1)^{n/2}|S^{n-1}||\ln\varepsilon| + C \quad \text{when} \quad p=n, \tag{2.8}$$

*with a constant $C$ independent of $\varepsilon \in (0,1)$.*

**Proof.** Denote

$$I(\varepsilon,R) = \min\Big\{\int_{B(0,R)} [\frac{1}{p}|\nabla u|^p + \frac{1}{\varepsilon^p}(1-|u|^2)^2]; u \in W_R\Big\},$$

where

$$W_R = \big\{u(x) = f(r)\frac{x}{|x|} \in W^{1,p}(B(0,R),\mathbb{R}^n); r=|x|, f(R)=1\big\}.$$

Then

$$\begin{aligned} I(\varepsilon,1) =& E_\varepsilon(u_\varepsilon,B)\\ =&\frac{1}{p}\int_B |\nabla u_\varepsilon|^p dx + \frac{1}{4\varepsilon^p}\int_B (1-|u_\varepsilon|^2)^2 dx\\ =&\varepsilon^{n-p}[\frac{1}{p}\int_{B(0,\varepsilon^{-1})} |\nabla u_\varepsilon|^p dy + \frac{1}{4}\int_{B(0,\varepsilon^{-1})} (1-|u_\varepsilon|^2)^2 dy]\\ =&\varepsilon^{n-p} I(1,\varepsilon^{-1}). \end{aligned} \tag{2.9}$$

Let $u_1$ be a solution of $I(1,1)$ and define

$$u_2 = \begin{cases} u_1, & \text{if } 0<|x|<1;\\ \frac{x}{|x|}, & \text{if } 1 \le |x| \le \varepsilon^{-1}. \end{cases}$$

Thus $u_2 \in W_{\varepsilon^{-1}}$ and when $p > n$,

$$\begin{aligned} I(1,\varepsilon^{-1}) \leq & \frac{1}{p}\int_{B(0,\varepsilon^{-1})} |\nabla u_2|^p + \frac{1}{4}\int_{B(0,\varepsilon^{-1})}(1-|u_2|^2)^2 \\ = & \frac{1}{p}\int_B |\nabla u_1|^p + \frac{1}{4}\int_B (1-|u_1|^2)^2 + \frac{1}{p}\int_{B(0,\varepsilon^{-1})\setminus B} |\nabla \frac{x}{|x|}|^p \\ = & I(1,1) + \frac{(n-1)^{p/2}|S^{n-1}|}{p}\int_1^{\varepsilon^{-1}} r^{n-p-1}dr \\ = & I(1,1) + \frac{(n-1)^{p/2}|S^{n-1}|}{p(p-n)}(1-\varepsilon^{p-n}) \leq C. \end{aligned}$$

Similarly, when $p = n$,

$$I(1,\varepsilon^{-1}) \leq I(1,1) + \frac{1}{n}(n-1)^{n/2}|S^{n-1}||\ln\varepsilon| + C.$$

Substituting these into (2.9) yields (2.7) and (2.8).

# 3 Location of zeros of minimizers

**Proposition 3.1** *Let $u_\varepsilon$ be a radial minimizer of $E_\varepsilon(u,B)$. Then there exists a constant $C$ independent of $\varepsilon \in (0,1]$ such that*

$$\frac{1}{\varepsilon^n}\int_B (1-|u_\varepsilon|^2)^2 \leq C. \tag{3.1}$$

**Proof.** When $p > n$, (3.1) can be derived by multiplying (2.7) by $\varepsilon^{p-n}$. When $p = n$, as in [6, eqn.(3.6)], we derive that

$$\int_B |\nabla u_\varepsilon|^n dx \geq (n-1)^{n/2}|S^{n-1}||\ln\varepsilon| - C,$$

where $C$ is independent of $\varepsilon$. Combining this with (2.8) we obtain (3.1).

