Electronic Journal of Differential Equations, Vol. 2002(2002), No. 50, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

METASTABILITY IN THE SHADOW SYSTEM FOR GIERER-MEINHARDT'S EQUATIONS

PIETER DE GROEN & GEORGI KARADZHOV

Abstract. In this paper we study the stability of the single internal spike solution of the shadow system for the Gierer-Meinhardt equations in one space dimension. It is well-known, that the linearization around this spike consists of a differential operator plus a non-local term. For parameter values in certain subsets of the 3D (p, q, r)–parameter space we prove that the non-local term moves the negative $O(1)$ eigenvalue of the differential operator to the positive (stable) half plane and that an exponentially small eigenvalue remains in the negative half plane, indicating a marginal instability (dubbed "metastability"). We also show, that for parameters (p, q, r) in another region, the $O(1)$ eigenvalue remains in the negative half plane. In all asymptotic approximations we compute rigorous bounds for the order of the error.

1. Introduction

Based on pioneering ideas of Turing [14] about pattern formation by interaction of diffusing chemical substances, Gierer & Meinhardt proposed and studied in [3] the following system of reaction diffusion equations on a spatial domain Ω

$$\begin{aligned} &U_t = \varepsilon^2 \Delta U - U + U^p H^{-q} && x \in \Omega, \quad t > 0, \\ &\tau H_t = D\Delta H - \mu H + U^r H^{-s} && x \in \Omega, \quad t > 0, \\ &\partial_n U = 0 = \partial_n H && x \in \partial\Omega, \quad t \geq 0, \end{aligned} \tag{1.1}$$

where U and H represent activator and inhibitor concentrations, ε and D their diffusivities, and where τ and μ are the reaction time rate and the decay rate of the inhibitor; D is assumed to be large and ε and τ small (positive). Ω is a bounded domain; we shall restrict our analysis to one space dimension and choose $\Omega := [-1, 1]$. Since in our analysis the factor μ is not important, we set $\mu := 1$; since the parameters q and s only appear together in the quotient $q/(s+1)$ and $s \geq 0$, we may choose $s = 0$ without loss of generality. The exponents $\{p > 1,\ q > 0,\ r > 1\}$

2000 *Mathematics Subject Classification.* 35B25, 35K60.
Key words and phrases. Spike solution, singular perturbations, reaction-diffusion equations, Gierer-Meinhardt equations.

Submitted April 17, 2002. Published June 2, 2002.
Partially supported by NATO Collaborative Research Grant No. 973981, by the Research Community "Advanced Numerical Methods for Mathematical Modeling" of the Fund for Scientific Research-Flanders, and by a FWO-BAS grant.

satisfy the inequality

$$\gamma_r := \frac{qr}{p-1} > 1. \tag{1.2}$$

The numerical study in [3] revealed strongly localized pulse solutions in the activator U against a nearly constant background of the inhibitor concentration for large values of D and $1/\varepsilon$. Irons & Ward [15] analyze by formal asymptotic expansions metastable spike solutions for the reduced "shadow"-system, derived from (1.1) in the limit for $D \to \infty$ and $\tau \to 0$. Doelman, Gardner & Kaper [2] consider the same problem, using "geometric singular perturbation theory" to derive criteria for stability of a solution with one spike. The reader is advised to consult those papers for a more general description of facts and results in this type of problems. As in [11], [10] and [15] we derive the simplified shadow system from (1.1) as follows. The average h of H satisfies the average of (1.1b), $\tau h_t = -h + \frac{1}{2}\int_{-1}^{1} U^r dx$. If τ is small and D is large, $H(x) \approx h$ is approximately constant and τh_t negligible, such that the averaged equation reduces to the constraint

$$h = \frac{1}{2}\int_{-1}^{1} U^r dx. \tag{1.3}$$

So we find from (1.1a) the shadow system for $(x,t) \in (-1,1) \times (0,\infty)$,

$$U_t = \varepsilon^2 U_{xx} - U + h^{-q}U^p, \quad U_x(-1,t) = 0 = U_x(1,t). \tag{1.4}$$

We can eliminate from this equation the direct dependence on h by (1.3).

The aim of this paper is a study of metastability of the single internal spike solution via the spectrum of the linearized operator. We consider the shadow system of (1.1) in one space dimension, construct a (stationary) solution to it along the lines of [13] and we show how the interesting spectral properties of the variation around this solution can be derived. We extend the results obtained by Wei in [17]. The differential operator associated with this variation consists of a selfadjoint differential operator L_ε and a non-local term (due to the constant inhibitor background). L_ε has one large ($O(1)$) negative (i.e. unstable) eigenvalue associated to an even eigenfunction and one exponentially small negative eigenvalue associated to an odd eigenfunction. Using several perturbational techniques, we derive conditions under which the non-local term pushes the spectrum to the right (i.e. it increases the stability) and maps the large negative eigenvalue into the positive half plane, see figure 1. Since the non-local term is zero on the odd functions, it does not affect the exponentially small eigenvalue. We remark, that Doelman et al. in [2] assume that the spatial domain is unbounded and consider this only a minor simplification. Although this simplification looks quite innocently, our study shows that it removes the asymptotically small terms in the expansions of the eigenvalues of the shadow equation, that are decisive for the (very weak) stability or instability.

The boundary conditions on the finite domain are extremely important. With Neumann boundary conditions, the only exact stationary spike solution, having exactly one interior maximum and no boundary maxima, is the symmetric spike with maximum at the center. If we want an off-center spike, we have to change the Neumann condition at $x = 1$ or at $x = -1$ into a mixed condition; either $U_x(1) + \varepsilon\delta U(1) = 0$ for a spike on $(0,1)$ or $U_x(1) - \varepsilon\delta U(1) = 0$ for a spike on $(-1,0)$ for some $\delta > 0$. Only using the exact spike, we can linearize around it in such a way that the quadratic terms do not contain secular terms due to the approximation of

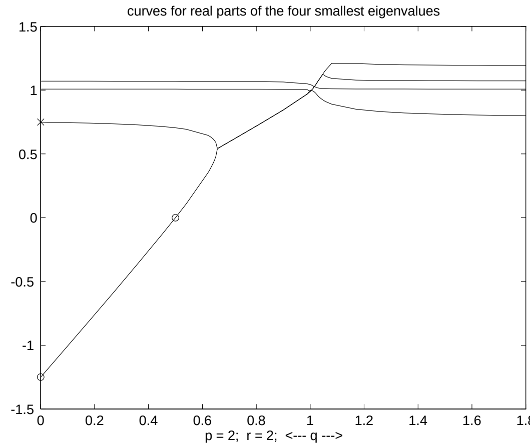

FIGURE 1. Plot of the four eigenvalues with smallest real part of the operator, restricted to the subspace of **even** functions, as a function of q if $p = 2$ and $r = 2$. As soon as q crosses the point $q = \frac{p-1}{r} = \frac{1}{2}$, condition (1.2) is met and the smallest eigenvalue crosses the imaginary axis. Near $q = 0.66$ the two smallest eigenvalues meet and go off into the complex plane.

the spike. Also this aspect disappears, when we study the problem on the whole real axis.

In section 2 we study stationary spike-solutions of (1.4), their asymptotics for small ε and the linearization of this equation around such a solution. In section 3 we study the spectrum of the differential operator L_ε using the (new) explicit solution 2.2 (see also [7]) of the shadow system. The eigenvalues are estimated using Rayleigh's quotient. In section 4 we make a detailed study of the influence of the nonlocal term on the eigenvalues as a function of the parameters p, q and r using perturbational methods. Finally, we study in section 5 the "metastability" of (1.3–1.4) along the lines of [6].

While linearizing (1.1) or (1.4) around a spike we should always keep in mind that the non-linear system is well-defined only for positive solutions and hence that only those variations are acceptable, that do not compromise the positivity. For the study of the spectrum of the linearized equation this does not matter. However, for a study of metastability, we have to restrict the perturbations to functions that are not larger than some (fixed positive) factor $\varrho < 1$ times the spike.

Essentially, the methods are applicable too in the case where D is large but not pushed to the limit $D \to \infty$, see [16].

2. A spike solution and linearization around it

2.1. **A stationary one-spike solutions and its asymptotics.** A typical (stationary) spike solution of (1.3–1.4) should be large on a small support around some interior point x_o and negligible elsewhere. Assuming that such a spike solution S has the form $S(x_o+\varepsilon\xi, t) = h^\gamma u(\xi)$, where ξ is the stretched variable and

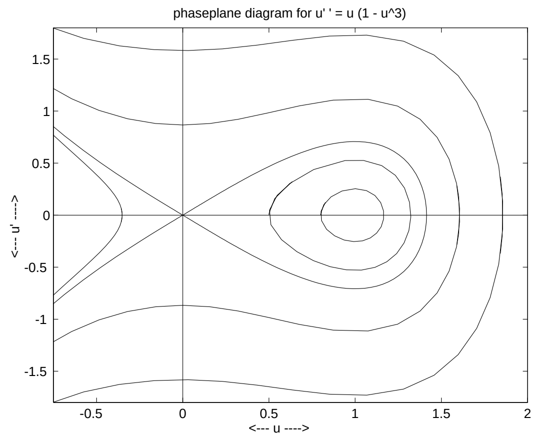

FIGURE 2. Phasediagram in the (u, u')-plane for the solution of $u'' - u + u^p = 0$ from (2.1) with $p = 3$.

$\gamma = q/(p-1-rq)$, we find for u the equation

$$u'' - u + u^p = 0, \quad u'((\pm 1 - x_o)/\varepsilon) = 0. \tag{2.1}$$

For $p > 1$ the system of equations for (u, u') in the phase plane has a saddle point at $(0, 0)$ and a center at $(1, 0)$. A unique homoclinic solution around the center connects the saddle to itself, see figure 2.

This homoclinic solution happens to have for all $p > 1$ the closed form[1]

$$w_p(\xi) := \big(\frac{p+1}{2}\big)^{1/(p-1)}(\cosh(\frac{p-1}{2}\xi))^{-2/(p-1)}, \tag{2.2}$$

and for large $|\xi|$ it has the asymptotic behaviour

$$w_p(\xi) = \alpha e^{-|\xi|}(1 + O(e^{-(p-1)|\xi|})), \quad \alpha := (2p+2)^{1/(p-1)}. \tag{2.3}$$

Inside the homoclic orbit we find positive periodic orbits; from the phase portrait it is clear that these are the only ones that may solve the Neumann boundary conditions in (2.1). Necessary and sufficient is that an integral multiple of its half period is equal to the interval length $2/\varepsilon$. Let φ be such a periodic solution that has minimum a and maximum b, then apparently

$$0 < a < 1 < b < t_p := \big(\frac{p+1}{2}\big)^{1/(p-1)}. \tag{2.4}$$

Multiplying the equation for φ by $\dot\varphi$ and integrating we find

$$\dot\varphi^2 = F(\varphi) - F(a), \quad F(a) = F(b) \quad \text{and} \quad F(\varphi) := \varphi^2 - \frac{2}{p+1}\varphi^{p+1}. \tag{2.5}$$

Minimum and maximum of the solution φ are related by the equation $F(a) = F(b)$ and its period is given by the integral

$$T_a := 2\int_a^b \frac{d\varphi}{\sqrt{F(\varphi) - F(a)}}. \tag{2.6}$$

[1]It is remarkable that we did not find this fact in any of the references consulted.

