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ON COMBINED ASYMPTOTIC EXPANSIONS IN SINGULAR PERTURBATIONS

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Abstract. A structured and synthetic presentation of Vasil'eva's combined expansions is proposed. These expansions take into account the limit layer and the slow motion of solutions of a singularly perturbed differential equation. An asymptotic formula is established which gives the distance between two exponentially close solutions. An "input-output" relation around a *canard* solution is carried out in the case of turning points. We also study the distance between two canard values of differential equations with given parameter. We apply our study to the Liouville equation and to the splitting of energy levels in the one-dimensional steady Schrödinger equation in the double well symmetric case. The structured nature of our approach allows us to give effective symbolic algorithms.

1. Introduction

The main motivation of this work is the study of the real steady Schrödinger equation

$$\varepsilon^2 \ddot{\psi} = (U(t) - E)\psi \,, \tag{1.1}$$

where the dot denotes the derivative with respect to the space variable t, the small parameter $\varepsilon > 0$ is related to the Planck constant ($\varepsilon = \hbar/\sqrt{2m}$), $E \in \mathbb{R}$ is the energy, and U is a symmetric non-degenerated double well potential. Precisely, U is assumed to be a C^∞ even function with three critical points: one local maximum at the origin and two global minima at $\pm t_0$, which are supposed to be quadratic. By a translation on U and E, we may suppose that U vanishes at $\pm t_0$. Hence the potential is of the form $U(t) = \varphi(t)^2$ where φ is itself a C^∞ even function, and satisfies furthermore $\varphi(0) > 0,\ \varphi(t_0) = 0,\ \varphi'(t_0) \neq 0$ and φ decreasing on $\mathbb{R}^+$. The simplest example, which has already been studied in [3, 15, 16] is

$$U(t) = (1 - t^2)^2. \tag{1.2}$$

The following description contains some statements which will be proven in subsection 3.3. The asymptotic behaviour of the solutions in the neighborhood of $+\infty$

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is as follows: there is a one dimensional subspace denoted by V (in the two dimensional space of solutions) of exponentially decaying solutions as $t \to +\infty$; the other solutions increase exponentially. The situation is similar at $-\infty$.

A natural question is to find the energy values for which these two subspaces coincide. This is equivalent to the fact that equation (1.1) has nontrivial solutions in $L^2(]-\infty,+\infty[)$ which leads to energy quantification. These values of E are related to the energy levels which correspond to *observable solutions* of [2].

We consider in this paper only solutions without zero in the neighborhood of $\pm t_0$. This corresponds to the first energy level and this implies that E/ε is infinitely close to $-\varphi'(t_0)$.

It appears that the posed problem has two solutions, denoted by $E^{\#}(\varepsilon)$ and $E^{\flat}(\varepsilon)$. They are *canard* values for the Riccati equation associated to (1.1):

$$\varepsilon v' = U(t) - E - v^2, \tag{1.3}$$

that is to say, values for which (1.3) has solutions with a particular asymptotic behaviour. Those solutions, denoted by $v^{\#}$ and $v^{\flat}$, border the slow curve $v = -\varphi(t)$ on $]-\infty,0[$, and the other slow curve $v = \varphi(t)$ on $]0,+\infty[$. The solution $v^{\#}$ (resp. $v^{\flat}$) takes the value $v = \infty$ (resp. 0) at $t = 0$ and is a *canard* solution both at $t = -t_0$ and at $t = t_0$.

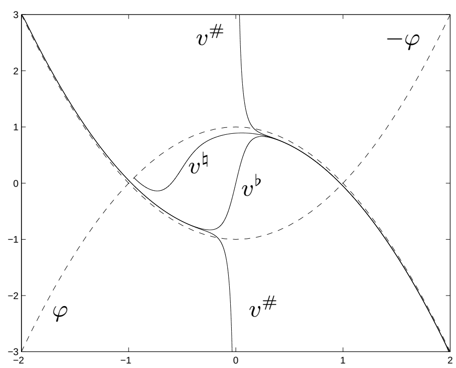

FIGURE 1. The solutions $v^{\#}$ and $v^{\flat}$, and a "great canard solution" $v^{\natural}$ for the potential (1.2) and $\varepsilon = 1/10$.

Concerning the potential (1.2), it is proven in [6] that these canard values are exponentially close to each other; precisely:

$$E^{\#}(\varepsilon) - E^{\flat}(\varepsilon) = \exp\Big(-\frac{1}{\varepsilon}\big(\frac{4}{3} + o(1)\big)\Big), \quad \varepsilon \to 0\,. \tag{1.4}$$

In the general case, the same method yields the analogous result, where the constant a that plays the role of $4/3$ is given by

$$a = 2\int_0^{t_0} \varphi(t)dt\,. \tag{1.5}$$

We present in this paper the following result.

Theorem 1.1. *There are a constant C and a real sequence $(a_n)_{n\geq 1}$ such that for any fixed integer $N \geq 1$ one has, as $\varepsilon \to 0$,*

$$E^{\#}(\varepsilon) - E^{\flat}(\varepsilon) = C\varepsilon^{1/2}\exp(-a/\varepsilon)\left(1 + a_1\varepsilon + \dots + a_{N-1}\varepsilon^{N-1} + O(\varepsilon^N)\right). \tag{1.6}$$

Comments: Before giving an idea of proof of this theorem, we describe below some experimental results and related conjectures.

In the case of the potential (1.2) we found $C = \frac{16\sqrt{2}}{\sqrt{\pi}}$ and $a_1 = -\frac{71}{96}$, which were already found in [3] (only a_1 has to be replaced by $-\frac{71}{48}$ because [3] uses the potential $U(t) = t^2(1-t)^2$). See also [15, 16] for a related work. Using Maple, we obtained the following terms $a_2 = -\frac{6299}{18432}$, $a_3 = -\frac{2691107}{5308416}$. Symbolic manipulations for other potentials led us to guess the following:

Conjecture 1. In the case of potential (1.2), all a_n are rational.

This conjecture has no particular physical relevance. Moreover it cannot be generalized to other potentials, since we found several polynomial potentials with rational coefficients for which the a_n are not rational, see subsection 3.4.

On the other hand, the conjectures that follow seem to us more interesting. Let $E^{\natural}$ be some parameter value such that the corresponding solution $v^{\natural}$ of (1.3) borders the slow curve $v = \varphi(t)$ from $-t_0$ to $+\infty$, *i.e.* a *great canard*. Such a value is defined up to exponentially small. Precisely, one shows as in (1.4) that, if $\overline{E} = \overline{E}(\varepsilon)$ is a given great canard value, then it is the same for $E^{\natural}$ if and only if

$$\overline{E}(\varepsilon) - E^{\natural}(\varepsilon) = O\Big(\exp\big(-\frac{1}{\varepsilon}(2a + o(1))\big)\Big).$$

We say in this case that $E^{\natural}$ *is defined within an exponential of type* $2a$.

Since $E^{\flat}$, $E^{\#}$ are unique, the differences $E^{\#} - E^{\natural}$, $E^{\natural} - E^{\flat}$ are known up to an exponential of type $2a$; as these quantities are exponentials of type a (*c.f.* [6]), they are known in relative value up to an exponential of type a. Therefore it is natural to expect an expansion in powers of ε in the expression of these differences. In any case, if such an expansion exists, it is necessarily unique, (*i.e.* independent of the chosen great canard value $E^{\natural}$). Indeed, we obtain as in theorem 1.1 the analogous formulae:

$$E^{\#}(\varepsilon) - E^{\natural}(\varepsilon) = C^{\#}\varepsilon^{1/2}\exp(-a/\varepsilon)\Big(1 + a_1^{\#}\varepsilon + \dots + a_{N-1}^{\#}\varepsilon^{N-1} + r_N^{\#}(\varepsilon)\Big), \tag{1.7}$$

$$E^{\natural}(\varepsilon) - E^{\flat}(\varepsilon) = C^{\flat}\varepsilon^{1/2}\exp(-a/\varepsilon)\Big(1 + a_1^{\flat}\varepsilon + \dots + a_{N-1}^{\flat}\varepsilon^{N-1} + r_N^{\flat}(\varepsilon)\Big), \tag{1.8}$$

where $r_N^{\#}(\varepsilon) = O(\varepsilon^N)$ and $r_N^{\flat}(\varepsilon) = O(\varepsilon^N)$. Using Maple we found $a_n^{\#} = a_n^{\flat} = a_n$ for $n \leq 4$ for several potentials. Hence we are led to the following:

Conjecture 2. With the assumptions on φ, we have $C^{\#} = C^{\flat} = \frac{C}{2}$ and for all $n \in \mathbb{N}^*$, $a_n^{\#} = a_n^{\flat} = a_n$.