**Proposition 3.2** *Let $u_\varepsilon$ be a radial minimizer of $E_\varepsilon(u,B)$. Then for any $\eta \in (0,1/2)$, there exist positive constants $\lambda,\mu$ independent of $\varepsilon \in (0,1)$ such that if*

$$\frac{1}{\varepsilon^n}\int_{B(0,1-\rho\varepsilon)\cap B^{2l\varepsilon}} (1-|u_\varepsilon|^2)^2 \leq \mu, \tag{3.2}$$

*where $B^{2l\varepsilon}$ is the ball of radius $2l\varepsilon$ with $l \geq \lambda$, then*

$$|u_\varepsilon(x)| \geq 1-\eta, \quad \forall x \in B(0,1-\rho\varepsilon)\cap B^{l\varepsilon}. \tag{3.3}$$

**Proof.** First we observe that there exists a constant $C_2 > 0$ which is independent of $\varepsilon$ such that for any $x \in B$ and $0 < \rho \le 1$,

$$|B(0, 1-\rho\varepsilon) \cap B(x,r)| \ge C_2 r^n.$$

To prove this proposition, we choose

$$\lambda = \frac{\eta}{2C_1}, \quad \mu = \frac{C_2}{C_1^n}(\frac{\eta}{2})^{n+2}, \tag{3.4}$$

where $C_1$ is the constant in (2.3). Suppose that there is a point $x_0 \in B(0, 1-\rho\varepsilon) \cap B^{l\varepsilon}$ such that $|u_\varepsilon(x_0)| < 1-\eta$. Then applying (2.3)) we have

$$\begin{aligned}
|u_\varepsilon(x) - u_\varepsilon(x_0)| &\le C_1\varepsilon^{-1}|x - x_0| \le C_1\varepsilon^{-1}(\lambda\varepsilon) \\
&= C_1\lambda = \frac{\eta}{2}, \quad \forall x \in B(x_0, \lambda\varepsilon),
\end{aligned}$$

hence $(1 - |u_\varepsilon(x)|^2)^2 > \frac{\eta^2}{4}, \quad \forall x \in B(x_0, \lambda\varepsilon)$. Thus

$$\begin{aligned}
\int_{B(x_0,\lambda\varepsilon)\cap B(0,1-\rho\varepsilon)} (1-|u_\varepsilon|^2)^2 &> \frac{\eta^2}{4}|B(0, 1-\rho\varepsilon) \cap B(x_0, \lambda\varepsilon)| \\
&\ge C_2\frac{\eta^2}{4}(\lambda\varepsilon)^n = C_2\frac{\eta^2}{4}(\frac{\eta}{2C_1})^n\varepsilon^n = \mu\varepsilon^n.
\end{aligned} \tag{3.5}$$

Since $x_0 \in B^{l\varepsilon} \cap B$, and $(B(x_0, \lambda\varepsilon) \cap B(0, 1-\rho\varepsilon)) \subset (B^{2l\varepsilon} \cap B(0, 1-\rho\varepsilon))$, (3.5) implies

$$\int_{B^{2l\varepsilon}\cap B(0,1-\rho\varepsilon)} (1-|u_\varepsilon|^2)^2 > \mu\varepsilon^n,$$

which contradicts (3.2) and thus (3.3) is proved.

Let $u_\varepsilon$ be a radial minimizer of $E_\varepsilon(u, B)$. Given $\eta \in (0, 1/2)$. Let $\lambda, \mu$ be constants in Proposition 3.2 corresponding to $\eta$. If

$$\frac{1}{\varepsilon^n}\int_{B(x^\varepsilon,2\lambda\varepsilon)\cap B(0,1-\rho\varepsilon)} (1-|u_\varepsilon|^2)^2 \le \mu, \tag{3.6}$$

then $B(x^\varepsilon, \lambda\varepsilon)$ is called the ball of type I. Otherwise it is called the ball of type II.

Now suppose that $\{B(x_i^\varepsilon, \lambda\varepsilon), i \in I\}$ is a family of balls satisfying

$$\begin{gathered}
x_i^\varepsilon \in B(0, 1-\rho\varepsilon), i \in I; \\
B(0, 1-\rho\varepsilon) \subset \cup_{i\in I} B(x_i^\varepsilon, \lambda\varepsilon); \\
B(x_i^\varepsilon, \lambda\varepsilon/4) \cap B(x_j^\varepsilon, \lambda\varepsilon/4) = \emptyset, i \ne j.
\end{gathered} \tag{3.7}$$

Denote $J_\varepsilon = \{i \in I; B(x_i^\varepsilon, \lambda\varepsilon)$ is a ball of type II$\}$.

**Proposition 3.3** *There exists an upper bound for the number of balls of type II. i.e., there exists a positive integer $N$ such that* $\operatorname{Card} J_\varepsilon \le N$.