From this picture it is clear that for every $\varepsilon > 0$ there is a unique stationary solution with a single spike in the interior (and without boundary spikes). We shall denote it in the sequel by φ_ε; its minimum we denote by a_ε and its maximum by b_ε. If $\varepsilon \to 0$, then $\varphi_\varepsilon \to w_p$, $a_\varepsilon \to 0$ and $b_\varepsilon \to t_p$ and

$$a_\varepsilon = \sqrt{(p-1)t_p(t_p - b_\varepsilon)}(1 + O(t_p - b_\varepsilon)^\sigma), \quad \sigma := \min(1, p-1). \tag{2.7}$$

Because the period has to fit in the interval, we can derive the asymptotics of a_ε and b_ε for $\varepsilon \to 0$ from the equation $T_a = 2/\varepsilon$:

Lemma 2.1.

$$a_\varepsilon = 2\alpha e^{-1/\varepsilon}(1 + O(e^{-\sigma/\varepsilon})) \quad \textit{and} \quad t_p - b_\varepsilon = O(e^{-2/\varepsilon}). \tag{2.8}$$

PROOF. We estimate the integral (2.6) by the corresponding integral for w_p, for which we already know the asymptotics, (2.3) and we show that the difference tends to the constant $\log 2$ for $a \to 0$,

$$\int_a^b \Big[\frac{1}{\sqrt{F(t) - F(a)}} - \frac{1}{\sqrt{F(t)}}\Big] dt = \log 2 + O(a^\sigma), \tag{2.9}$$

To do so, we split the interval of integration into three parts, $[a, c]$, $[c, d]$, and $[d, b]$, where $a < c < d < b$ and where c and d are constants, not depending on a and bounded away from the zeros of F, a and b. Obviously, the integral over the middle segment is of order $F(a) = O(a^2)$.

In the third part on $[d, b]$ we may estimate F from below by

$$F(t) - F(b) \geq \delta^2(b - t), \quad F(t) \geq \delta^2(t_p - t), \quad \text{if} \quad d \leq t \leq b$$

for some $\delta > 0$. Hence we estimate the integral from above as follows,

$$\begin{aligned}
&\int_d^b \Big[\frac{1}{\sqrt{F(t) - F(b)}} - \frac{1}{\sqrt{F(t)}}\Big] dt \\
&= \int_d^b \frac{F(a)\, dt}{\sqrt{F(t) - F(b)}\sqrt{F(t)}\Big(\sqrt{F(t) - F(b)} + \sqrt{F(t)}\Big)} \\
&\leq \int_d^b \frac{F(a) dt}{\delta^3\sqrt{t - b}\sqrt{t - t_p}\big(\sqrt{t - b} + \sqrt{t - t_p}\big)} \\
&= \frac{\delta^{-3}F(a)}{t_p - b}\int_d^b \Big[\frac{1}{\sqrt{t - b}} - \frac{1}{\sqrt{t - t_p}}\Big] dt \\
&= O(\sqrt{t_p - b}) = O(a).
\end{aligned}$$

In the first part on $[a, c]$ we substitute $s^2 = F(t) = t^2 - \frac{2}{p+1}t^{p+1}$, assuming $c < 1$. This implies

$$s = t + O(t^p) \quad \text{or} \quad t = s + O(s^p) \quad \text{and} \quad dt = \frac{sds}{t - t^p} = ds\big(1 + O(s^{p-1})\big)$$

Writing $\widetilde{a}^2 := F(a)$ and $\widetilde{c}^2 := F(c)$ we find

$$\int_a^c \Big[\frac{1}{\sqrt{F(t) - F(b)}} - \frac{1}{\sqrt{F(t)}}\Big] dt = \int_{\widetilde{a}}^{\widetilde{c}} \Big[\frac{1}{\sqrt{s^2 - \widetilde{a}^2}} - \frac{1}{\sqrt{s^2}}\Big]\Big(1 + O(s^{p-1})\Big) ds.$$

The main term evaluates to

$$\int_{\widetilde{a}}^{\widetilde{c}} \Big[\frac{1}{\sqrt{s^2-\widetilde{a}^2}} - \frac{1}{\sqrt{s^2}}\Big] ds = \log\big(1+\sqrt{1-\widetilde{a}^2/\widetilde{c}^2}\big) = \log 2 + O(a^2).$$

The remainder is bounded by a constant times the following integral, in which we substitute $s = \widetilde{a}\sigma$,

$$\int_{\widetilde{a}}^{\widetilde{c}} \Big[\frac{1}{\sqrt{s^2-\widetilde{a}^2}} - \frac{1}{\sqrt{s^2}}\Big] s^{p-1} ds = \int_1^{\widetilde{c}/\widetilde{a}} \frac{\widetilde{a}^{p-1}\sigma^{p-2} d\sigma}{\sqrt{\sigma^2-1}(\sigma+\sqrt{\sigma^2-1})}.$$

The integrand in the right-hand side has a square root singularity at $\sigma = 1$ and behaves like σ^{p-4} for large σ, hence the integral is of the order $O(a^{\min(2,p-1)})$. Finally, the integral for w_p extends to t_p, hence we have to subtract from (2.9) the contribution

$$\int_b^{t_p} \frac{dt}{\sqrt{F(t)}} = O\Big(\int_b^{t_p} \frac{dt}{\sqrt{t_p-t}}\Big) = O(\sqrt{t_p-b}) = O(a).$$

Summing up, we have proved

$$\int_a^b \frac{dt}{\sqrt{F(t)-F(a)}} = \int_a^{t_p} \frac{dt}{\sqrt{F(t)}} + \log 2 + O(a^\sigma). \tag{2.10}$$

Since the integral in the right-hand side is the inverse function of w_p, its asymptotics (2.3) imply (2.8). □

As we expect we can use the function w_p as an approximation to φ_ε.

Lemma 2.2. *For each $\varepsilon > 0$ equation* (2.1) *has a unique spike solution φ_ε with a single internal maximum, located at the center of the interval. It approximates w_p,*

$$|\varphi_\varepsilon(\xi) - w_p(\xi)| \le ce^{-1/\varepsilon}, \tag{2.11}$$

uniformly for $|\xi| \le 1/\varepsilon, 0 < \varepsilon < \varepsilon_{\rm o}$ and it satisfies on $[-1/\varepsilon, 1/\varepsilon]$ the equivalence $\varphi_\varepsilon \asymp w_p$.

Proof: The equivalence is an immediate consequence of (2.11) and the fact that both φ_ε and w_p are bounded from below by $\alpha \approx e^{-1/\varepsilon}$. To prove (2.11) we start with the evident property $\varphi_{\rm o}(\xi) = w_p(\xi)$. For any $1 < b < t_p$ we can define the positive solution $\varphi(\xi, b)$ with global maximum at zero by the integral:

$$\int_{\varphi(\xi,b)}^b \frac{dt}{\sqrt{F(t)-F(b)}} = |\xi| \quad \text{or} \quad \varphi_\varepsilon(\xi) = \varphi(\xi, b_\varepsilon).$$

Thus $\varphi(\xi, b)$ is defined for $|\xi| \le \xi(b)$, where

$$\xi(b) := \int_{a(b)}^b \frac{dt}{\sqrt{F(t)-F(b)}} \quad \text{and} \quad F(a(b)) = F(b),\ a(b) < 1 < b.$$

In particular, $\xi(b) \to \infty$ as $b \to t_p$. We have

$$\psi(\xi, b) := \varphi(\xi, b) - w_p(\xi) = \varphi(\xi, b) - \varphi(\xi, t_p) = \int_{t_p}^b \frac{\partial\varphi}{\partial b}(\xi, b) db$$

and the function $h(\xi, b) := \frac{\partial\varphi}{\partial b}(\xi, b)$ satisfies the problem ($' = d/d\xi$)

$$h'' = (1 - p\varphi^{p-1})h, \quad h(0, b) = 1, \quad h'(0, b) = 0.$$

In particular $h(\xi, b)$ is bounded on any ε-independent rectangle $[-\eta_1, \eta_1]\times[t_p-\delta, t_p]$. Therefore

$$|\psi(\xi, b_\varepsilon)| \le ce^{-2/\varepsilon}, \quad \text{if} \quad |\xi| \le \eta_1, 0 < \varepsilon < \varepsilon_{\rm o}.$$

Since ψ is an even function it remains to consider the interval $[\eta_1, 1/\varepsilon]$. Now we shall use the maximum principle for the boundary problem:

$$\psi'' = q_\varepsilon \psi, \qquad q(\xi, \varepsilon) := \frac{f(\varphi_\varepsilon) - f(w_p)}{\varphi_\varepsilon - w_p},$$
$$\psi(\eta_1, b_\varepsilon) = O(e^{-1/\varepsilon}), \quad \psi(1/\varepsilon, b_\varepsilon) = O(e^{-1/\varepsilon}),$$

where $f(t) = t - t^p$. The function f is decreasing on $[0, t_{\rm o})$ $t_{\rm o} := p^{-1/(p-1)} < 1$, hence q is non-negative if $\varphi_\varepsilon \le t_{\rm o}$ and $w_p \le t_{\rm o}$. If we choose η_1 such that those conditions are satisfied by φ_ε and w_p in this point, they are true on all of $[\eta_1, 1/\varepsilon]$ because of the monotony of both functions. Then the maximum principle implies

$$|\psi(\xi, b_\varepsilon)| \le ce^{-1/\varepsilon}$$

in the interval $[\eta_1, 1/\varepsilon]$. □

Thus we have only one steady state solution S which has a single internal spike at zero:

$$S(x, \varepsilon) = H(\Phi)\Phi(x, \varepsilon), \; H(\Phi) := \Big[\tfrac{1}{2}\int_{-1}^{1} \Phi^r(\xi, \varepsilon)d\xi\Big]^{-\alpha_r}, \tag{2.12}$$

where $\Phi(x, \varepsilon) := \varphi_\varepsilon(x/\varepsilon)$ and $\alpha_r := \frac{q}{rq-p+1}$. For all $x \in [-1, 1]$ it satisfies the estimates

$$|\Phi(x, \varepsilon) - w_p(x/\varepsilon)| \le ce^{-1/\varepsilon}, \quad |\Phi'(x, \varepsilon) - w_p'(x/\varepsilon)| \le ce^{-1/\varepsilon}. \tag{2.13}$$

Using this internal spike solution Φ we can define two steady state solutions with one boundary spike. Namely,

$$\Phi_+(x, \varepsilon) := \Phi(\tfrac{x}{2} - \tfrac{1}{2}, \tfrac{\varepsilon}{2}), \quad \text{and} \quad \Phi_-(x, \varepsilon) := \Phi(\tfrac{x}{2} + \tfrac{1}{2}, \tfrac{\varepsilon}{2}), \quad 0 < \varepsilon < \varepsilon_{\rm o} \tag{2.14}$$

have spikes at $x = 1$ and at $x = -1$ respectively. Both functions are smooth periodic functions with period 4.