This result seems to be amazing insofar as the equation has no symmetry according to the change of variable $u \mapsto 1/u$. Concerning potential (1.2), in the approach of [3], this symmetry between the coefficients $a^{\#}$ and $a^{\flat}$ seems to follow directly from the fact that U is even. Since this approach may be generalized to other polynomial symmetric potentials, and due to its formal nature, conjecture 2 is highly believable.

We must point out here that, if the method in [3] may work for polynomial — possibly analytic – potentials, it is by no means applicable to general C^{∞} potentials, as this approach makes a wide use of complex analysis. Furthermore, our approach seems to be applicable to the case of minima that are not quadratic. On the

other hand, the technique of [3] allows a deep insight of the analytic structure of $E^\#$ and $E^\flat$, which is out of the range of our "real" methods. Anyway, numerical computations (see section 3.4 suggest the following:
Conjecture 3. In the case of potential (1.2), the mean value $\frac{1}{2}(E^\# + E^\flat)$ is a great canard value.

More general conjectures for analytic potentials are available, but need to deal with the singularities of the potential in the complex plane. As we chose to keep a real viewpoint in this article, we do not formulate them. A way to show part of this conjecture would be to prove that the asymptotic expansion $1 + \sum_{n\geq 1} a_n\varepsilon^n$ is Gevrey-1 as well as the remainder terms, in other words, that there are $A, C, \varepsilon_0 > 0$ such that for all $n \in \mathbb{N}^*$ and all $\varepsilon \in]0, \varepsilon_0[$ one has

$$|a_n| \leq A\,C^n\,n!, \quad |r_n^\#(\varepsilon)| \leq A\,C^n\,n!\,\varepsilon^n, \quad |r_n^\flat(\varepsilon)| \leq A\,C^n\,n!\,\varepsilon^n, \tag{1.9}$$

where $r_n^\#$ and $r_n^\flat$ are defined (within an exponential of type 4/3) by (1.7) and (1.8). This point seems to be more accessible and allows to prove that the solution corresponding to $\frac{1}{2}(E^\# + E^\flat)$ borders the slow curve $v = 1 - t^2$ on a interval $]\alpha, +\infty[$ containing 0 (*i.e.* a canard solution longer than $v^\#$ and $v^\flat$).

Actually, the classical results of Gevrey analysis and a study of potential (1.2) in the complex domain allow to prove that $\frac{1}{2}(E^\# + E^\flat)$ is a great canard value if:

- for all $n \in \mathbb{N}^*$, one has $a_n^\# = a_n^\flat = a_n$
- The expansion $1 + \sum_{n\geq 1} a_n\varepsilon^n$ as well as the remainders $r_n^\#$ and $r_n^\flat$ are Gevrey-1 of type 3/4, *i.e.* for all $\delta > 0$ there is $A > 0$ such that (1.9) is satisfied with $C = \frac{3}{4} + \delta$.

These questions of Gevrey analysis are beyond the scope of the present article and will be the topic of another study.

We now return to theorem 1.1. The principle of the proof is to consider an associated Riccati equation to (1.1) (different from(1.3) for technical reasons). To each family of exponentially decaying solutions of (1.1) described above correspond two solutions (analogous to $v^\#$ and $v^\flat$) denoted by $u^\#$ and $u^\flat$ of the Riccati equation with $E^\#$ and $E^\flat$ as values of the energy.

Using the differential equation satisfied by $y = u^\# - u^\flat$ written in a linear form, we express $E^\# - E^\flat$ in terms of integrals containing $u^\flat$ and $u^\#$; see subsection 3.3. This requires an accurately estimate of $u^\#$ and $u^\flat$, not only for the slow motion, but also for the fast one.

We used for this purpose the combined asymptotic expansions introduced by A.B. Vasil'eva and V.F. Butuzov [13]. In spite of the large use of these expansions in asymptotic analysis, we believe useful to present them in a structured and simplified version. Indeed, the approach of Vasil'eva and Butuzov is more general and therefore with some technical difficulties. Sporadic presentations of these expansions are done [12], [14], without complete proofs.

The algebraic properties of combined expansions are described in section 2.4. Among them, are proved general compatibility results with respect to usual operations (algebraic, analytic and differential). Some elementary results related to exponentially decaying functions, which are used later, are included in section 2.3.

The existence of combined expansions for solutions of singularly perturbed differential equations is proved in section 2.5. Next, an application for the estimation of the difference of two solutions in a slow-fast differential equation is presented. The case without turning point is illustrated on the Liouville equation in section

3.1. We briefly describe the turning point case, together with a canard solution, and the input-output relation [1] is carried out. Section 3.3 is devoted to the proof of theorem 1.1 and the numerical results mentioned above.

This paper is written in the framework of nonstandard analysis in its IST version, introduced by Edward Nelson [10]. However, the reader can easily "translate" all the statements and proofs into standard mathematical language. Notions and notations related to nonstandard analysis are collected in section 2.

2. Foundations

2.1. **Notations.** $\mathbb{R}^d$ is equipped with the maximum norm. $T=[t_1,t_2]$ denotes the standard interval in $\mathbb{R}$, $\varepsilon>0$ is infinitesimally small, and X denotes the nonstandard interval $X=[0,(t_2-t_1)/\varepsilon]=\{x\in\mathbb{R}\ ;\ t_1+\varepsilon x\in T\}$.
The symbol $\pounds$ denotes any finite real number (generally functions of t or of x). Two occurrences of this symbol have not necessarily the same value.
The symbol $\oslash$ denotes any infinitesimally small quantity.
The symbol @ denotes a positive, finite and non infinitesimally small quantity .
The symbols $\forall^{st}$ and $\exists^{st}$ stand for the expressions "for all standard" and "there is a standard".
The notation $x\simeq y$ means "$x-y$ is infinitesimally small".
$]]a,b]$ denotes the external set of points in $]a,b]$ which are not infinitesimally close to a.

Given a function f of class C^1 on an open subset U in $\mathbb{R}^d$, we introduce the notation

$$\Delta_i f(x;h_i):=\begin{cases}\frac{1}{h_i}(f(x_1,\dots,x_i+h_i,\dots,x_d)-f(x_1,\dots,x_d)), & \text{if } h_i\neq 0\\ \frac{\partial f}{\partial x_i}(x_1,\dots,x_d) & \text{if } h_i=0\end{cases}$$

We will use the following formula: if $x=(x_i)_{i\in\{1,\dots,d\}}$ and $h=(h_i)_{i\in\{1,\dots,d\}}$ are such that $x+(h_1,\dots,h_k,0,\dots,0)$ belongs to U, for any $k\in\{1,\dots,d\}$, then

$$f(x+h)=f(x)+\sum_{k=1}^{d}h_k\tilde{\Delta}_k f(x,h)\,, \tag{2.1}$$

with $\tilde{\Delta}_k f(x,h):=\Delta_k f(x+(h_1,\dots,h_{k-1},0,\dots,0);h_k)$.

2.2. **Expansion.** Given a finite quantity q, we say that q has an *ε-expansion* if there is a standard sequence $(q_n)_{n\in\mathbb{N}}$ such that for all standard integer $N\geq 1$, we have

$$q=\sum_{n=0}^{N-1}q_n\varepsilon^n+\pounds\varepsilon^N\,. \tag{2.2}$$

The sequence (q_n) is of course unique in this case and we simply write

$$q\sim\sum_{n\geq 0}q_n\varepsilon^n\,.$$

When q is a function defined on an internal or external set E, the relation (2.2) must be satisfied *for every element of E, standard or not.* In classical terms, the expansion in ε-(resp. for every standard element of E) notion corresponds to the uniform asymptotic expansion (pointwise asymptotic expansion). Uniform expansion on any compact subset of some domain D would correspond to ε-expansion on the *S-interior of D*, which is the external set of limited points of D that are not i-close to the boundary of D.

Given an integer k and a function f defined on a standard open subset U in $\mathbb{R}^d$ into $\mathbb{R}^p$, we say that f is of *class* S^k *on* U if f is of class C^k on U, if f has a shadow ${}^\circ f$ of class C^k on U and if

$$\forall j \le k\ ,\quad f^{(j)} \text{ is S-continuous and } \ {}^\circ(f^{(j)}) = ({}^\circ f)^{(j)}\,.$$

By "S-continuous" we mean: $x \simeq y \Rightarrow f(x) \simeq f(y)$. This corresponds to "uniformly S-continuous" in other texts. We recall that the shadow of a function f defined on a standard set is the only standard function ${}^\circ f$ that takes at any standard x the standard part of $f(x)$.