**Proof.** Since (3.7) implies that every point in $B$ can be covered by finite, say $m$ (independent of $\varepsilon$) balls, from (3.6) and the definition of balls of type II, we have

$$\begin{aligned}\mu\varepsilon^n Card J_\varepsilon &\le \sum_{i\in J_\varepsilon}\int_{B(x_i^\varepsilon,2\lambda\varepsilon)\cap B(0,1-\rho\varepsilon)}(1-|u_\varepsilon|^2)^2\\ &\le m\int_{\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,2\lambda\varepsilon)\cap B(0,1-\rho\varepsilon)}(1-|u_\varepsilon|^2)^2\\ &\le m\int_B(1-|u_\varepsilon|^2)^2\le mC\varepsilon^n\end{aligned}$$

and hence for some $n$, Card $J_\varepsilon \le \frac{mC}{\mu}\le N$. □

Proposition 3.3 is an important result since the number of balls of type II is always finite as $\varepsilon$ becomes sufficiently small.

Similar to the argument of [1, Theorem IV.1], we have

**Proposition 3.4** *There exist a subset $J\subset J_\varepsilon$ and a constant $h\ge\lambda$ such that*

$$\begin{gathered}\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,\lambda\varepsilon)\subset\cup_{i\in J}B(x_j^\varepsilon,h\varepsilon),\\ |x_i^\varepsilon-x_j^\varepsilon|>8h\varepsilon,\quad i,j\in J,\quad i\ne j.\end{gathered}\tag{3.8}$$

**Proof.** If there are two points $x_1,x_2$ such that (3.8) is not true with $h=\lambda$, we take $h_1=9\lambda$ and $J_1=J_\varepsilon\setminus\{1\}$. In this case, if (3.8) holds we are done. Otherwise we continue to choose a pair points $x_3,x_4$ which does not satisfy (3.8) and take $h_2=9h_1$ and $J_2=J_\varepsilon\setminus\{1,3\}$. After at most $N$ steps we may choose $\lambda\le h\le\lambda 9^N$ and conclude this proposition.

Applying Proposition 3.4, we may modify the family of balls of type II such that the new one, denoted by $\{B(x_i^\varepsilon,h\varepsilon);i\in J\}$, satisfies

$$\begin{gathered}\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,\lambda\varepsilon)\subset\cup_{i\in J}B(x_i^\varepsilon,h\varepsilon),\\ \text{Card}\,J\le\text{Card}\,J_\varepsilon,\\ |x_i^\varepsilon-x_j^\varepsilon|>8h\varepsilon, i,j\in J, i\ne j.\end{gathered}$$

The last condition implies that every two balls in the new family are disjoint.

Now we prove the main result of this section.

**Theorem 3.5** *Let $u_\varepsilon$ be a radial minimizer of $E_\varepsilon(u,B)$. Then for any $\eta\in(0,1/2)$, there exists a constant $h=h(\eta)$ independent of $\varepsilon\in(0,1)$ such that $Z_\varepsilon=\{x\in B;|u_\varepsilon(x)|<1-\eta\}\subset B(0,h\varepsilon)$. In particular the zeros of $u_\varepsilon$ are contained in $B(0,h\varepsilon)$.*

**Proof.** Suppose there exists a point $x_0\in Z_\varepsilon$ such that $x_0\overline{\in}B(0,h\varepsilon)$. Then all points on the circle $S_0=\{x\in B;\ |x|=|x_0|\}$ satisfy $|u_\varepsilon(x)|<1-\eta$ and hence by virtue of Proposition 3.2 and (2.4), all points on $S_0$ are contained in balls of type II. However, since $|x_0|\ge h\varepsilon, S_0$ can not be covered by a single ball of type

II. $S_0$ can be covered by at least two balls of type II. However this is impossible. $\square$

This theorem plays the key role in proving the uniqueness of radial minimizers. Furthermore, it implies that all the zeros of the radial minimizer locate near the singularity 0 of $\frac{x}{|x|}$, which is not mentioned in [6] when $p = n > 2$.

Using Theorem 3.5 and (2.4), we can see that

$$|u_\varepsilon(x)| \geq \min(\frac{10}{11}, 1 - \eta), \quad x \in B(0, h(\eta)\varepsilon). \tag{3.9}$$

# 4 Proof of Theorem 1.2

Fix $\varepsilon \in (0, 1)$. Suppose $u_1(x) = f_1(r)\frac{x}{|x|}$ and $u_2(x) = f_2(r)\frac{x}{|x|}$ are both radial minimizers of $E_\varepsilon(u, B)$ on $W$, then they are weak radial solutions of (2.1) (2.2). Thus

$$\begin{aligned}&\int_B (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla\phi dx \\ &= \frac{1}{\varepsilon^p}\int_B [(u_1 - u_2) - (u_1|u_1|^2 - u_2|u_2|^2)]\phi dx.\end{aligned}$$