2.2. **Linearization around the one-spike solution.** To study stability of the spike solution, we consider the first variation of the shadow system (1.4)–(1.3) around this solution. It is convenient to rewrite this system as one equation:

$$\begin{gathered} u_t = \varepsilon^2 u_{xx} - u + g(u), \\ u_x(\pm 1, t) = 0, \; u(x, 0) = u_{\rm o}(x), \; u_{\rm o}'(\pm 1) = 0, \end{gathered} \tag{2.15}$$

where

$$g(u) = [\tfrac{1}{2}\int_{-1}^{1} u^r(\xi, t)d\xi]^{-q} u^p. \tag{2.16}$$

Let v be the variation around S; set $u(x, t) = S(x, \varepsilon) + v(x, t)$, then v satisfies

$$v_t = \varepsilon^2 v_{xx} - v + g(S + v) - g(S),$$
$$v_x(\pm 1, t) = 0, \quad v(x, 0) = v_{\rm o}(x) := u_{\rm o}(x) - S(x, \varepsilon),$$

or written in operator form

$$v_t + Av = f[v], \quad v(x, 0) = v_{\rm o}(x), \tag{2.17}$$

where f is the quadratic term

$$f[v] := \int_0^1 (1-\sigma)\partial_\sigma^2 g(S+\sigma v)\,d\sigma \tag{2.18}$$

where ∂_σ^2 denotes the second derivative of $\sigma \mapsto g(S+\sigma v)$ w.r.t. σ and where A is the spatial linear operator,

$$Av := -\varepsilon^2 v'' + v - p\Phi^{p-1}v + qr\Phi^p\Big(\int_{-1}^1 \Phi^r(\xi,\varepsilon)d\xi\Big)^{-1}\int_{-1}^1 \Phi^{r-1}(\xi,\varepsilon)v(\xi)d\xi, \tag{2.19}$$

defined on the Sobolev space $H^2(-1,1)$ with boundary conditions $v'(\pm 1) = 0$.

For the study of the spectrum and the study of stability using this spectrum it is convenient to stretch the spatial variable by $x = \varepsilon\xi$ and to define the operators $A_\varepsilon = L_\varepsilon + B_\varepsilon$ on the stretched interval $[-1/\varepsilon, 1/\varepsilon]$, where ($\dot u := du/d\xi$)

$$L_\varepsilon u := -\ddot u + u - p\varphi_\varepsilon^{p-1}u, \quad \mathcal{D}(L_\varepsilon) := \{u \in H^2([-\tfrac{1}{\varepsilon}, \tfrac{1}{\varepsilon}]) \big| \dot u(\pm\tfrac{1}{\varepsilon}) = 0\}. \tag{2.20}$$

Its limit $L_{\rm o}$ for $\varepsilon \to 0$ has domain $H^2(\mathbb{R})$. The non-local operator B_ε (of rank one) is defined by

$$B_\varepsilon = d_{r,\varepsilon}\langle \cdot, \varphi_\varepsilon^{r-1}\rangle \varphi_\varepsilon^p, \tag{2.21}$$

where

$$d_{r,\varepsilon} = \frac{\gamma_r(p-1)}{\beta_{r,\varepsilon}} = \frac{qr}{\beta_{r,\varepsilon}}, \quad \beta_{r,\varepsilon} = \int_{-1/\varepsilon}^{1/\varepsilon} \varphi_\varepsilon^r(\xi,\varepsilon)d\xi$$

and $\gamma_r > 1$ is given by $\gamma_r = qr/(p-1)$, see (1.2).

The derivative of φ_ε and powers of φ_ε are to be used in estimating eigenvalues etc. Below we give some useful relations. For all $m > 0$ we have

$$L_\varepsilon \varphi_\varepsilon^m = (1-m^2)\varphi_\varepsilon^m + \kappa_m \varphi^{m+p-1} + m(m-1)F(a_\varepsilon)\varphi_\varepsilon^{m-2}, \tag{2.22}$$

where $\kappa_m := m(2m+p-1)/(p+1) - p$. The third term is uniformly of the order $O(e^{-\min(m,2)/\varepsilon})$. In particular, the remainder is zero for $m = 1$:

$$L_\varepsilon \varphi_\varepsilon = (1-p)\varphi_\varepsilon^p. \tag{2.23}$$

It is also easy to check that

$$L_\varepsilon\big(\frac{\varphi_\varepsilon}{p-1} + \frac{\xi\dot\varphi_\varepsilon}{2}\big) = -\varphi_\varepsilon \quad \text{and} \quad L_\varepsilon \dot\varphi_\varepsilon = 0. \tag{2.24}$$

Finally, we can evaluate the integral $\beta_{m,0} := \int_{-\infty}^\infty w_p^m(\xi)d\xi$ in terms of the Gamma function,

$$\beta_{m,0} = \big(\frac{p+1}{2}\big)^{m/(p-1)} \frac{2}{p-1} \frac{\sqrt{\pi}\,\Gamma(\frac{m}{p-1})}{\Gamma(\frac{m}{p-1} + \frac{1}{2})}. \tag{2.25}$$

3. The spectrum of the differential operator

The eigenvalues of a selfadjoint differential operator $L := -d^2/dx^2 + Q(x)$ with domain $\mathcal{D}(L)$ of functions on a bounded or unbounded interval $I \subset \mathbb{R}$ satisfy the minimax property, see [12, Theorem XIII.1, p. 76]. If L has isolated eigenvalues $\lambda_{\rm o} \le \lambda_1 \le \lambda_2 \le \cdots$, ordered in increasing sense and counted according their multiplicity (and below the continuous spectrum if present), these satisfy

$$\lambda_k = \inf_{E\subset\mathcal{C},\ \dim(E)\ge k+1}\ \max_{u\in E,\ \|u\|=1} \langle Lu, u\rangle, \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product and where $\mathcal{C}$ is the domain of the operator.

The operator L_ε of our study (with "potential" $Q := 1 - p\varphi^{p-1}$) is a selfadjoint differential operator bounded from below and it has a discrete spectrum consisting of eigenvalues of multiplicity one for each $\varepsilon > 0$: $\lambda_o(\varepsilon) < \lambda_1(\varepsilon) < \lambda_2(\varepsilon) < \cdots$ with corresponding eigenfunctions $\psi_o(\cdot,\varepsilon), \psi_1(\cdot,\varepsilon), \psi_2(\cdot,\varepsilon), \cdots$. Its spectrum converges for $\varepsilon \to 0$ (and for all selfadjoint boundary conditions) to the spectrum of L_o, see e.g. [1, ch. 9]. We shall calculate the rate of convergence.

The "limiting" operator L_o (on the whole real axis) has the continuous spectrum $[1,\infty)$ and may have discrete eigenvalues below this interval (see [8, p. 140]). Simple calculations show:

$$
\begin{aligned}
&\psi_o := w_p^{\frac{p+1}{2}}, && L_o\psi_o = -\tfrac{(p-1)(p+3)}{4}\,\psi_o, && p > 1,\\
&\psi_1 := \dot w_p\,, && L_o\psi_1 = 0, && p > 1, \qquad (3.2)\\
&\psi_2 := w_p^{\frac{3-p}{2}} - \tfrac{1}{2}\tfrac{p+3}{p+1} w_p^{\frac{p+1}{2}}, && L_o\psi_2 = \tfrac{(p-1)(5-p)}{4}\,\psi_2, && 1 < p < 3.
\end{aligned}
$$

Since ψ_o, ψ_1 and ψ_2 have zero, one and two zeros respectively, and since the zeros of the eigenfunctions of second order ordinary differential operators interlace, $\lambda_o := -\frac{1}{4}(p-1)(p+3)$, $\lambda_1 := 0$ and (if $p < 3$) $\lambda_2 := \frac{1}{4}(p-1)(5-p)$ are the three smallest eigenvalues of L_o. In order to show that L_o does not have a second isolated eigenvalue for $p > 3$, we substitute $\psi(\xi) = \vartheta(\frac{p-1}{2}\xi)$ in the eigenvalue equation $L_o\psi = \lambda\psi$ using the explicit form of w_p from (2.2). This yields the equation

$$
M_p\vartheta := -\ddot\vartheta - 2p(p+1)(p-1)^{-2}\cosh^{-2}(\eta)\,\vartheta = \Big(\frac{2}{p-1}\Big)^2(\lambda-1)\vartheta = \mu\vartheta. \qquad (3.3)
$$

Since the "potential" in M_p is an increasing function of p, its eigenvalues are increasing functions of p by the minimax theorem (3.1). Since $\lambda_2 \to 1$ if $p \to 3$ from below, the second eigenvalue of M_p tends to zero for $p \nearrow 3$ and gets absorbed into the continuous spectrum if $p \geq 3$. So L_o has only two eigenvalues below 1 if $p \geq 3$.

In order to compute the exponentially small rate of convergence of the smallest eigenvalues $\lambda_o(\varepsilon)$ and $\lambda_1(\varepsilon)$ (and $\lambda_2(\varepsilon)$ if $p < 3$) of L_ε, we can use the technique of [5] and [6]. We compute (formally) approximate eigenfunctions and project them onto the true eigenfunctions; the residuals yields estimates for the eigenvalues.

The asymptotic expansion of $\lambda_k(\varepsilon)$ will be calculated explicitly for $k = 1$; the expansions for the other eigenvalues are not needed explicitly and are completely analogous. We use the same technique as in [5] and [6]. We compute an approximate eigenfunction w of unit norm $\|w\| = 1$ of the operator L_ε and we show that

$$
\langle L_\varepsilon w, w\rangle = \nu_\varepsilon(1 + O(\widetilde R_\varepsilon)) \quad\text{and}\quad \|L_\varepsilon w\|^2 = O(\widetilde R_\varepsilon R_\varepsilon), \qquad (3.4)
$$

where both $\widetilde R_\varepsilon = o(1)$ and $R_\varepsilon = o(1)$ as $\varepsilon \to 0$.