We say that a function $f : U \subset \mathbb{R}^d \to \mathbb{R}^p$ *has a regular* ε*-expansion in* U if f is of class S^∞ on U and if f has an ε-expansion on U as well as all its derivatives of standard order. Be aware that not only the expansion is "regular".

It is known [4] that, if f has a regular ε-expansion in U, then the expansion of f commutes with the derivation: there exists a standard sequence $(f_n)_{n\in\mathbb{N}}$ of C^∞ functions such that

$$\forall x \in U\ \forall^{st} k \in \mathbb{N}\ \forall^{st} N \in \mathbb{N}^*,\quad f^{(k)}(x) = \sum_{n=0}^{N-1} f_n^{(k)}(x)\varepsilon^n + \pounds\varepsilon^N\,. \tag{2.3}$$

We will also use the following.

Proposition 2.1. (1) *If f and g have regular ε-expansions, then the same holds for f' and $\int f$, for $f+g$, for fg, for $f\circ g$ and for Δf.*

(2) *If f has a regular ε-expansion, and let $y = y(x,c)$ denote the solution of the b.v.p.*

$$y' = f(x,y)\,,\quad y(x_0) = c\,.$$

Then y has a regular ε-expansion with respect to x and c.

proof. (1) Since these results are well known for usual ε-expansions, we only check the property "regular". For f' and $\int f$ it is obvious. For fg, use Leibnitz formula. For $f\circ g$, use $(f\circ g)' = f'\circ g \times g'$. For Δf, use $\Delta f(x;h) = \frac{1}{h}\int_0^h f'(x+u)du$ for dimension 1 and similar formulae for higher dimension.
(2) For x, use the result for $f\circ g$: f and y ε-expandable implies that y' is, too. For c, the variation equation yields the formula

$$\frac{\partial y}{\partial c}(x,c) = \exp\Big(\int_{x_0}^{x} \frac{\partial f}{\partial y}(\xi, y(\xi,c))d\xi\Big)$$

shows that $\frac{\partial y}{\partial c}$ has an ε-expansion. It is then clear that expansion w.r.t ε and derivation w.r.t. c commute. Conclude by induction. $\Box$ A free Maple package for these computations is available at http://www.univ-lr.fr/Labo/MATH/DAC.

2.3. **Functions with exponential decay and with S-exponential decay.**

The Laplace method will be used later in the following form:

Proposition 2.2. *Let $t_1 < 0 < t_2$ be two standard real numbers and f, g two S^∞ functions admitting a regular ε-expansion in $T = [t_1, t_2]$. Assume that $f_0(0) = f_0'(0) = 0$, $a := f_0''(0)/2 > 0$, for all $t \in T\setminus\{0\}$, $f_0(t) > 0$. Then*

$$I = \int_{t_1}^{t_2} \exp\big(\frac{-f(t)}{\varepsilon}\big)g(t)\,dt$$

has an $\eta-$shadow expansion, with $\eta = \sqrt{\varepsilon}$. Furthermore, If $g_0(0) \neq 0$ we have $I = @\eta$. Namely, $I = \sqrt{\frac{\pi}{a}}\, g(0)\, \eta + \oslash\eta$.

Proof. There is a standard $k > 0$ such that for all t in T one has $f_0(t) \geq kt^2$. Thus, for every $t \in T$ one has $f(t) \geq kt^2 - C\varepsilon$, with $C = \sup_{t\in T} |f_1(t)| + 1$ for example. Using the change of variable $t = \eta\tau$ and Taylor expansion $f_0(t) = at^2 + \sum_{i=3}^{2N+1} a_i t^i + £t^{2N+2}$, one has, for τ limited,

$$f_0(\eta\tau)/\varepsilon = a\tau^2 + \sum_{i=3}^{2N+1} a_i \tau^i \eta^{i-2} + £\varepsilon^N .$$

When we expand each f_n and g_n using Taylor's formula, we find that the function

$$G(\tau) = \exp\Big(a\tau^2 - \frac{f(\eta\tau)}{\varepsilon}\Big) g(\eta\tau)$$

admits an expansion in powers of τ (for limited τ) whose coefficients, denoted $G_n(\eta)$, admit an $\eta-$shadow expansion with valuation at least $n-2$. Indeed, with the notation $f_j(t) = \sum_{i\geq 0} a_{ij} t^i$ and $g_j(t) = \sum_{i\geq 0} b_{ij} t^i$ we have

$$\begin{aligned} G(\tau) = \exp\Big(& -\sum_{i=3}^{2N+1} a_i \tau^i \eta^{i-2} - \sum_{j=1}^{N} \sum_{i=0}^{2(N-j)+1} a_{ij} \tau^i \eta^{2j+i-2}\Big) \\ &\times \sum_{\substack{0\leq j\leq N-1 \\ 0\leq i\leq 2(N-j)-1}} b_{ij} \tau^i \eta^{2j+i} + \eta^{2N} £ . \end{aligned}$$

With this notation, the new integrand is $\eta \exp(-a\tau^2) G(\tau)$. Since $\eta \exp(-k\tau^2 + C)(\sup_{t\in T} |g_0(t)| + 1)$ bounds this integrand on $\tilde{T} := \{\tau \in \mathbb{R}\ ;\ \eta\tau \in T\}$, the dominated convergence theorem implies:

$$\int_{t_1}^{t_2} \exp(-f(t)/\varepsilon) g(t)\, dt = \sum_{n=0}^{2N} \eta\, G_n(\eta) \int_{-\infty}^{\infty} e^{-a\tau^2} \tau^n d\tau + £\varepsilon^N .$$

To conclude this proof, it suffices to rearrange these terms. □

Definition 2.1. Let $I = [0, x^*]$ with $x^* \in \mathbb{R}^+$ not necessarily limited. A function $f : I \to \mathbb{R}$ is said to have *S-exponential decay* (notation $f(x) = £e^{-@x}$) if there are standard constants c, $C > 0$ such that

$$\forall x \in I, \quad |f(x)| < Ce^{-cx}.$$

In the case of a standard and bounded function on $\mathbb{R}^+$, this notion coincides of course with the usual exponential decay at infinity. The following properties of these functions will be used in the article; their proofs are straightforward.

Proposition 2.3. *(1) If f has S-exponential decay then, for every standard polynomial P, the product Pf has S-exponential decay too.*
(2) Let a be a continuous function on $\mathbb{R}^+$, with limited values, and x_0 be a non-negative limited real number such that, for every $x > x_0$, $a(x)$ is appreciably negative. Let b be a continuous function on $\mathbb{R}^+$ with S-exponential decay. If y is a solution of the differential equation

$$y' = a(x)y + b(x)$$

with $y(0)$ limited, then y has an S-exponential decay. In other words, if $y' = \pounds y + \pounds$ for $x \le x_0$, and $y' = -@y + \pounds e^{-@x}$ for $x > x_0$ with $y(0) = \pounds$, then $y(x) = \pounds e^{-@x}$. As a consequence, y' itself has S-exponential decay.

2.4. Combined expansions: Algebraic properties.

Definition 2.2. *Consider again $T = [t_1, t_2]$. A function $\varphi : T \to \mathbb{R}^d$ admits a combined expansion if there are two standard sequences of C^∞ functions $(\varphi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}}$, $\varphi_n : T \to \mathbb{R}^d$, $\psi_n : \mathbb{R}^+ \to \mathbb{R}^d$ such that*

- *For all $N \in \mathbb{N}$ and all $t \in T$,*

$$\varphi(t) = \sum_{n=0}^{N-1} \big(\varphi_n(t) + \psi_n(\frac{t - t_1}{\varepsilon})\big)\varepsilon^n + \pounds\varepsilon^N \tag{2.4}$$

- *ψ_n has exponential decay at infinity.*

The sequence $(\varphi_n)_{n\in\mathbb{N}}$ is called the *slow part* and $(\psi_n)_{n\in\mathbb{N}}$ the *fast part* of the combined expansion. The dimension d will allow us in the next sections to consider two different solutions of an ordinary differential equation as a function on $\mathbb{R}^2$ with a combined expansion. The variable t will be considered itself as a third component.