Taking $\phi = u_1 - u_2 = (f_1 - f_2)\frac{x}{|x|}$, we have

$$\begin{aligned}&\int_B (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla(u_1 - u_2) dx \\ &= \frac{1}{\varepsilon^p}\int_B (f_1 - f_2)^2 dx - \frac{1}{\varepsilon^p}\int_B (f_1 - f_2)^2(f_1^2 + f_2^2 + f_1 f_2) dx \\ &= \frac{1}{\varepsilon^p}\int_{B\setminus B(0,h\varepsilon)} (f_1 - f_2)^2[1 - (f_1^2 + f_2^2 + f_1 f_2)] dx \\ &\quad + \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)} (f_1 - f_2)^2 dx - \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)} (f_1 - f_2)^2(f_1^2 + f_2^2 + f_1 f_2) dx.\end{aligned}$$

Letting $\eta < 1 - \frac{1}{\sqrt{2}}$ in (3.9), we have $f_1, f_2 \geq 1/\sqrt{2}$, on $B \setminus B(0, h\varepsilon)$ for any given $\varepsilon \in (0, 1)$. Hence

$$\int_B (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla(u_1 - u_2) dx \leq \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)} (f_1 - f_2)^2 dx.$$

Applying [12, eqn.(2.11)], we can see that there exists a positive constant $\gamma$ independent of $\varepsilon$ and $h$ such that

$$\gamma\int_B |\nabla(u_1 - u_2)|^2 dx \leq \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)} (f_1 - f_2)^2 dx, \tag{4.1}$$

which implies

$$\int_B |\nabla(f_1 - f_2)|^2 dx \leq \frac{1}{\gamma\varepsilon^p}\int_{B(0,h\varepsilon)} (f_1 - f_2)^2 dx. \tag{4.2}$$

Denote $G = B(0, h\varepsilon)$. Applying [8, Theorem 2.1], we have $\|f\|_{\frac{2n}{n-2}} \leq \beta\|\nabla f\|_2$, where $\beta = 2(n-1)/(n-2)$. Taking $f = f_1 - f_2$ and applying (4.2), we obtain $f(|x|) = 0$ as $x \in \partial B$ and

$$[\int_B |f|^{\frac{2n}{n-2}} dx]^{\frac{n-2}{n}} \leq \beta^2 \int_B |\nabla f|^2 dx \leq \beta^2\gamma^{-1}\int_G |f|^2 dx\varepsilon^{-p}.$$

Using Holder's inequality, we derive

$$\int_G |f|^2 dx \leq |G|^{1-\frac{n-2}{n}}[\int_G |f|^{\frac{2n}{n-2}} dx]^{\frac{n-2}{n}} \leq |B|^{1-\frac{n-2}{n}} h^2\varepsilon^{2-p}\frac{\beta^2}{\gamma}\int_G |f|^2 dx.$$

Hence for any given $\varepsilon \in (0,1)$,

$$\int_G |f|^2 dx \leq C(\beta, |B|, \gamma, \varepsilon)h^2 \int_G |f|^2 dx. \tag{4.3}$$

Denote $F(\eta) = \int_{B(0,h(\eta)\varepsilon)} |f|^2 dx$, then $F(\eta) \geq 0$ and (4.3) implies that

$$F(\eta)(1 - C(\beta, |B|, \gamma, \varepsilon)h^2) \leq 0. \tag{4.4}$$

On the other hand, since $C(\beta, |B|, \gamma, \varepsilon)$ is independent of $\eta$, we may take $0 < \eta < 1 - \frac{1}{\sqrt{2}}$ so small that $h = h(\eta) \leq \lambda 9^N = 9^N \frac{\eta}{2C_1}$ (which is implied by (3.4)) satisfies

$$0 < 1 - C(\beta, |B|, \gamma, \varepsilon)h^2$$

for the fixed $\varepsilon \in (0,1)$, which and (4.4) imply that $F(\eta) = 0$. Namely $f = 0$ a.e. on $G$, or

$$f_1 = f_2, \quad a.e. \quad on \quad B(0, h\varepsilon).$$

Substituting this into (4.1), we know that $u_1 - u_2 = C$ a.e. on $B$. Noticing the continuity of $u_1, u_2$ which is implied by Proposition 2.1, and $u_1 = u_2 = x$ on $\partial B$, we can see at last that

$$u_1 = u_2, \quad \text{on } \overline{B}.$$

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