The generalized Fourier expansion of w in the true eigenfunctions $\{\psi_k | k = 0, 1, \cdots\}$ of L_ε is

$$
w = \sum_{k=0}^{\infty} c_k\psi_k \quad\text{with}\quad \sum_{k=0}^{\infty} |c_k|^2 = \|w\|^2 = 1. \qquad (3.5)
$$

Since all eigenvalues of L_ε except $\lambda_1(\varepsilon)$ are uniformly bounded away from $\lambda_1(0) = 0$ by a distance $d > 0$, we find from (3.4)

$$
1 - |c_1|^2 = \sum_{k=0,\ k\neq 1}^{\infty} |c_k|^2 \ \leq\ d^{-2} \sum_{k=0,\ k\neq 1}^{\infty} \lambda_k^2 |c_k|^2 \ \leq\ d^{-2}\|L_\varepsilon w\|^2 = O(\widetilde R_\varepsilon R_\varepsilon),
$$

implying that $|c_1|^2 = 1 + O(\widetilde{R}_\varepsilon R_\varepsilon)$. The estimate for the inner product in (3.4) now implies that

$$\langle L_\varepsilon w, w\rangle - \nu_\varepsilon = |c_1|^2\lambda_1(\varepsilon) - \nu_\varepsilon + \sum_{k=0,\ k\neq 1}^{\infty} \lambda_k|c_k|^2 = O(\widetilde{R}_\varepsilon(\nu_\varepsilon + R_\varepsilon))$$

and hence that

$$\lambda_1(\varepsilon) = \nu_\varepsilon + O(\widetilde{R}_\varepsilon(\nu_\varepsilon + R_\varepsilon)). \tag{3.6}$$

Let $\psi_1(\cdot,\varepsilon)$ be the true eigenfunction of L_ε corresponding to $\lambda_1(\varepsilon)$. We look for an approximate eigenfunction of the form $\psi_1(\cdot,\varepsilon) \approx \dot\varphi_\varepsilon +$ boundary layer corrections. Within the interval $[-1/\varepsilon, 1/\varepsilon]$ the tails of $\dot\varphi_\varepsilon$ are exponentially small by (2.8) and (2.11),

$$\dot\varphi_\varepsilon(\pm\tfrac{1}{\varepsilon}) = 0 \quad\text{and}\quad \ddot\varphi_\varepsilon(\pm\tfrac{1}{\varepsilon}) = 2\alpha e^{-1/\varepsilon}(1 + O(e^{-\sigma/\varepsilon})). \tag{3.7}$$

We construct boundary layer terms at both endpoints by standard matched asymptotic expansions. Suitable boundary layer corrections at the right and left endpoints are

$$\begin{aligned} h(\xi) &:= -\ddot\varphi_\varepsilon(\tfrac{1}{\varepsilon})\varrho(\varepsilon\xi)\exp\big(\xi - \tfrac{1}{\varepsilon}\big),\\ k(\xi) &:= \ddot\varphi_\varepsilon(-\tfrac{1}{\varepsilon})\varrho(-\varepsilon\xi)\exp\big(-\xi - \tfrac{1}{\varepsilon}\big),\\ \widetilde{\psi}_1 &:= \dot\varphi_\varepsilon + h + k \end{aligned} \tag{3.8}$$

where ϱ is a monotonous $\mathcal{C}^\infty$ cut-off function satisfying $\varrho(x) = 1$ if $x \geq 1/2$ and $\varrho(x) = 0$ if $x \leq 1/4$. From the definition it is clear that $\widetilde{\psi}_1$ satisfies the boundary conditions at $\xi = \pm 1/\varepsilon$ and

$$L_\varepsilon h = -p\varphi_\varepsilon^{p-1}h + \ddot\varphi_\varepsilon(\tfrac{1}{\varepsilon})\big(\varepsilon^2\varrho'' + 2\varepsilon\varrho'\big)\exp\big(\xi - \tfrac{1}{\varepsilon}\big). \tag{3.9}$$

In the first term the decay of φ_ε towards the boundary compensates (partially if $p < 2$) the increase of the boundary layer term h giving the order $O(e^{-(1+\sigma)/\varepsilon})$ with $\sigma := \min(1, p-1)$. Since the support of ϱ' is in $[1/4, \frac{1}{2}]$, the second term in the r.h.s. is of the order $O(e^{-3/2\varepsilon})$. For $L_\varepsilon k$ we have an analogous estimate. Hence,

$$\|L_\varepsilon\widetilde{\psi}_1\|^2 = O\left(e^{-3/\varepsilon} + e^{-2(1+\sigma)/\varepsilon}\right). \tag{3.10}$$

Since $L_\varepsilon\widetilde{\psi}_1 = L_\varepsilon(h+k)$ and $\dot{\widetilde{\psi}}_1(\pm 1/\varepsilon) = 0$, we can evaluate the quadratic form (3.4), using integration by parts:

$$\begin{aligned} \langle L_\varepsilon\widetilde{\psi}_1, \widetilde{\psi}_1\rangle &= \langle L_\varepsilon(h+k), \widetilde{\psi}_1\rangle = \Big[-(\dot h + \dot k)\widetilde{\psi}_1\Big]_{-1/\varepsilon}^{1/\varepsilon} + \langle h, L_\varepsilon h\rangle + \langle k, L_\varepsilon k\rangle\\ &= -8\alpha^2 e^{-2/\varepsilon}\big(1 + O(e^{-\sigma/\varepsilon})\big) \end{aligned} \tag{3.11}$$

By (3.6) we may conclude, see [15, eq. (43)]

$$\lambda_1(\varepsilon) = -8\alpha^2\zeta e^{-2/\varepsilon}\big(1 + O(e^{-\sigma/\varepsilon})\big), \quad \zeta^{-1} := \int_{-1/\varepsilon}^{1/\varepsilon} \dot\varphi_\varepsilon^2(\xi)d\xi. \tag{3.12}$$

The factor 8 is due to the factor 2 in (2.8), which makes the approximate eigenfunction two times as large as the function w_p used in [15].

The asymptotics of $\lambda_{\mathrm{o}}(\varepsilon)$ and $\lambda_2(\varepsilon)$ is derived in an analogous way. By analogy to (3.2) and (3.8) suitable approximate eigenfunction corresponding to $\lambda_{\mathrm{o}}(\varepsilon)$ and $\lambda_2(\varepsilon)$ are

$$\widetilde{\psi}_{\mathrm{o}} := \varphi_\varepsilon^{\frac{p+1}{2}} \quad\text{and}\quad \widetilde{\psi}_2 := \varphi_\varepsilon^{\frac{3-p}{2}} - \tfrac{1}{2}\,\tfrac{p+3}{p+1}\,\varphi_\varepsilon^{\frac{p+1}{2}}. \tag{3.13}$$

Their asset is, that they satisfy the boundary conditions exactly; however, $L_\varepsilon\psi - \lambda\psi$ is small but nonzero, due to (2.22). As an alternative, we can use the functions given in (3.2) plus boundary layer corrections, which produce error terms likewise. A lower bound on λ_2 for $p > 3$ is produced likewise as before, using the fact that φ_ε is well approximated by w_p according to lemma 2.2. So we have proved the following statement.

Theorem 3.1. *The three smallest eigenvalues of L_ε satisfy the asymptotic formulae for $p > 1$ and for $\varepsilon \to 0$*

$$\begin{aligned}
&\lambda_{\mathrm{o}}(\varepsilon) = -\tfrac{1}{4}(p-1)(p+3) + O\big(e^{-2/\varepsilon}\big)\\
&\lambda_1(\varepsilon) = -8\alpha^2\zeta e^{-2/\varepsilon}\big(1 + O(e^{-\sigma/\varepsilon})\big)\\
&\lambda_2(\varepsilon)\begin{cases} = \tfrac{1}{4}(p-1)(5-p) + O\left(e^{-(3-p)/\varepsilon}\right) & \text{if } \ p < 3\\ \geq 1 + O(e^{-2/\varepsilon}) & \text{if } \ p \geq 3.\end{cases}
\end{aligned} \tag{3.14}$$

where $\sigma := \min(1, p-1)$, see (2.7).

Remark 3.2. When we linearize around the single boundary spike steady state solutions (cf. (2.14)), we can do the same analysis as before by considering the respective operator L_ε on $H^2(0, 2/\varepsilon)$ (using the translational invariance). In comparison to the previous case this means that we have to consider on the interval $[-2/\varepsilon, 2/\varepsilon]$ the subset of even functions only and hence, that the odd-indexed eigenvalues are not present. Hence there is no exponentially small negative eigenvalue in this case: $\lambda_{\mathrm{o}}(\varepsilon)$ persists and the next eigenvalue is near $\frac{1}{4}(p-1)(5-p)$ if $p < 3$ or above 1 otherwise.

Remark 3.3. Likewise we can linearize the problem around w_p at any off-central location instead of φ_ε, like Ward & all do, [15, 16, 17]; we need not use an exact stationary solution of (1.4). For the eigenvalue analysis this makes little difference. All above results can be translated to this case, if we replace $1/\varepsilon$ in the estimates by the distance from the center of the (approximate) spike to the boundary.

4. Perturbation by the non-local term

In this section we consider how the non-local operator B_ε, defined in (2.21), perturbs the eigenvalue of L_ε with smallest real part. Both the operators L_ε and B_ε are invariant under change of sign $\xi \mapsto -\xi$ and hence leave the subspaces of even and odd functions invariant. Since B_ε maps odd functions to zero, the part of the spectrum of L_ε associated with odd eigenfunctions remains unchanged. *Hence we restrict our analysis exclusively to even functions in this section.* In particular, this removes the negative exponentially small eigenvalue $\lambda_1(\varepsilon)$ with asymptotics given by (3.14) from the spectrum under study, because it is associated with an odd eigenfunction. Except for λ_{o}, all eigenvalues of (the *even* part) of L_ε are positive and bounded from below by $\mu := \min(\lambda_2, 1)$. The addition of the non-local perturbation B_ε changes the eigenvalues; as seen from fig. 1 they may go off into the complex plane. Our goal is to find conditions on the parameters p, q, r so that the spectrum of A_ε (on even functions) lies in the right half -plane. Condition (1.2) is necessary. In [17] and [16] it is shown, that positivity is true if $r = p{+}1$ or if $r = 2$ and $1 < p < 5$. We show that positivity is true also if $r = (p+3)/2$ and, by perturbational techniques, in a wide area around these cases. Finally we consider a negative result, that the smallest eigenvalue remains in the negative half plane if

r is slightly larger than $(p-1)/q$ and $q>2$. For consistency of the exposition we repeat some arguments from the papers cited above. For ease of notation we use the spectral resolution of L_ε (restricted to the subspace of *even* functions).

$$L_\varepsilon u = \lambda_\mathrm{o}\langle u,\psi_\mathrm{o}\rangle\psi_\mathrm{o} + \int_{\mu_\varepsilon}^\infty \nu\, dE_\nu u, \tag{4.1}$$

where ψ_o is the normalized ground state of L_ε and μ_ε is the next eigenvalue $\mu_\varepsilon = \lambda_2(\varepsilon)$ or the bottom of the continuous spectrum (if $\varepsilon = 0$ and $p\ge 3$).

4.1. **Preliminaries.** First we show that $\lambda_\mathrm{o}(\varepsilon)$ is not in the spectrum of A_ε. Suppose the contrary, that λ_o is an eigenvalue with eigenfunction ψ, $A_\varepsilon\psi = \lambda_\mathrm{o}(\varepsilon)\psi$. Then $\langle\psi,\varphi_\varepsilon^{r-1}\rangle \neq 0$, because otherwise $B_\varepsilon\psi = 0$ and $L_\varepsilon\psi = \lambda_\mathrm{o}(\varepsilon)\psi$, such that ψ is a multiple of ψ_o, which is a positive and cannot have zero inner product with the positive function φ_ε. Because λ_o is an eigenvalue of L_ε, selfadjointness implies

$$0 = \langle (L_\varepsilon - \lambda_\mathrm{o}(\varepsilon))\psi,\psi_\mathrm{o}\rangle = -\langle B_\varepsilon\psi,\psi_\mathrm{o}\rangle = -d_{r,\varepsilon}\langle\psi,\varphi^{r-1}\rangle\langle\psi_\mathrm{o},\varphi^p\rangle \neq 0.$$

This contradiction shows that $\lambda_\mathrm{o}(\varepsilon)$ is not an eigenvalue for A_ε. In order to prove, in addition, that λ_o is in the resolvent set we must show that the inverse of $A_\varepsilon - \lambda_\mathrm{o}(\varepsilon)$ is bounded. Since this is a closed operator defined on the whole Hilbert space $L^2(-1/\varepsilon,1/\varepsilon)$, it suffices to show that we can solve the equation $(A_\varepsilon - \lambda_\mathrm{o}(\varepsilon))\psi = g$ for arbitrary g. To this end we apply Fourier's method and write g, φ_ε and ψ as a multiple of ψ_o and a rest orthogonal to it,

$$\begin{aligned} g &= c\psi_\mathrm{o} + g_\mathrm{o}, & & \langle g_\mathrm{o},\psi_\mathrm{o}\rangle = 0,\\ \psi &= l\psi_\mathrm{o} + h, & \text{where}\quad & \langle h,\psi_\mathrm{o}\rangle = 0,\\ \varphi_\varepsilon^p &= c_\mathrm{o}\psi_\mathrm{o} + h_\mathrm{o}, & & \langle h_\mathrm{o},\psi_\mathrm{o}\rangle = 0, \end{aligned}$$

with $c_\mathrm{o}\neq 0$. This splits the equation into an equation for h and a scalar one for l:

$$(L_\varepsilon - \lambda_\mathrm{o}(\varepsilon))h = g_\mathrm{o} - \frac{c}{c_\mathrm{o}}h_\mathrm{o}, \qquad l c_\mathrm{o}\langle\psi_\mathrm{o},\varphi^{r-1}\rangle + \langle h,\varphi^{r-1}\rangle c_\mathrm{o} = c.$$

Since both have a unique solution, we conclude that $\lambda_\mathrm{o}(\varepsilon)$ is not in the spectrum of A_ε.