Proposition 2.4. (1) *Combined expansions are unique.*

(2) *A vector function has a combined expansion if, and only if, each of its components has.*

(3) *Let f be a C^∞ standard function from an open subset U of $\mathbb{R}^d$ to $\mathbb{R}^p$ and φ a function from T to $\mathbb{R}^d$ having a combined expansion (φ_n, ψ_n). Suppose that for all $t \in T$, $\varphi_0(t) \in U$ and for all $x \in \mathbb{R}^+$, $\varphi_0(t_1) + \psi_0(x) \in U$. Then $f \circ \varphi$ is well defined and has a computable combined expansion.*

(4) *If φ has a combined expansion (φ_n, ψ_n), then $\Phi : [t_1, t_2] \to \mathbb{R}^d, t \mapsto \int_{t_1}^t \varphi(\tau)d\tau$ has a combined expansion (Φ_n, Ψ_n) given, for every x in X and t in T, by*

$$\Phi_0(t) = \int_{t_1}^t \varphi_0(\tau)d\tau\ , \quad \Psi_0(x) = 0$$

and for $n \ge 1$:

$$\Phi_n(t) = \int_{t_1}^t \varphi_n(\tau)d\tau + \int_0^{+\infty} \psi_{n-1}(x)\,dx \quad \Psi_n(x) = -\int_x^{+\infty} \psi_{n-1}(\xi)\,d\xi\,.$$

In particular, we have

$$\int_{t_1}^{t_2} \varphi(t)dt \sim \int_{t_1}^{t_2} \varphi_0(t)dt + \sum_{n\ge1} \Big(\int_{t_1}^{t_2} \varphi_n(t)dt + \int_0^{+\infty} \psi_{n-1}(x)dx\Big)\varepsilon^n.$$

The word "computable" in statement 3 means that algorithms exist, but are not described in the present article. They allow to calculate the expansion of $f \circ \varphi$. Actually, the reader may find a free Maple package already mentioned at http://www.univ-lr.fr/Labo/MATH/DAC, see procedure called subsDAC (only for $f : \mathbb{R} \to \mathbb{R}$).

Proof. (1) By contradiction. If a function admits two different combined expansions, then their difference is a non trivial combined expansion (φ_n, ψ_n) of 0. Let n_0 be the first index such that φ_{n_0} or ψ_{n_0} is not the zero function. Using the transfer principle, n_0 is standard. Taking $N = n_0 + 1$ and multiplying by ε^{-n_0}, one obtains, for any t in T, that $\varphi_{n_0}(t) + \psi_{n_0}\big(\frac{t-t_1}{\varepsilon}\big) \simeq 0$. Since ψ_{n_0} has exponential decay, for any standard t in $T \setminus \{t_1\}$, $\varphi_{n_0}(t) \simeq 0$ and consequently $\varphi_{n_0}(t) = 0$. By transfer,

this remains valid for every t in $T \setminus \{t_1\}$, and by continuity for $t = t_1$. Now, for a standard x in $\mathbb{R}^+$ one obtains $\psi_{n_0}(x) \simeq 0$, therefore $\psi_{n_0}(x) = 0$, and this remains valid for any real x by transfer. This leads to the contradiction.
(2) This statement is obvious.
(3) Denote by $\hat{\varphi}^i$ the components of $\hat{\varphi} := \sum_{n\geq 0} \varphi_n \varepsilon^n$ (and similarly for $\hat{\psi}$). Formula (2.1) yields

$$f(\hat{\varphi} + \hat{\psi}) = f(\hat{\varphi}) + \sum_{k=1}^{d} \tilde{\Delta}_k f(\hat{\varphi}, \hat{\psi})(\hat{\psi}^k).$$

The first term $f(\hat{\varphi})$ gives the slow part $(f_n)_{n\in\mathbb{N}}$, since the image of an usual asymptotic expansion by a C^∞ mapping is an asymptotic expansion (expand f at $\varphi_0(t)$ with Taylor formula).

For the fast part, given by the sum $\sum_{k=1}^d (\hat{\psi}^k)\tilde{\Delta}_k f(\hat{\varphi}, \hat{\psi})$, one first expands each function $\varphi_i^j(t) = \varphi_i^j(t_1 + \varepsilon x)$ up to order $(N-1)$ by Taylor formula. Using the same Taylor expansion of order $(N-1)$ to the function $\tilde{\Delta}_k f$ at $(\varphi_0(t_1), \psi_0(x))$ and multiplying it by the expansion of $(\hat{\psi}^k)$, one obtains an ε-expansion. These coefficients g_n are polynomial in x and $\psi_i^j(x)$ and derivatives of $\tilde{\Delta}_k f$ at $(\varphi_0(t_1), \psi_0(x))$. As ψ_0 is bounded on $\mathbb{R}^+$, each of these derivatives is a bounded function of x. Moreover, each of the monomial terms of g_n contains at least one term ψ_i^j. Therefore, these functions g_n have exponential decay. Concerning the remainder term

$$R_N(x) := \sum_{k=1}^{d} \tilde{\Delta}_k f(\hat{\varphi}, \hat{\psi})(\hat{\psi}^k) - \sum_{n=0}^{N-1} g_n(x)\varepsilon^n,$$

if each of the former Taylor expansions is written with a remainder term of the form $\frac{1}{N!}\varphi_i^{j\,(N)}(\tau_{ij})\varepsilon^N$, $\tau_{ij} \in T$ (similarly for $\tilde{\Delta}_k f$), we see that R_N is polynomial in x, ε, $\varphi_i^{j\,(N)}(\tau_{ij})$, $\psi_i^j(x)$ and in the differentials of $\tilde{\Delta}_k f$ at some points (α, β) with α i-close to $\varphi_0(T)$ and β i-close to $\psi_0(\mathbb{R})$. Moreover, each monomial term of R_N contains at least one term ψ_i^j and a power of ε greater or equal to N. Thus, $R_N\varepsilon^{-N}$ has S-exponential decay, hence is limited on T. (4) Formula (2.4) gives:

$$\begin{aligned}
\int_{t_1}^{t} \varphi &= \int_{t_1}^{t} \Big(\sum_{n=0}^{N-1} \varphi_n(\tau)\varepsilon^n + \sum_{n=0}^{N-1} \psi_n\big(\frac{\tau - t_1}{\varepsilon}\big)\varepsilon^n + \pounds\varepsilon^N \Big)\, d\tau \\
&= \sum_{n=0}^{N-1} \Big(\int_{t_1}^{t} \varphi_n(\tau) d\tau\ \varepsilon^n + \int_0^{+\infty} \psi_{n-1}(\xi)\, d\xi \Big) \\
&\quad - \sum_{n=0}^{N-1} \int_{\frac{t-t_1}{\varepsilon}}^{+\infty} \psi_n(\xi)\, d\xi\, \varepsilon^{n+1} + \pounds\varepsilon^N.
\end{aligned}$$

Since ψ_n has exponential decay, $\Psi_{n+1}(x) := -\int_x^{+\infty} \psi_n(s)\, ds$ has exponential decay. In particular, since t_2 is standard, we conclude that $\int_{(t_2-t_1)/\varepsilon}^{+\infty} \psi_n(x) dx = e^{-@/\varepsilon} = \pounds\varepsilon^N$. □

Remarks on statement 3. (1) We detail here the computation of these expansions in the cases of dimensions d and p equal to 1. With f C^∞ standard, $t = t_1 + \varepsilon x$ and $\varphi(t) = \sum_{i=0}^n (\varphi_i(t) + \psi_i(x))\varepsilon^i + \pounds\varepsilon^{n+1}$, we look for an expression of $f \circ \varphi$ in the form

$$f(\varphi(t))=\sum_{i=0}^{n}(f_i(t)+g_i(x))\varepsilon^i+\pounds\varepsilon^{n+1}. \tag{2.5}$$

The slow expansion is the usual asymptotic expansion of a composition of two expansions, given by: $f_0(t)=f(\varphi_0(t))$, and for $n\geq 1$:

$$f_n(t)=\sum_{1\leq p_i\leq n,\ 1\leq k\leq n\ p_1+\dots+p_k=n}\frac{1}{k!}f^{(k)}(\varphi_0(t))\varphi_{p_1}(t)\dots\varphi_{p_k}(t)\,. \tag{2.6}$$

Here and in the sequel, we will use bold letters for multi-indices. We denote by E_n the set of finite sequences of positive integers that are smaller than or equal to n, i.e.