In this way the problem is reduced to investigate the spectrum of A_ε on the resolvent set of L_ε. There we can use the formula of Sherman-Morrison for computing the inverse,

$$\begin{aligned}(A_\varepsilon - \lambda)^{-1} &= \left(1 + (L_\varepsilon-\lambda)^{-1}B_\varepsilon\right)^{-1}(L_\varepsilon-\lambda)^{-1}\\ &= \Big(1 - \frac{(L_\varepsilon-\lambda)^{-1}B_\varepsilon}{h_\varepsilon(\lambda)}\Big)(L_\varepsilon-\lambda)^{-1},\end{aligned} \tag{4.2}$$

where the function $h_\varepsilon(\lambda)$ in the denominator by (2.23) and (4.1) satisfies

$$\begin{aligned} h_\varepsilon(\lambda) &:= 1 + d_{r,\varepsilon}\langle (L_\varepsilon-\lambda)^{-1}\varphi_\varepsilon^p,\varphi_\varepsilon^{r-1}\rangle\\ &= 1 - \frac{d_{r,\varepsilon}}{p-1}\langle (L_\varepsilon-\lambda)^{-1}L_\varepsilon\varphi_\varepsilon,\varphi_\varepsilon^{r-1}\rangle\\ &= 1 - \frac{qr}{p-1} - \frac{d_{r,\varepsilon}}{p-1}\langle (L_\varepsilon-\lambda)^{-1}\varphi_\varepsilon,\varphi_\varepsilon^{r-1}\rangle\\ &= 1 - \frac{qr}{p-1} - \frac{qr\lambda}{\beta_{r,\varepsilon}(p-1)}\left\{\frac{\langle\varphi_\varepsilon,\psi_\mathrm{o}\rangle\langle\psi_\mathrm{o},\varphi_\varepsilon^{r-1}\rangle}{\lambda_\mathrm{o}-\lambda} + \int_{\mu_\varepsilon}^\infty \frac{\langle dE_\nu\varphi_\varepsilon,\varphi_\varepsilon^{r-1}\rangle}{\nu-\lambda}\right\}. \end{aligned} \tag{4.3}$$

This formula shows that h_ε is analytic on the resolvent set of L_ε and that any zero of h_ε outside the spectrum of L_ε is an eigenvalue of A_ε with associated eigenfunction

$\psi := (L_\varepsilon - \lambda)^{-1}\varphi_\varepsilon^p$. Its value at $\lambda = 0$ is given by $h_\varepsilon(0) = 1 - \frac{qr}{p-1} = 1 - \gamma_r < 0$. Since $h(\lambda) \searrow -\infty$ if λ is real and $\lambda \searrow \lambda_o$, $h(\lambda)$ has a negative real zero if $\gamma_r < 1$, i.e. if condition (1.2) is not satisfied. Moreover, this formula implies

$$|h_\varepsilon(\lambda) - 1 + \frac{qr}{p-1}| \le \frac{d_{r,\varepsilon}\|\varphi_\varepsilon\|\|\varphi_\varepsilon^{r-1}\|}{p-1}\frac{|\lambda|}{\min(|\lambda-\mu_\varepsilon|,|\lambda_o-\lambda|)}, \tag{4.4}$$

such that a circle around 0 of radius $O(1-\frac{p-1}{qr})$ (for $qr+p-1\searrow 0$) must be contained in the resolvent set of A_ε.

4.2. **The selfadjoint case** $r = p+1$**.** Since L_ε is a selfadjoint operator and $B_\varepsilon u = d_{r,\varepsilon}\langle u, \varphi_\varepsilon^{r-1}\rangle\varphi_\varepsilon^p$, the operator A_ε is selfadjoint if $r = p+1$ and has real eigenvalues. For real λ the derivative of h satisfies

$$h_\varepsilon'(\lambda) = d_{r,\varepsilon}\|(L_\varepsilon - \lambda)^{-1}\varphi_\varepsilon^p\|^2 \ge 0, \quad \lambda \in \mathbb{R}.$$

Hence h is monotonically increasing and can not have negative zeros, because $h_\varepsilon(0) < 0$.

To find the magnitude of perturbations around this case that conserve the positivity, we derive a lower bound for the smallest eigenvalue. Because the resolution of the identity of L_ε depends smoothly on $\varepsilon \in [0, \varepsilon_o]$ for all λ in a compact subset of the resolvent of L_o, we may compute a lower estimate for $\varepsilon = 0$ and extend the result by continuity to $[0, \varepsilon_o(p,q,r)]$. Let α be the smallest eigenvalue of A_o. Evidently $\alpha < \mu_o$. We estimate α from below by an upper estimate of the function h_o given in (4.3). Using (2.23), we see that for $r = p+1$ the integral in its right-hand side may be written as

$$-\int_{\mu_o}^\infty \frac{\langle dE_\nu\varphi_o, \varphi_o^{r-1}\rangle}{\nu-\lambda} = \int_{\mu_o}^\infty \frac{\nu}{p-1}\frac{\langle dE_\nu\varphi_o,\varphi_o\rangle}{\nu-\lambda} = \int_{\mu_o}^\infty \frac{p-1}{\nu}\frac{\langle dE_\nu\varphi_o^p,\varphi_o^p\rangle}{\nu-\lambda} \tag{4.5}$$

and that both variants are positive for real $\lambda < \mu_o$ and can be estimated from above by

$$\frac{M}{\mu_o-\lambda} := \min\Big(\frac{(p-1)(\|\varphi_o^p\|^2 - \langle\psi_o,\varphi_o^p\rangle^2)}{\mu_o(\mu_o-\lambda)}, \frac{\mu_o(\|\varphi_o\|^2 - \langle\psi_o,\varphi_o\rangle^2)}{(p-1)(\mu_o-\lambda)}\Big)$$

Inserting this in (4.3) we find a simple upper estimate for $h_o(\lambda)$ which is monotone on (λ_o, μ_o); hence it has a zero in this interval, that can be computed from a simple quadratic equation. For some values of q this lower bound is plotted in figure 3. Since the integral (4.5) is positive, we obtain from (4.3) an upper bound for α by omitting the integral.

Having the above control on the smallest positive eigenvalue of the selfadjoint operator A_ε in the case $r = p+1$, we can consider it for neighbouring values of r as a perturbation of the selfadjoit one. Namely, if $A = L + B$, where L is selfadjoint and B has relatively small norm, then from the formulae

$$\begin{aligned}(A-\lambda)^{-1} &= (1 + (L-\lambda)^{-1}B)^{-1}(L-\lambda)^{-1},\\ \|(L-\lambda)^{-1}\| &\le 1/\operatorname{dist}(\lambda,\sigma(L)),\end{aligned} \tag{4.6}$$

where $\sigma(L)$ is the spectrum of L, it follows that if λ is in the spectrum of A then the distance from λ to the spectrum of L is at most the norm of B. In particular, $\Re\lambda \ge \alpha - \|B\|$, where α is the smallest real part of the spectrum of L. In our particular case

$$B = u \mapsto \langle u, d_{r,\varepsilon}\varphi_\varepsilon^{r-1} - d_{p+1,\varepsilon}\varphi_\varepsilon^p\rangle\varphi_\varepsilon^p$$

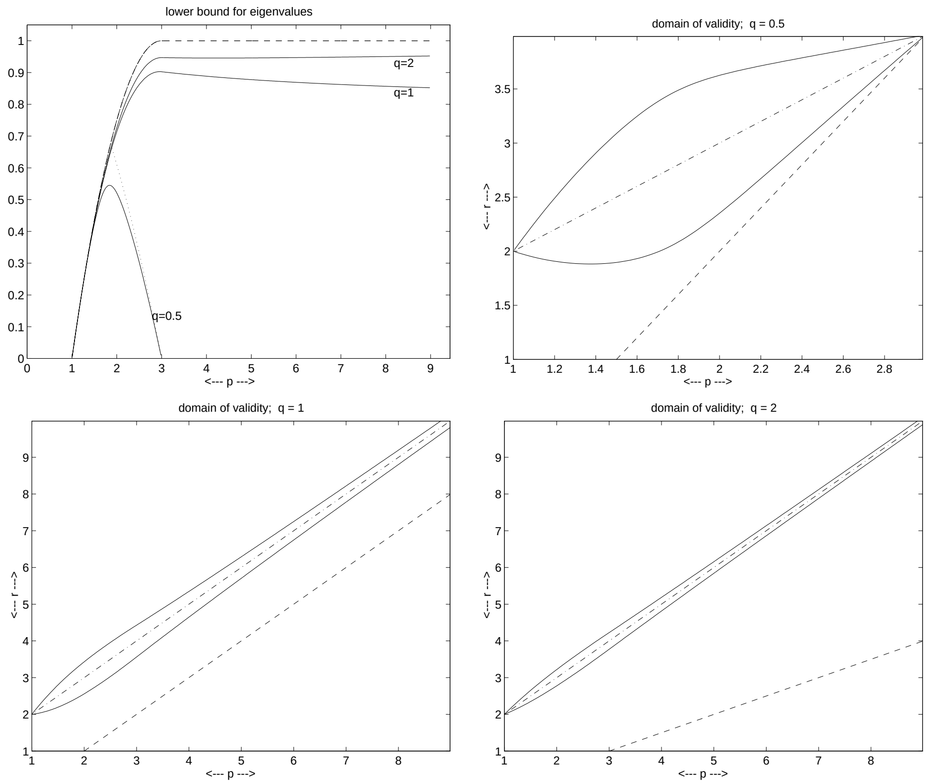

FIGURE 3. Upper left: Lower and upper bounds for the smallest eigenvalue of A_{o} for q=0.5, q=1 and q=2. The lower bound is a monotone function of q (as is the eigenvalue itself). The dashed curve is μ_{o} (the second eigenvalue or the bottom of the continuous spectrum), which provides an upper bound for all q. The dotted line provides a sharper upper bound for q=0.5. The other pictures give the corresponding domains in $p\times r$-plane around the selfadjoint line (dash-dotted), where the perturbations are smaller than the lower bound and the real part of the smallest eigenvalue remains positive. The width of this domain is (more or less) inversely proportional to q. The dashed line represents condition (1.2).

and we can estimate the real part of any eigenvalue λ of A_ε:

$$\Re\lambda \geq \alpha_\varepsilon - \|\varphi_\varepsilon^p\| \|d_{r,\varepsilon}\varphi_\varepsilon^{r-1} - d_{p+1,\varepsilon}\varphi_\varepsilon^p\|,$$

hence all eigenvalues of A_ε will have positive real part for all $0 < \varepsilon < \varepsilon_{\mathrm{o}}(p,q,r)$ if (p,q,r) satisfy

$$\|w_p^p\| \, \|d_{r,\mathrm{o}}w_p^{r-1} - d_{p+1,\mathrm{o}}w_p^p\| < \alpha_{\mathrm{o}}. \tag{4.7}$$

4.3. **The case $p = 2r-3$.** Using Gershgorin's theorem we prove that spectrum of A_ε (restricted to the subspace of *even* functions) is in the positive half plane for a large domain in parameter space (p, q, r) around the plane $p = 2r-3$. As before, we consider first the case $\varepsilon = 0$.