$$E_n:=\cup_{d=0}^{+\infty}\{1,\dots,n\}^d.$$

For $\mathbf{p}=(p_1,\dots,p_d)\in E_n$, we denote its length by $\#(\mathbf{p}):=d$ and its size by $|\mathbf{p}|:=p_1+\dots+p_d$. We denote by $\Phi_{\mathbf{p}}$ the product $\Phi_{\mathbf{p}}=\varphi_{p_1}\dots\varphi_{p_k}$ (with the usual convention $\Phi_\emptyset=1$). For instance (2.6) becomes, with these notation,

$$\forall n\geq 0,\quad f_n(t)=\sum_{\mathbf{p}\in E_n,\ |\mathbf{p}|=n}\frac{1}{\#(\mathbf{p})!}f^{(\#(\mathbf{p}))}(\varphi_0(t))\Phi_{\mathbf{p}}\,.$$

For the fast expansion, given some $n\in\mathbb{N}$, with the notation $\varphi(t)=\phi(t)+\psi(x)$, $\psi:=\sum_{i=0}^{n}\psi_i\varepsilon^i$ (hence $\phi(t)=\sum_{i=0}^{n}\varphi_i(t)\varepsilon^i+\pounds\varepsilon^{n+1}$ for all $t\in T$) we have

$$f(\phi(t)+\psi(x))=f(\phi(t))+\Delta f(\phi(t);\psi(x))\psi(x)\,. \tag{2.7}$$

The first term $f(\phi(t))$ yields the slow part already calculated, and the second part (which has S-exponential decay) corresponds to the fast part. We then use the Taylor formula in the form

$$\Delta f(u;v)=\sum_{\substack{i,j\geq 0\\ i+j\leq n}}\Delta_{ij}(u_0,v_0)(u-u_0)^i(v-v_0)^j+\pounds(u-u_0)^{n+1}+\pounds(v-v_0)^{n+1} \tag{2.8}$$

with $\Delta_{ij}:=\frac{1}{i!j!}\frac{\partial^{i+j}}{\partial u^i\partial v^j}\Delta f$. We apply it to $u:=\phi(t)$, $u_0:=\varphi_0(t_1)$, $v:=\psi(x)$, $v_0:=\psi_0(x)$.

Using Taylor formula for ϕ at point $t=t_1$, we get

$$u=\sum_{k=0}^{n}u_k(x)\varepsilon^k+\pounds x^{n+1}\varepsilon^{n+1}\quad\text{with}\quad u_k(x):=\sum_{j=0}^{k}\frac{1}{j!}\varphi_{k-j}^{(j)}(t_1)x^j$$

(notice that $u_0(x)$ is constant equal to $u_0=\varphi_0(t_1)$). In addition, we simplify the notation

$$f_{ij}(x):=\Delta_{ij}(\varphi_0(t_1),\psi_0(x))\,.$$

Altogether, using $u-u_0=\pounds x\varepsilon$ and $v-v_0=\pounds\varepsilon$, (2.8) gives

$$\Delta f(u;v)=\sum_{i,j\geq 0,\ i+j\leq n}f_{ij}(x)\Big(\sum_{k=1}^{n}u_k(x)\varepsilon^k\Big)^i\Big(\sum_{l=1}^{n}\psi_l\varepsilon^l\Big)^j+\pounds(1+x^{n+1})\varepsilon^{n+1} \tag{2.9}$$

Taking into account the last term $\psi(x)$ in (2.7), whose S-exponential decay implies that $\pounds(1+x^{n+1})\varepsilon^{n+1}\psi(x)=\pounds\varepsilon^{n+1}$ on T, we obtain the coefficients of the fast expansion of (2.5):

$$g_0(x) = f(\varphi_0(t_1) + \psi_0(x)) - f(\varphi_0(t_1))\,,$$
$$g_n(x) := \sum f_{ij}(x)\, u_{k_1}(x) \dots u_{k_i}(x)\, \psi_{l_1}(x) \dots \psi_{l_j}(x)\, \psi_m(x),$$

for $n \geq 1$, where the summation is taking on $i, j, m \geq 0$, $1 \leq k_1, \dots, k_i \leq n$, $1 \leq l_1, \dots, l_j \leq n$, and $k_1 + \dots + k_i + l_1 + \dots + l_j + m = n$. More concisely, using the notation below (2.6),

$$g_n = \sum_{\substack{\mathbf{k},\mathbf{l} \in E_n,\ m \geq 0 \\ |\mathbf{k}|+|\mathbf{l}|+m=n}} f_{\#(\mathbf{k})\ \#(\mathbf{l})}\, \mathbf{u_k}\ \Psi_\mathbf{l}\ \psi_m\,.$$

For the implementation of this formula, we refer the reader to the Maple package already mentioned.

(2) We insist on the fact that the fast expansion of $f \circ \varphi$ depends of the slow and fast expansions of φ, whereas the slow expansion of $f \circ \varphi$ depends only of the slow expansion of φ. Consider, for example, the product of two real combined expansions φ and $\tilde{\varphi}$. Using capital letters for the resulting combined expansion, we have

$$\Phi_n(t) = \sum_{k=0}^{n} \varphi_k(t) \tilde{\varphi}_{n-k}(t)\,.$$

To obtain $\Psi_n(x)$, we consider the other terms,

$$\Big(\sum_{\nu \geq 0} \varphi_\nu(t_1 + \varepsilon x)\varepsilon^\nu\Big)\Big(\sum_{\nu \geq 0} \tilde{\psi}_\nu(x)\varepsilon^\nu\Big) + \Big(\sum_{\nu \geq 0} \psi_\nu(x)\varepsilon^\nu\Big)\Big(\sum_{\nu \geq 0} \tilde{\varphi}_\nu(t_1 + \varepsilon x)\varepsilon^\nu\Big) + \Big(\sum_{\nu \geq 0} \psi_\nu(x)\varepsilon^\nu\Big)\Big(\sum_{\nu \geq 0} \tilde{\psi}_\nu(x)\varepsilon^\nu\Big),$$

and expand each term $\varphi_\nu(t_1 + \varepsilon x)$ with Taylor formula. Then, $\Psi_n(x)$ will be the n-th term of the obtained expansion in powers of ε.

2.5. **Combined expansions in singular perturbation theory.** Consider the singularly for the perturbed real differential equation

$$\varepsilon \dot{u} = f(t, u) \tag{2.10}$$

with the following hypotheses.

H1 The function f is S^∞ and has a regular ε-expansion in a standard open subset U of $\mathbb{R}^2$.

H2 There is a slow curve $u = u_0(t)$ in U defined and C^∞ on a standard compact interval $T = [t_1, t_2]$, i.e.

$$\forall t \in [t_1, t_2]\ ,\ (t, u_0(t)) \in U \text{ and } f_0(t, u_0(t)) = 0\,. \tag{2.11}$$

Let c_0 be such that the segment $\{t_1\} \times [u_0(t_1), c_0]$ is in U (in the case $c_0 < u_0(t_1)$ one may replace $[u_0(t_1), c_0]$ by $[c_0, u_0(t_1)]$ in the sequel).

H3 (attractiveness) The function $a(t, u) := \frac{\partial f_0}{\partial u}(t, u)$ is bounded above by some standard negative constant on U.

Notice that for (H3) the following suffices (use compactness and take a smaller U if necessary):

$$\forall u \in [u_0(t_1), c_0],\ a(t_1, u) < 0 \quad \text{and} \quad \forall t \in [t_1, t_2],\ a(t, u_0(t)) < 0\,. \tag{2.12}$$

In this situation, it is well known [9] that, if the initial condition $u(t_1) = c$ is such that $c \simeq c_0$, then the solution, after a possible boundary layer at t_1, borders the slow curve until t_2.

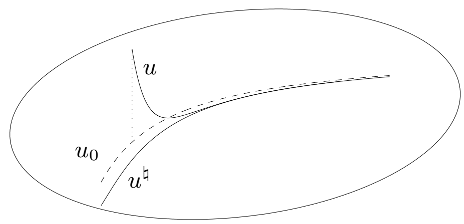

FIGURE 2. Here, $f(t, u) = -(u + 1/t)$, $t_1 = 1$, $t_2 = 3$, $c = 0$, and $\varepsilon = 1/7$.

It is known too that the slow part of u has a unique ε-expansion:

$$\exists^{st}!(u_n)_{n\in\mathbb{N}}\ \forall t \in]]t_1, t_2]\ \forall^{st} n \in \mathbb{N},\ u(t) = \sum_{n=0}^{N-1} u_n(t)\varepsilon^n + \pounds\varepsilon^N .$$

This expansion is said to be *slow* and corresponds to the outer expansion in classical asymptotics. Moreover, u has a regular expansion on $]]t_1, t_2]$, since its k-th derivative $u^{(k)}$ can be expressed with the derivatives of f and u of order $j \le k - 1$.

It is also possible to consider the *inner* expansion but we will use another expansion, see Theorem 2.5 below. The inner expansion may be defined as follows. In the neighborhood of t_1, the change of variables $t = t_1 + \varepsilon x$ yields

$$\tilde{u}'(x) = f(t_1 + \varepsilon x, \tilde{u}(x)) \tag{2.13}$$

which is a regular perturbation of $u'(x) = f_0(t_1, u(x))$. Every solution of (2.13) with an initial condition c having an ε-expansion admits an ε-expansion, $\tilde{u}(x) \sim \sum_{n\ge0} \tilde{u}_n(x)\varepsilon^n$. This expansion is said to be *fast*; the coefficients u_n are solutions of

$$\begin{aligned} \tilde{u}_0' &= f_0(t_1, \tilde{u}_0), \quad \tilde{u}_0(0) = c_0 \\ \tilde{u}_n' &= \tilde{a}(x)\tilde{u}_n + \Phi_n(x, \tilde{u}_0(x), \dots \tilde{u}_{n-1}(x)), \quad \tilde{u}_n(0) = c_n \end{aligned} \tag{2.14}$$

where f_n and c_n are the coefficients of the ε-expansions of f and c. The function $\tilde{a}$ is given by $\tilde{a}(x) = \frac{\partial f_0}{\partial u}(t_1, \tilde{u}_0(x))$ and Φ_n is obtained by considering the $n-$th term (in ε) of the Taylor expansion of $f(t_1 + \varepsilon x, \tilde{u}(x))$ at the point $(t_1, \tilde{u}_0(x))$ and by removing the term containing $\tilde{u}_n$.

This fast expansion $u(t) \sim \sum_{n\ge0} \tilde{u}(\frac{t-t_1}{\varepsilon})\varepsilon^n$ is valid a priori for $t = t_1 + \varepsilon\pounds$, but the slow one is valid for $t = t_1 + @$.

Using the permanence principle, the validity domains of these expansions can be respectively extended to $[t_1, t_3]$, ($(t_3 - t_1)/\varepsilon$ infinitely large) and $[t_4, t_2]$, ($t_4 \simeq t_1$). These intervals are disjoint and none of the expansions is valid for certain intermediate t.

The following theorem shows that the solution u has a combined expansion in the sense of section 2. It is the sum of the slow expansion (corresponding to the slow phase) and the fast expansion (corresponding to the limit layer) and it is valid in $[t_1, t_2]$. This expansion is intrinsically different of the one given by (2.14).

Theorem 2.5. *We consider the differential equation* (2.10) *where the function* f *satisfies (H1)–(H3). Then, for every number* $c \sim \sum_{n\geq 0} c_n\varepsilon^n$*, the solution* u *of* (2.10) *with the initial condition* $u(t_1) = c$ *is defined and admits a combined expansion in* $[t_1, t_2]$

$$u(t) \sim \sum_{n\geq 0} u_n(t)\varepsilon^n + \sum_{n\geq 0} y_n(x)\varepsilon^n\,, \quad t = t_1 + \varepsilon x\,. \tag{2.15}$$

Proof. To fix ideas we assume that $c_0 \geq u_0(t_1)$. For simplicity we also reduce to $t_1 = 0$ and $u_0(t_1) = 0$. Denote by K the compact set

$$K := (\{0\} \times [0, c_0]) \cup \{(t, u_0(t))\ ;\ 0 \leq t \leq t_2\}\,.$$

By (H3) the function a is appreciably negative on K. Therefore the solution u is defined at least until t_2 and the shadow of its graph on $[0, t_2]$ is K. The idea is to compare u with another solution $u^\natural$ which borders the slow curve on a larger interval. For this purpose, note that, by continuity, the hypotheses 2 and 3 of the statement are still valid if $t_1(= 0)$ is replaced by $-\delta$ (for $\delta > 0$ standard and sufficiently small). We can then consider $u^\natural$ as the solution of (1.1) with the initial condition $u^\natural(-\delta) = u_0(-\delta)$.

The attractiveness of the slow curve implies that this solution is defined and has a regular ε-expansion in $[0, t_2]$, with C^∞ coefficients u_n. This allows to isolate the slow part of the expansion (2.15). Set $\tilde{y} := u - u^\natural$; this leads to

$$\varepsilon\dot{\tilde{y}} = g(t, \tilde{y})\tilde{y}\ , \tag{2.16}$$

where $g(t, \tilde{y}) := \Delta_2 f(t, u^\natural(t); \tilde{y})$ and $\Delta_2 f$ is defined just above (2.1).

Since $u^\natural$ has itself a regular expansion, g has, too; we denote by g_i its coefficients. Moreover g remains appreciably negative on a standard neighborhood V of

$$L := (\{0\} \times [0, c_0]) \cup ([0, t_2] \times \{0\})\,.$$

Indeed, the shadow of g, denoted by g_0, satisfies $g_0(t, 0) = a(t, u_0(t))$ for any $t \in [0, t_2]$ and $g_0(0, y) = \frac{1}{y}\int_0^y a(0, v)dv$ for any $y \in]0, c_0]$. Let $t = \varepsilon x$ and $y(x) := \tilde{y}(\varepsilon x)$. Then

$$y'(x) = g(\varepsilon x, y(x))y(x)\ , \quad y(0) = c - u^\natural(0) \tag{2.17}$$

where $'$ denotes the derivation with respect to x. It will be shown that this solution admits an ε-expansion with coefficients y_n exponentially decreasing.

First of all, y is decreasing, hence y' is limited on $X := [0, t_2/\varepsilon]$. Therefore y is S-continuous, hence has a shadow, denoted by y_0. This shadow satisfies

$$y_0'(x) = g_0(0, y_0(x))\ y_0(x)\ , \quad y_0(0) = c_0\,. \tag{2.18}$$

This implies that y_0 is decreasing on $\mathbb{R}^+$ and has exponential decay at infinity, by statement 2 of proposition 2.3. By definition one has a priori $y(x) \simeq y_0(x)$ only for limited values of x, but this remains true for all $x \in X$, as both functions are i-small for x i-large.

Examining now the formal solutions. We write the coefficients of g in the form

$$g_i(t, y) = \sum_{j,k\geq 0} g_i^{jk}(x)t^j(y - y_0(x))^k$$

with

$$g_i^{jk}(x) := \frac{1}{j!k!}\frac{\partial^{j+k} g_i}{\partial t^j \partial y^k}(0, y_0(x))\,.$$

We omit in the sequel the dependance in x of g_i^{jk} and y_l. With these notation, one has

$$\sum_{i\geq 0} y_i'\varepsilon^i = \sum_{i,j,k\geq 0} g_i^{jk} x^j \varepsilon^{i+j}\Big(\sum_{\nu\geq 1} y_\nu \varepsilon^\nu\Big)^k \sum_{l\geq 0} y_l\,\varepsilon^l. \tag{2.19}$$

Symbolic identification yields a linear differential equation for y_n; namely

$$y_n' = \varepsilon\text{-terms of order } n \text{ in right-hand of the above equaiton.}$$

The terms in this expression are of the form $g_i^{jk} x^j y_{\nu_1}\dots y_{\nu_k} y_l,\ \nu_p \geq 1$. Later, we will write separately those terms containing y_n; namely $(g_0^{00} + g_0^{01} y_0) y_n$. It is convenient to introduce the following indexed set:

$$M_n := \bigcup_{k=0}^{n} \left\{\mu = (i, j, l, \nu_1, \dots, \nu_k) \in \{0, \dots, n\}^{k+3}\ ;\ \nu_p \geq 1\right\}\ , \tag{2.20}$$

and the notation, for $\mu = (i, j, l, \nu_1, \dots, \nu_k) \in M_n$:

$$|\mu| := i + j + l + \nu_1 + \dots + \nu_k \quad \text{and} \quad \mathbf{y}_\mu := g_i^{jk} x^j y_{\nu_1}\dots y_{\nu_k} y_l. \tag{2.21}$$

In summary, y_n satisfies

$$y_n' = \sum_{\mu\in M_n, |\mu|=n} \mathbf{y}_\mu\,. \tag{2.22}$$

We notice that (this will be used later) that, if $m < n$, then M_n contains M_m. In other words, one has

$$\forall m < n,\ y_m' = \sum_{\mu\in M_n, |\mu|=m} \mathbf{y}_\mu\,. \tag{2.23}$$

Let us now end up with the formal part of the proof of theorem 2.5. We have to show that y_n has exponential decay on $\mathbb{R}^+$. For that purpose, we rewrite (2.22) in the form

$$y_n' = (g_0^{00} + g_0^{01} y_0) y_n + \sum_{\mu\in M_n^*, |\mu|=n} \mathbf{y}_\mu\,. \tag{2.24}$$

where M_n^* is equal to M_n except the "special terms" $(0, 0, m)$ and $(0, 0, 0, m)$. Remark that the g_i^{jk} depends on x only though $y_0(x)$. In particular, they are standard and bounded functions on $\mathbb{R}^+$. By induction on n, if for all $m < n$, y_m has exponential decay, then it is the same for any $\mathbf{y}_\mu,\ \mu \in M_n^*$. As $g_0^{00} + g_0^{01} y_0$ is standard and bounded above by a standard negative constant, statement 2 of proposition 2.3 applies and yields that y_n has exponential decay on $\mathbb{R}^+$.