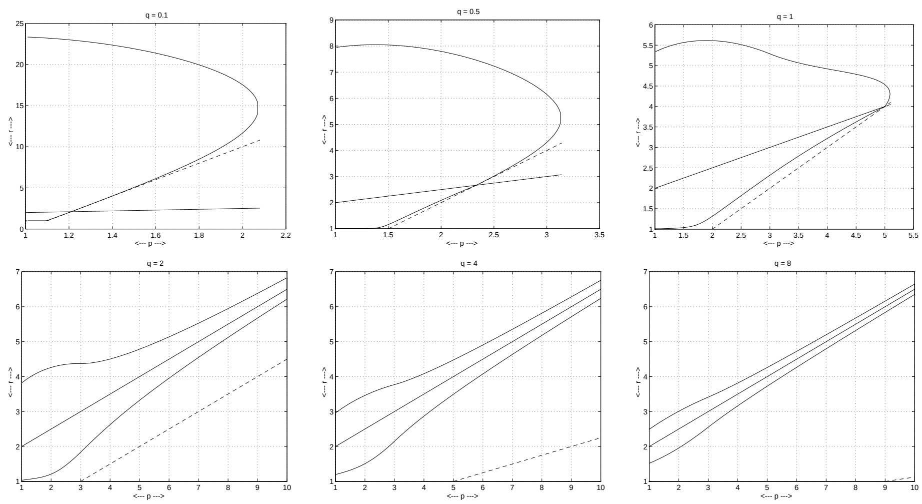

FIGURE 4. Domains in the $p{\times}r$-plane, where lemma 4.1 implies positivity of the real part of any eigenvalue, pictured for $q = 0.1,\ 0.5,\ 1,\ 2,\ 4$ and 8. The dashed line represents condition (1.2); below it the smallest real part is negative, see. The straight line, emanating from the point $p = 1$ and $r = 2$ is the line $2r = p + 3$, at which positivity is proved in remark 4.2, provided (1.2) is satisfied.

Let (λ, ψ) with $\Re\lambda < \mu_{\rm o}$ be an eigenpair of $A_{\rm o}$ and let $\psi = \xi\psi_{\rm o} + \eta g$ be the orthogonal decomposition with $\langle \psi_{\rm o}, g\rangle = 0$ and $\|g\| = 1$, then λ is eigenvalue and $(\xi, \eta) \in \mathbb{R}^2$ the corresponding eigenvector of the matrix

$$M_g = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} := \begin{pmatrix} \langle L_{\rm o}\psi_{\rm o}, \psi_{\rm o}\rangle + \langle B_\varepsilon \psi_{\rm o}, \psi_{\rm o}\rangle & \langle B_\varepsilon g, \psi_{\rm o}\rangle \\ \langle B_\varepsilon \psi_{\rm o}, g\rangle & \langle L_{\rm o} g, g\rangle + \langle B_\varepsilon g, g\rangle \end{pmatrix}. \tag{4.8}$$

With constants c_1, c_2, d_1 and d_2 defined by

$$c_1 := \langle \varphi_{\rm o}^p, \psi_{\rm o}\rangle, \quad c_2 := \langle \psi_{\rm o}, \varphi_{\rm o}^{r-1}\rangle, \quad d_1^2 := \|\varphi_{\rm o}^p\|^2 - c_1^2 \quad \text{and} \quad d_2^2 := \|\varphi_{\rm o}^{r-1}\|^2 - c_2^2$$

the elements of M_g are given by

$$\begin{aligned} m_{11} &= \lambda_{\rm o} + d_{r,\varepsilon} c_1 c_2, \quad m_{12} = d_{r,\varepsilon} c_1 \langle g, \varphi_{\rm o}^{r-1}\rangle, \\ m_{21} &= d_{r,\varepsilon} c_2 \langle \varphi_{\rm o}^p, g\rangle, \quad m_{22} = \langle L_{\rm o} g, g\rangle - d_{r,\varepsilon} \end{aligned}$$

and satisfy for any (normalized) function g orthogonal to $\psi_{\rm o}$ the inequalities

$$|m_{12}| \le d_{r,\varepsilon} c_1 d_2, \quad |m_{21}| \le d_{r,\varepsilon} c_2 d_1, \quad \Re m_{22} \ge \mu_{\rm o} - d_{r,\varepsilon} d_1 d_2 \tag{4.9}$$

Lemma 4.1. *If both m_{11} and $\mu_{\rm o} - d_{r,\varepsilon} d_1 d_2$ are positive, then all eigenvalues of $A_{\rm o}$ have positive real part if*

$$m_{11}(\mu_{\rm o} - d_{r,\varepsilon} d_1 d_2) > (d_{r,\varepsilon})^2\, c_1 c_2 d_1 d_2 \tag{4.10}$$

Proof. From Gershgorin's circle theorem (see [4], theorem 7.2.1) we know that the eigenvalues of M_g are in the union of the disks around m_{11} and m_{22} with radii $|tm_{12}|$ and $|m_{21}/t|$ respectively for any $t \in \mathbb{R}$. If both diagonal elements have positive real part, the Gershgorin circles around them are contained in the positive half plane if their radii are smaller than those real parts. The optimal choice for t, valid for all g, leads to condition (4.10). □

Remark 4.2. If $2r = p + 3$, the non-local term B is zero on the orthogonal complement of ψ_o. This implies that $m_{12} = 0$ and $m_{22} \geq \mu_o$. Hence, positivity of the spectrum reduces to the condition $m_{11} > 0$. Using relation (2.23) we find

$$\begin{aligned} m_{11} &= \lambda_o + d_{r,\varepsilon}\langle\varphi_o^p, \psi_o\rangle\langle\psi_o, \varphi_o^{(p+1)/2}\rangle \\ &= \lambda_o - \frac{qr\lambda_o\langle\varphi_o, \psi_o\rangle\|\varphi_o^{(p+1)/2}\|}{\beta_{r,\varepsilon}(p-1)} \\ &= \lambda_o(1 - \frac{qr}{p-1}) > 0, \end{aligned}$$

implying positivity in the region satisfying (1.2). □

Since the significance of condition (4.10) is difficult to grasp, we have computed the region in $p\times r$-plane where it is satisfied and plotted its boundary for several values of q in figure 4. Clearly, the line $2r = p + 3$ is always inside the computed domain. Moreover, we see that r may take large values if q is small. If q and/or r are large, the non-local term is large too. Its contribution to m_{11} in (4.9) is always positive. However, in the lower estimate, that we have to use for m_{22}, it gives a negative contribution. It can be made small only if the components of φ_ε^p and $\varphi_\varepsilon^{r-1}$ in the orthogonal complement of ψ_o are small.

For $\varepsilon > 0$ and small we may conclude as before, that all eigenvalues of A_ε will have positive real part for all $0 < \varepsilon < \varepsilon_o(p, q, r)$ if (4.10) is satisfied.

4.4. **A negative result.** As we pointed out in the beginning of this section, condition (1.2) is necessary for positivity of the real parts of all eigenvalues. It is however not sufficient; we show that h_ε has a negative zero near the line $qr = p - 1$ and $q > 3$. As before, we do the analysis for $\varepsilon = 0$ and extend the result to $\varepsilon > 0$ by continuity. We can compute the next term in the expansion (4.3) of h_o, writing

$$h_o(\lambda) = 1 - \frac{qr}{p-1} - \frac{d_{r,o}\lambda g(\lambda)}{p-1}, \qquad g(\lambda) := \langle(L_o - \lambda)^{-1}\varphi, \varphi^{r-1}\rangle.$$

By (2.24) we can compute the value of g at zero

$$g(0) = \langle L_o^{-1}\varphi, \varphi^{r-1}\rangle = \langle\frac{\varphi}{p-1} + \frac{\xi\varphi'}{2}, \varphi^{r-1}\rangle = (\frac{1}{2r} - \frac{1}{p-1})\beta_{r,o}. \tag{4.11}$$

and for the remainder R with $g(\lambda) = g(0) + \lambda R(\lambda)$ we have

$$R(\lambda) = \langle[(L_o - \lambda)^{-1} - L_o^{-1}]\varphi, \varphi^{r-1}\rangle/\lambda = \langle(L_o - \lambda)^{-1}L_o^{-1}\varphi, \varphi^{r-1}\rangle. \tag{4.12}$$

From (4.11) we see that $g(0) > 0$ if $p > 2r + 1$ and, hence, that h has a negative slope at $\lambda = 0$; since $h(0) < 0$ this may imply that $h(\lambda)$ may be positive for some $\lambda < 0$ and hence that h has two negative zeros, if the remainder R is not too bad. Using the spectral resolution (4.1) of L_o (with $\mu_o = 1$ because $p > 2r+1 \geq 3$) we get for real λ in the interval $[-1, 0]$

$$R(\lambda) = \frac{\langle\varphi_o, \psi_o\rangle\langle\psi_o, \varphi_o^{r-1}\rangle}{\lambda_o(\lambda_o - \lambda)} + \int_1^\infty \frac{\langle dE_\nu\varphi_o, \varphi_o^{r-1}\rangle}{\nu(\nu - \lambda)} \leq \frac{c_1c_2}{|\lambda_o|(|\lambda_o| - 1)} + d_1d_2 =: a_o,$$