Concerning the remainder terms, let us write $y = Y_n + r_n\varepsilon^{n+1}$ with $Y_n(x) = \sum_{i=0}^n y_i(x)\varepsilon^i$. Obviously Y_n has S-exponential decay, and we have

$$Y_n(x) = y_0(x) + \pounds\varepsilon \quad \forall x \in X = [0, \frac{t_2}{\varepsilon}]\,, \tag{2.25}$$

as the y_i are standard bounded. In particular $(\varepsilon x, Y_n(x))$ is in the neighborhood V for any $x \in X$; therefore, $g(\varepsilon x, Y_n(x))$ is bounded above by a standard negative constant. The idea is to write the differential equation satisfied by r_n in a linear form, considering y and Y_n as known (we already know that y and Y_n are defined and infinitely close to y_0 on X).

We have

$$Y_n' + r_n'\varepsilon^{n+1} = \left(g(\varepsilon x, Y_n) + \Delta_2 g(\varepsilon x, Y_n; y - Y_n) r_n \varepsilon^{n+1}\right)(Y_n + r_n\varepsilon^{n+1}),$$

hence $r_n' = a_n(x) r_n + b_n(x)$ with

$$\begin{aligned} a_n(x) &= g(\varepsilon x, Y_n(x)) + \Delta_2 g(\varepsilon x, Y_n(x); y(x) - Y_n(x)) y(x), \\ b_n(x) &= \left(g(\varepsilon x, Y_n(x)) Y_n(x) - Y_n'(x)\right)/\varepsilon^{n+1}. \end{aligned}$$

For any x in X, we have $a_n(x) = -@ + e^{-@x}£$. It follows that, if x_0 is chosen standard sufficiently large, then $a_n(x)$ is bounded above by a standard negative constant for all $x \in [x_0, \frac{t_2}{\varepsilon}]$.

It thus suffices to show that b has S-exponential decay, then apply again statement 2 of proposition 4 and deduce that r_n has S-exponential decay, hence is bounded. For this purpose, we first write

$$g(\varepsilon x, Y_n) = \sum_{i=0}^{n} g_i(\varepsilon x, Y_n)\varepsilon^i + £\varepsilon^{n+1}\,.$$

Secondly, using Taylor formula at $t = \varepsilon x = 0$:

$$g_i(\varepsilon x, Y_n) = \sum_{j=0}^{n-i} \frac{1}{j!}\,\frac{\partial^j g_i}{\partial t^j}(0, Y_n)(\varepsilon x)^j + £(\varepsilon x)^{n-i+1}\,.$$

Finally, the Taylor formula at $y = y_0$ is applied to each function $\frac{\partial^j g_i}{\partial t^j}$, $0 \le i < n, 0 \le j < n - i$:

$$\frac{1}{j!}\,\frac{\partial^j g_i}{\partial t^j}(0, Y_n) = \sum_{k=0}^{n-i-j} g_i^{jk}\Big(\sum_{\nu=1}^{n} y_\nu \varepsilon^\nu\Big)^k + £(Y_n - y_0)^{n-i-j+1}\,.$$

Altogether, we obtain:

$$\begin{aligned} g(\varepsilon x, Y_n) Y_n = &\sum_{i,j,k \ge 0, i+j+k \le n} g_i^{jk} x^j \varepsilon^{i+j}\,\Big(\sum_{\nu=1}^{n} y_\nu \varepsilon^\nu\Big)^k \sum_{l=0}^{n} y_l \varepsilon^l \\ &+ \Big(£ + \sum_{i=0}^{n} £x^{n-i+1} + \sum_{i,j \ge 0, i+j \le n} £x^j \big(\frac{Y_n - y_0}{\varepsilon}\big)^{n-i-j+1}\Big) Y_n \varepsilon^{n+1}\,. \end{aligned}$$

By (2.25), the sum, between brackets, multiplying $Y_n\varepsilon^{n+1}$ is of the form $£ + £x^{n+1}$. Therefore, with the notations (2.20) and (2.21), we obtain

$$g(\varepsilon x, Y_n) Y_n = \sum_{\mu \in M_n'} \mathbf{y}_\mu\ \varepsilon^{|\mu|} + (£ + £x^{n+1}) Y_n \varepsilon^{n+1}$$

with $M_n' = \{\mu = (i, j, l, n_1, \dots, n_k) \in M_n\ ;\ i + j + k \le n\}$. Notice that M_n' contains all the $\mu \in M_n$ such that $|\mu| \le n$.

Moreover, using (2.23), we have

$$Y_n' = \sum_{\mu \in M_n, |\mu| \le n} \mathbf{y}_\mu\ \varepsilon^{|\mu|}\,.$$

Therefore

$$g(\varepsilon x, Y_n)Y_n - Y_n' = \sum_{\mu\in M_n', |\mu|>n} \mathbf{y}_\mu \varepsilon^{|\mu|} + (\pounds + \pounds x^{n+1})Y_n\varepsilon^{n+1}\,.$$

Each term $\mathbf{y}_\mu = g_i^{jk} x^j y_{\nu_1}\dots y_{\nu_k} y_l$ contains at least one factor y_l, hence has S-exponential decay, and Y_n has S-exponential decay too. To sum up,

$$g(\varepsilon x, Y_n)Y_n - Y_n' = (\pounds + \pounds x^{n+1})e^{-@x}\varepsilon^{n+1} = \pounds e^{-@x}\varepsilon^{n+1}\,.$$

This shows that b has S-exponential decay. □

Remarks: (1) The more general hypothesis "f admits an ε-expansion" instead of "f standard" is useful for the following:

- This allows to treat problems where the initial instant t_1 is not standard but has only an ε-expansion. Indeed, if φ is standard and C^∞ — or has an ε-expansion — and if t_1 has an ε-expansion,then $\tilde\varphi : [0, t_2 - t_1] \to \mathbb{R}^d$, $s \mapsto \varphi(t_1 + s)$ has an ε-expansion and is S^∞.
- Furthermore, the Schrödinger equation(we will study) may contain a non-standard parameter (canard value).

(2) Theorem 2.5 will be applied in more general situations, for instance when the equation has a turning point in $]t_1, t_2[$, or when the starting point t_1 is not standard. Therefore we present the following result.

Proposition 2.6. *(1) Theorem 2.5 remains valid if (H3) is replaced by the following hypotheses:*

(i) *There is a S^∞ solution of* (2.10)*, close to the slow curve $u = u_0(t)$ on a standard open interval containing* $[t_1, t_2]$.

(ii) *For every $u \in [u_0(t_1), c_0]$ one has $a(t_1, u) < 0$.*

(iii) *For every $t \in]t_1, t_2]$ one has $A_0(t) < 0$, where A_0 is given by* (2.26).

(2) Theorem 2.5 remains valid if t_1 is only ε-expandable instead of standard.

Proof. (1) Assume that there is already a S^∞ canard solution $u^\natural$ close to the slow curve on a standard open interval containing $[t_1, t_2]$. In that case, the solutions u and $u^\natural$ are exponentially close to each other as soon as t is appreciably greater than t_1 and as far as the "accumulated stability" is positive: more precisely, if A_0 is given by

$$A_0(t) = \int_{t_1}^{t} a(\tau, u_0(\tau))d\tau \tag{2.26}$$

(recall that $a(t,u) = \frac{\partial f_0}{\partial u}(t,u)$). Then

$$u(t) - u^\natural(t) = \exp\left((A_0(t) + \oslash)/\varepsilon\right).$$

As far as A_0 is appreciably negative (the "accumulated stability" would be defined as $-A_0$). Since $u^\natural$ is S^∞ and defined on $[t_1 - \delta, t_2 + \delta]$ for some $\delta > 0$ standard, it admits an ε-expansion on $[t_1, t_2]$. Theorem 2.5 applied to $[t_1, t_1 + \delta]$, $\delta > 0$ standard sufficiently small, yields a combined expansion for u on $[t_1, t_1 + \delta]$. As u is exponentially close to $u^\natural$, this combined expansion remains valid on $[t_1 + \delta, t_2]$. This proves the first part.