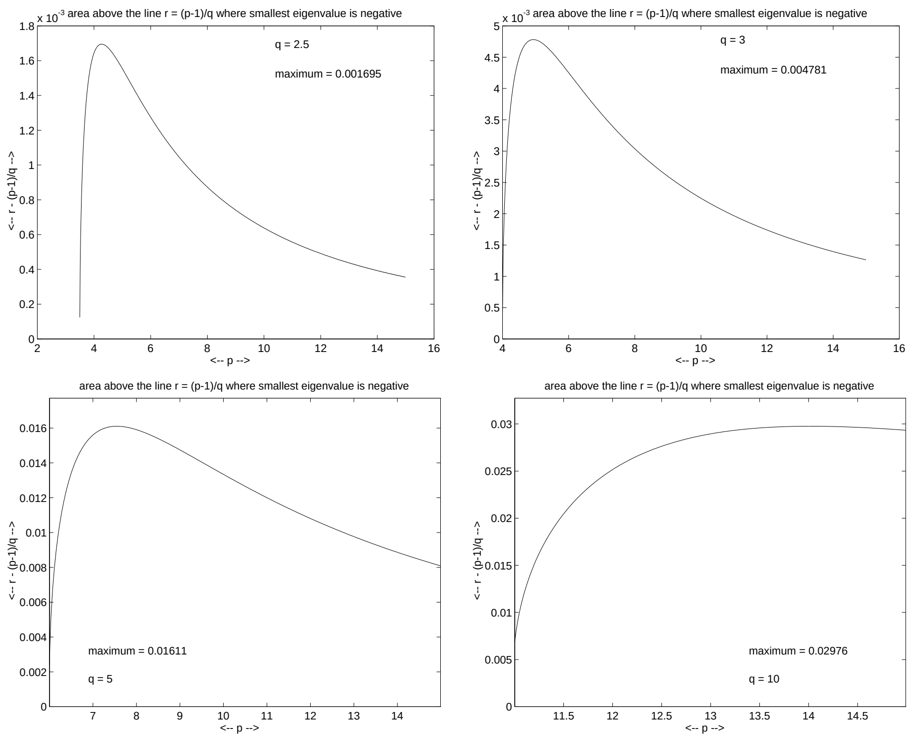

FIGURE 5. Region above the line $qr = p - 1$ with $q > 3$, where the operator A_ε (restricted to the subspace of even functions) has two real negative eigenvalues. Because the region is quite small, we rescale the domain and plot vertically $r - \frac{p-1}{q}$ versus horizontally p for the values $q = 2.5,\ 3,\ 5$, and 10.

where c_j and d_j are given in (4.9). Hence, we can estimate h_o from below for $\lambda \in [-1, 0]$ by the quadratic

$$h_\text{o}(\lambda) \geq 1 - \frac{qr}{p-1} - \frac{q\lambda(p-1-2r)}{2(p-1)^2} - \frac{d_{r,\text{o}} a_\text{o} \lambda^2}{p-1}, \tag{4.13}$$

for which we can easily check, whether it has positive values on $[-1, 0]$. In figure 5 we have plotted the small strip above the line $qr = p - 1$ with $q > 3$, where this estimate proves the existence of two negative eigenvalues. An obvious drawback of the method is, that it detects a negative result only if there are two real negative eigenvalues and that the method fails if both eigenvalues coalesce and go off into a complex pair with a negative real part as can be seen in figure 6, where the (real parts of) the four smallest eigenvalues (of a numerical discretisation) are plotted in the neighbourhood of the critical point, where $rq = p - 1$.

5. Contraction around the steady state $S(x, \varepsilon)$

In this section we study metastability of the spike solution S of (1.3–1.4) as given in (2.12). We assume that the parameters (p, q, r) are such that all eigenvalues of

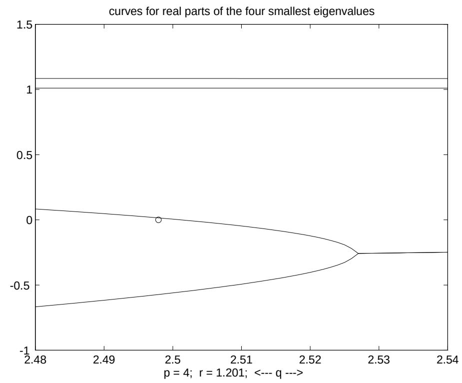

FIGURE 6. Plot of (the real parts of) the four eigenvalues with smallest real part as a function of q if $p = 2.5$ and $r = 1.201$ near the critical value of q, where $p - 1 = 2r$.

A_ε are located in the right half plane, except for the exponentially small negative eigenvalue $\lambda_1(\varepsilon)$ associated with an odd (or antisymmetric) eigenfunction.
NORMS. Besides the standard L^2-norm for functions on the interval $[-1, 1]$ denoted by $\|\cdot\|$, we use in this section the "energy norm" $\|\cdot\|_1$, and the "operator norm" $\|\cdot\|_2$, which are associated naturally to a problem with a small parameter like (2.1) and are defined by:

$$\|u\|_1^2 := \|u\|^2 + \|\varepsilon u'\|^2 \quad \text{and} \quad \|u\|_2^2 := \|u\|^2 + \|\varepsilon^2 u''\|^2.$$

For fixed positive a and uniformly for all $\varepsilon \in (0, \varepsilon_o]$ these norms satisfy the equivalences

$$\langle (A + a)u, u\rangle^{1/2} \asymp \|u\|_1 \quad \text{and} \quad \|(A + a)u\| \asymp \|Au\| + a\|u\| \asymp \|u\|_2 \quad (a > 0) \tag{5.1}$$

We denote by $\|\cdot\|_\infty$ the supremum norm (on the continuous functions); it satisfies the Sobolev inequality

$$\|u\|_\infty \le \sqrt{2/\varepsilon}\|u\|_1. \tag{5.2}$$

Finally, C denotes a generic positive constant that may differ at each occurrence.

Metastability is generally used in a vague way in association with small eigenvalues of some linearized operator. The idea is, that a solution S is to be called metastable, if a small perturbation of the initial condition yields a solution that remains in the vicinity of S during a long time interval, whose length grows beyond bound, if $\varepsilon \to 0$. To be precise:

Definition 5.1. The spike solution $S(\cdot, \varepsilon)$, $0 < \varepsilon < \varepsilon_o$, is called metastable in the norm $|\cdot|$ if there exist (monotonous) order functions $\delta(\varepsilon) \to 0$, $\delta_o(\varepsilon) \to 0$ and $T(\varepsilon) \to \infty$ as $\varepsilon \to 0$, such that if a solution U of (1.4) initially satisfies $|U(\cdot, 0) - S| < \delta_o(\varepsilon)$, then it satisfies $|U(\cdot, t) - S| < \delta(\varepsilon)$ for all $0 < t < T(\varepsilon)$, $0 < \varepsilon < \varepsilon_o$.

Obviously, we are not satisfied with pure existence of such order functions and we want to find explicit estimates on their decay or growth.

We study perturbations around the steady state spike solution S. As in [6] we use a contraction method to show, that a solution starting near S stays confined to a small tube around S for an (exponentially) long time lapse. The perturbation satisfies eq. (2.17), which reads:

$$v_t + Av = f[v], \quad v(x,0) = v_{\rm o}(x),$$

where the quadratic term f is given by (2.18) and the (linear) operator A is defined by (2.19). Obviously, this operator A has the same spectral properties as its (stretched) cousin A_ε has in sections 3 and 4. Under the positivity condition, stated above, A is a sectorial operator, see [9]: there exists an angle $\chi \in (0, \pi/2)$, such that the resolvent set of A contains the sector

$$\Lambda := \{\lambda \in \mathbb{C} \big| \chi \le |\arg(\lambda - \lambda_1(\varepsilon))| \le \pi,\ \lambda \ne \lambda_1(\varepsilon)\}.$$

In this sector the resolvent satisfies for some constant M, not depending on ε for all small ε, the estimate

$$\|(A-\lambda)^{-1}\| \le \frac{M}{|\lambda - \lambda_1(\varepsilon)|} \quad \text{for all } \lambda \in \Lambda.$$

Associated to A is the semigroup

$$e^{-At} := \frac{1}{2\pi i}\int_\Gamma (A-\lambda)^{-1} e^{-\lambda t} d\lambda, \quad t > 0, \tag{5.3}$$

where Γ is a suitable contour in the resolvent set Λ.

Lemma 5.2. *For all $t > 0$, all $\varepsilon \in (0, \varepsilon_0]$ and for some constant $c > 0$ not depending on t and ε this semigroup satisfies:*

$$\|e^{-At}\| \le Ce^{-\lambda_1(\varepsilon)t}, \tag{5.4}$$

$$\|Ae^{-At}\| \le C\,(|\lambda_1| + t^{-1})e^{-\lambda_1(\varepsilon)t}, \tag{5.5}$$

$$\|e^{-At}\,u\|_1 \le Ce^{-\lambda_1(\varepsilon)t} \begin{cases} \|u\|_1, \\ (1+t^{-1/2})\,\|u\|, \end{cases} \tag{5.6}$$

$$\|e^{-At}\,u\|_2 \le C\,e^{-\lambda_1(\varepsilon)t}\,\|u\|_2\,. \tag{5.7}$$

Proof. We choose in (5.3) the contour $\Gamma := \{-\delta + \lambda_1 + \varrho e^{\pm i\chi} | \varrho \in \mathbb{R}^+\}$ for some $\delta > 0$ to be determined later on. Because $\cos\chi > 0$, we may estimate the resolvent for any point in this set by

$$\|(A + \delta - \lambda_1 + \varrho e^{\pm i\chi})^{-1}\| \le \frac{M}{|\delta + \varrho e^{\pm i\chi}|} \le \frac{M}{\sqrt{(1-\cos\chi)(\delta^2+\varrho^2)}}.$$

Hence,

$$\|e^{-At}\| \le \frac{Me^{\delta t - \lambda_1 t}}{\pi\sqrt{1-\cos\chi}}\int_0^\infty \frac{e^{-\varrho t\cos\chi}}{\sqrt{\varrho^2+\delta^2}}d\varrho.$$

Clearly, the choice $\delta = 1/t$ yields the desired constant for a proof of (5.4). To check (5.5) we use the same contour and δ in the t-derivative of (5.3):

$$\begin{aligned}\|Ae^{-At}\| &= \|\frac{1}{2\pi i}\int_\Gamma (A-\lambda)^{-1}e^{-\lambda t}\lambda d\lambda\| \\ &\le \frac{Me^{\delta t-\lambda_1 t}}{\pi\sqrt{1-\cos\chi}}\int_0^\infty \frac{e^{-\varrho t\cos\chi}}{\sqrt{\varrho^2+\delta^2}}(|\delta| + |\lambda_1| + \varrho)d\varrho.\end{aligned}$$

To prove (5.7) we use the equivalence (5.1)

$$\|e^{-At}u\|_2 \asymp \|e^{-At}u\| + \|e^{-At}Au\| \le Ce^{-\lambda_1(\varepsilon)t}(\|u\| + \|Au\|) \le Ce^{-\lambda_1(\varepsilon)t}\|u\|_2.$$

The inequalities (5.6) results from interpolation between (5.7) and (5.4). $\square$

Remark 5.3. Let $\overline{A}$ be the restriction of A to the orthogonal complement of the (true) eigenfunction $\psi_1(x/\varepsilon,\varepsilon)$ associated with λ_1 and let $2\mu_1$ be equal to the real part of the eigenvalue of $\overline{A}$ with smallest real part, then the semigroup generated by this restriction satisfies the estimates

$$\|e^{-\overline{A}t}\|_j \le Ce^{-\mu_1 t}\|u\|_j,\ j=0,\ 1 \text{ or } 2, \tag{5.8}$$

$$\|\overline{A}e^{-\overline{A}t}\| \le Ct^{-1}e^{-\mu_1 t}, \tag{5.9}$$

proved in analogous way.

Now we estimate the nonlinear term $f[v]$ in (2.17), using $\langle\cdot,\cdot\rangle$ for the inner product in $L^2(-1,1)$. From $g(u) = 2^q\langle u^r,1\rangle^{-q}u^p$ we find the explicit formula

$$\begin{aligned} f[v] = 2^q\int_0^1 (1-\sigma)\Big\{ & p(p-1)\langle(S+\sigma v)^r,1\rangle^{-q}(S+\sigma v)^{p-2}v^2 + \\ & -2pqr\langle(S+\sigma v)^r,1\rangle^{-q-1}\langle(S+\sigma v)^{r-1},v\rangle(S+\sigma v)^{p-1}v + \\ & +q(q+1)\langle(S+\sigma v)^r,1\rangle^{-q-2}\langle(S+\sigma v)^{r-1},v\rangle^2(S+\sigma v)^p + \\ & -qr(r-1)\langle(S+\sigma v)^r,1\rangle^{-q-1}\langle(S+\sigma v)^{r-2},v^2\rangle(S+\sigma v)^p\Big\}d\sigma, \end{aligned} \tag{5.10}$$

provided $(S+\sigma v)$ is strictly positive. Using (2.11), (2.12) and (2.25)

$$\int_{-1}^1 \Phi^r(\xi,\varepsilon)d\xi \asymp \int_{-1}^1 w_p^r(\xi/\varepsilon)d\xi \asymp \varepsilon$$

implying

$$\int_{-1}^1 S^r d\xi \asymp \varepsilon^{1-r\alpha_r} = \varepsilon^{(1-p)/(qr+1-p)},$$

and assuming that $v(\cdot,t)\in H^1(-1,1)$ is bounded in modulus by $\varrho S(\cdot,\varepsilon)$ for some $\varrho\in(0,\frac12)$, we find the estimate

$$|f[v](x,\cdot)| \le C\varrho^2\varepsilon^{q(p-1)/(qr-p+1)}S^p(x,\varepsilon) \tag{5.11}$$

if $|v(x,\cdot)|\le\varrho S(x,\varepsilon)$ and $\varrho\le 1/2$. To estimate the difference $f[v]-f[w]$ we write

$$f[v]-f[w] = \int_0^1 (1-\sigma)\partial_\sigma^2 g(S+w+\sigma(v-w))d\sigma + \int_0^1 \partial_\sigma[\partial_\tau g(S+\sigma w+\tau(v-w))_{|\tau=0}]d\sigma$$

Assuming that $v(\cdot,t)$ and $w(\cdot,t)\in H^1(-1,1)$ are bounded in modulus by $\varrho S(\cdot,\varepsilon)$ and that the difference satisfies $|v(\cdot,t)-w(\cdot,t)|\le\tau S(\cdot,\varepsilon)$ we find by analogy to (5.10)

$$|f[v]-f[w]| \le C\tau\varrho\varepsilon^{q(p-1)/(qr-p+1)}S^p(x,\varepsilon) \tag{5.12}$$

Now we have the problem, that bounds weighted by S are not compatible with the L^2-bounds in lemma 5.2. We choose here for a rough way out by using the maximum and the minimum of S for estimates from above and below:

$$0 < c_1\varepsilon^{-\alpha_r}e^{-1/\varepsilon} \le S(x,\varepsilon) \le c_2\varepsilon^{-\alpha_r} \tag{5.13}$$

for some positive constants c_1 and c_2. This certainly will influence the magnitude of the region of contraction but not the existence of such a region. Using (5.2), we get the estimates

$$\|f[v]\| \le cR(\varepsilon)\|v\|_1^2, \quad \text{and} \quad \|f[v]-f[w]\| \le cR(\varepsilon)(\|v\|_1+\|w\|_1)\|v-w\|_1, \tag{5.14}$$

where

$$R(\varepsilon) := \varepsilon^{\alpha_r - 1} e^{2/\varepsilon}. \tag{5.15}$$

To apply the contraction method we consider the integral equation, equivalent to (2.17),

$$v = Gv, \quad \text{where} \quad Gv(\cdot, t) = e^{-At} v_{\rm o} + \int_0^t e^{-A(t-s)} f[v(\cdot, s)] ds.$$

On the time interval $[0, T]$ we have for all t the estimates:

$$\|Gv(\cdot, t)\|_1 \le c e^{|\lambda_1| T} \|v_{\rm o}\|_1 + c R(\varepsilon)(1 + T) e^{|\lambda_1| T} \sup_{0<s<T} \|v(\cdot, s)\|_1^2,$$

and

$$\begin{aligned} &\|Gv(\cdot, t) - Gw(\cdot, t)\|_1 \le c e^{|\lambda_1| T} \\ &\quad \times \Big\{ \|v_{\rm o} - w_{\rm o}\|_1 + R(\varepsilon)(1 + T) \sup_{0<s<T} (\|v(\cdot, s)\|_1 + w\|(\cdot, s)\|_1) \|v(\cdot, s) - (\cdot, s)\|_1 \Big\}. \end{aligned}$$

Hence, the operator G maps the cylinder of functions v on $(-1, 1) \times [0, T]$, that satisfy $\|v(\cdot, 0)\|_1 \le \varrho$, into itself and is a contraction there, provided $\varrho R(\varepsilon)(1+T) \le \gamma$ for some sufficiently small positive constant γ. This proves the metastability of the single internal spike solution $S(x, \varepsilon)$ in the Sobolev norm $\|\cdot\|_1$ in the sense of definition 5.1:

Theorem 5.4 (Metastability of the single internal spike)**.** *There exist positive constants c, γ and $\varepsilon_{\rm o}$, depending on p, q and r only, such that the solution U of the shadow equation* (1.4) *exists for all times $0 < t < 1/|\lambda_1(\varepsilon)|$ and satisfies*

$$\|U(\cdot, t) - S\|_1 \le \varrho, \qquad 0 < t < 1/|\lambda_1(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_{\rm o}, \quad 0 < \varrho < \varrho_\varepsilon .$$

for all initial conditions $U_{\rm o} \in H^1(-1, 1)$ in the vicinity of S, that satisfy the compatibility conditions $U_{\rm o}'(-1) = U_{\rm o}'(1) = 0$ and satisfy the bound

$$\|U_{\rm o} - S\|_1 \le c\varrho, \qquad \varrho_\varepsilon := \gamma |\lambda_1| / R(\varepsilon), \quad 0 < \varepsilon < \varepsilon_{\rm o}.$$

Remark 5.5 (Local stability of the single boundary spike)**.** We already know from remark 3.2 that the operator A will not have small negative eigenvalue, if we linearize around a single boundary spike steady state solution then. Thus in the region of parameters (p, q, r), where the large negative eigenvalue is shifted to the positive halfplane by the non-local term, the real part of the spectrum of A is positive and we have a positive lower bound $\Re\lambda > \alpha(p, q, r) > 0$. Therefore we have local stability:
There exist positive constants c, γ and $\varepsilon_{\rm o}$, depending on p, q and r only, such that the solution U of the shadow equation (1.4) exists for all times $t > 0$ and satisfies

$$\|U(\cdot, t) - S\|_1 \le \varrho e^{-\alpha(p,q,r) t}, \quad 0 < \varepsilon < \varepsilon_{\rm o}, \quad 0 < \varrho < \varrho_\varepsilon .$$

for all initial conditions $U_{\rm o} \in H^1(-1, 1)$ in the vicinity of S, that satisfy the compatibility conditions $U_{\rm o}'(-1) = U_{\rm o}'(1) = 0$ and satisfy the bound

$$\|U_{\rm o} - S\|_1 \le c\varrho, \quad \varrho_\varepsilon := \gamma / R(\varepsilon), \quad 0 < \varepsilon < \varepsilon_{\rm o}.$$

Remark 5.6. Since the exponentially small negative eigenvalue has a one dimensional eigenspace consisting of the derivative of the spike profile and since A is "stable" on the orthogonal complement, we can show the existence of an 1-D unstable manifold tangent to this eigenspace at the origin, like in [6] or in [8] (page

113). Any small perturbation is attracted to this unstable manifold exponentially fast and hence develops into (exponentially) slow motion of the spike.

References

[1] E.A. Coddington & N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

[2] A. Doelman, R.A. Gardner & T.J. Kaper, *Large stable pulse solutions in reaction diffusion equations*, Indiana Univ. Math. J., **50** (2001), pp. 443–407.

[3] A. Gierer & H. Meinhardt, *A theory of biological pattern formation*, Kybernetik, **12**, pp. 30–39, 1972.

[4] G.H. Golub & C.F. Van Loan, *Matrix Computations*, 2^{nd} ed., The Johns Hopkins University Press, Baltimore, 1989.

[5] P.P.N. de Groen, *The nature of resonance in a singular perturbation problem of turning point type*, SIAM J. Math. Anal., **11**, pp. 1–22, 1980.

[6] P. de Groen & G.E. Karadzhov, *Exponentially Slow Travelling Waves on a Finite Interval for Burgers' Type Equation*, Electronic Journal of Differential Equations, Vol. 1998(1998), No. 30, pp. 1–38.

[7] P. de Groen & G.E. Karadzhov, *Metastability of the steady state spike solutions in the shadow system for Gierer-Meinhardt's equations*, Comptes Rendus de l'Académie Bulgare des Sciences, Tome 55 (2002), pp. 13–18.

[8] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, Berlin, 1981.

[9] T. Kato, *Perturbation theory for linear operators*, Springer verlag, New York, 1966.

[10] C.-S. Lin, W.-M. Ni & I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Diff. Eq., **72** pp. 1–27 (1988).

[11] Wei-Ming Ni, *Diffusion, cross-diffusion, and their spike layer steady states*, Notices of the AMS, **45** pp. 9–18 (1998).

[12] M. Reed & B. Simon, *Methods of modern mathematical physics #4, Analysis of operators* , Academic Press, London, 1978. - 396 p. - ISBN 0125850042.

[13] I. Takagi, *Point-condensation for a reaction-diffusion system*, J. of Diff. Eq., **61** pp. 208–249 (1986).

[14] A. Turing, *The chemical basis of morhogenesis*, Phil. Trans. Roy. Soc. London, Series B, **237** pp. 37–72, 1952.

[15] D. Iron & M.J. Ward, *A metastable spike solution for a non-local reaction-diffusion model*, SIAM J. Applied Mathematics, **60**, pp. 778–802 (2000).

[16] D. Iron, M.J. Ward & J. Wei, *The stability of spike solutions to the one-dimensional Gierer-Meinhardt model*, Physica D: Nonlinear Phenomena, **150** pp. 25–62 (2001). http://www.math.ubc.ca/people/faculty/ward/ward.html, preprint.

[17] J. Wei, *On single interior spike solutions of Gierer-Meinhardt system: uniqueness and spectrum estimates.*, Europeam J. of Applied Math. **10**, pp 353–378 (1999).

Pieter de Groen

Vrije Universiteit Brussel, Pleinlaan 2, B–1050 Brussels, Belgium

E-mail address: pdegroen@vub.ac.be

Georgi E. Karadzhov

Bulgarian academy of Sciences, Institute for Mathematics and Informatics, Sofia, Bulgaria.

E-mail address: geremika@math.bas.bg