(2) Put $\alpha = t_1 -^\circ t_1$, $t = s + \alpha$, $u(t) = v(s)$. Then v satisfies $\varepsilon\dot v = g(s, v)$ with $g(s,v) := f(s + \alpha, v)$. The function g has a regular e-expansion; hence v has a combined expansion $v(s) = \sum v_n(s)\varepsilon^n + \sum y_n(x)\varepsilon^n$, $x := (s -^\circ t_1)/\varepsilon$. Taylor

formula shows that $\sum v_n(t-\alpha)\varepsilon^n$ has a regular ε-expansion $\sum u_n(t)\varepsilon^n$. This gives the slow part. The fast part is the same, since x is also equal to $(t-t_1)/\varepsilon$. □

3. Applications

3.1. **Towards a transasymptotic expansion.** The results of section 2 yield an estimate of the distance between two slow solutions of a slow-fast differential equation. In this section, we will study this distance, first for equations without parameter, and then for equations with parameter. Consider again equation (2.10),$\varepsilon\dot u = f(t,u)$, with the hypotheses (H1)–H(3) in section 2.5 (f is C^∞ and has a regular ε-expansion and a slow curve $u = u_0(t)$; we assume as above that this slow curve is attractive on a standard compact $T = [t_1, t_2]$; the situation with a turning point will be detailed at the end of this section).

Let $u^\natural$ be a solution close to the slow curve on a standard open interval containing $[t_1, t_2]$ (for example, the solution with initial condition $u^\natural(t_1-\delta) = u_0(t_1-\delta)$ where $\delta > 0$ is standard sufficiently small). Consider now an initial condition $c^\#$ having an ε-expansion and sufficiently close to the slow curve such that the solution $u^\#$, issuing from $c^\#$ at t_1, has a boundary layer at t_1 and borders the slow curve, at least, until t_2 (i.e. hypotheses 2 and 3 in 2.5 for $c_0 = {}^\circ c^\#$). These two solutions are exponentially close to each other as soon as $t \gg t_1$. Namely, it is easy to see that

$$u^\#(t) - u^\natural(t) = \exp\Big(\frac{1}{\varepsilon}\Big(\int_{t_1}^t \frac{\partial f_0}{\partial u}(\tau, u_0(\tau))d\tau + \oslash\Big)\Big).$$

A more precise result will be given in this section. As the essential task is to introduce the notations, the statement will be given after its proof.

If $t = t_1 + \varepsilon x$, theorem 2.5 provides two combined expansions:

$$u^\#(t) \sim \sum_{n\ge 0}(u_n(t) + y_n^\#(x))\varepsilon^n, \quad u^\natural(t) \sim \sum_{n\ge 0} u_n(t)\varepsilon^n\ ,$$

where $y_0^\#$ is the solution of the differential equation

$$y_0' = \frac{\partial f_0}{\partial u}(t_1, u_0(t_1) + y_0)y_0 \tag{3.1}$$

with $y_0^\#(0) = c_0^\# - u_0(t_1)$. Moreover, $\tilde y := u^\# - u^\natural$ satisfies the differential equation (2.16) rewritten below in a linear form:

$$\varepsilon\dot{\tilde y} = a(t)\tilde y \quad \text{with} \quad a(t) := g(t, u^\#(t) - u^\natural(t)) \tag{3.2}$$

with the notation $g(t,y) = \Delta_2 f(t, u^\natural(t); y)$ of 2.5. Hence with $c^\natural = u^\natural(t_1)$ one has:

$$u^\#(t) - u^\natural(t) = (c^\# - c^\natural)\exp\Big(\frac{1}{\varepsilon}\int_{t_1}^t a(\tau)d\tau\Big). \tag{3.3}$$

According to the statement 3 of the proposition 2.4, the function a admits a combined asymptotic expansion $a(t) \sim \sum_{n\ge 0}(a_n(t) + b_n(x))\varepsilon^n$ whose first terms are clarified below

$$a_0(t) = \frac{\partial f_0}{\partial u}(t, u_0(t)), \quad a_1(t) = u_1(t)\frac{\partial^2 f_0}{\partial u^2}(t, u_0(t)) + \frac{\partial f_1}{\partial u}(t, u_0(t)), \tag{3.4}$$

$$b_0(x) = \Delta_2 f_0(t_1, u_0(t_1); y_0^\#(x)) - \frac{\partial f_0}{\partial u}(t_1, u_0(t_1)), \tag{3.5}$$

where $y_0^\#$ given by (3.1). According to the statement 4 of this same proposition 2.4, the primitive of a has an ε-expansion

$$\int_{t_1}^{t} a(\tau)d\tau \sim \sum_{n\geq 0} A_n(t)\varepsilon^n \tag{3.6}$$

for $t \in]]t_1, t_2]$, the first terms of which are:

$$A_0(t) = \int_{t_1}^{t} a_0(\tau)d\tau\,, \quad A_1(t) = \int_{t_1}^{t} a_1(\tau)d\tau + \int_{0}^{+\infty} b_0(\xi)d\xi\,. \tag{3.7}$$

Since $c^\# - c^\natural$ has also an ε-expansion (its first term is $c_0^\# - c_0^\natural$), formula (3.3) shows the following result.

Theorem 3.1. *With the notation and hypotheses above, there is a standard sequence of functions $(r_n)_{n\in\mathbb{N}^*}$ such that the difference $u^\# - u^\natural$ satisfies for any $t \in]]t_1, t_2]$:*

$$\big(u^\#(t) - u^\natural(t)\big)\exp\left(-\frac{A_0(t)}{\varepsilon}\right) \sim \sum_{n\geq 0} r_n(t)\varepsilon^n\,. \tag{3.8}$$

with $r_0(t) = (c_0^\# - c_0^\natural)\exp(A_1(t))$.

Remarks: (1) Concerning values of t close to t_1, an analogous formula is available with combined expansions. We do not mention it for simplicity.

(2) As mentioned in the introduction, the functions r_n can explicitly be computed with an algorithm based on the previous proof (subject to compute the primitives of the occurring functions). Instead of detailing this algorithm in the general case, we find it is more useful to illustrate it on the Liouville equation, see below. We refer to the Maple package already mentioned for the general case.

(3) If a standard t is fixed, formula (3.8) gives immediately an asymptotic expression of $u^\#(t) - u^\natural(t)$ with standard constants. If t has an ε-expansion, such an asymptotic expression is also possible, but one has to expand each term $A_i(t),\ i = 0, 1$ and $r_j(t),\ j \geq 1$ with Taylor formula.

(4) As for combined asymptotic expansions in 2.5 (proposition 2.6 of remark 2), this result remains valid if the slow curve is not attractive on the whole interval $[t_1, t]$. The existence of a solution close the slow curve on an open interval containing $[t_1, {}^\circ t]$ and the assumption that A_0 is negative on $]t_1, {}^\circ t]$ are enough.

(5) Formula (3.3) allows to find an ε-expansion of $c^\#$ from an ε-expansion of t and $u^\#(t) - u^\natural(t)$. More precisely, we have the following.

Corollary 3.2. *Denote by $I =]c_{\min}, c_{\max}[$ the set of all numbers $c \in \mathbb{R}$ which satisfy hypothesis 3 of subsection 2.5. Let $t^* \in]t_1, t_2]$ ε-expandable with ${}^\circ t^* > t_1$. Denote by $u^\# = u^\#(t, c)$ the solution of (2.10) with boundary condition $u^\#(t_1) = c$. Consider the function*

$$\varphi : I \to \mathbb{R},\ c \mapsto \big(u^\#(t^*) - u^\natural(t^*)\big)\exp\Big(-\frac{A_0(t^*)}{\varepsilon}\Big),$$

Then φ is an S-diffeomorphism from the S-interior of I to its image. (In classical terms: $\varphi = \varphi_\varepsilon$ is a diffeomorphism from I to $\varphi(I)$ and for any fixed $c \in I$ there is a constant $M > 0$ independent of ε such that $\frac{1}{M} \leq \varphi'_\varepsilon(c) \leq M$.) As a consequence, given $\alpha \in \varphi(I)$ with $c^\#(\alpha) := \varphi^{-1}(\alpha) \in]]c_{\min}, c_{\max}[[$, if α is ε-expandable, then $c^\#(\alpha)$ is ε-expandable and by theorem 2.5, $u^\#(\cdot, c^\#(\alpha))$ has a combined expansion on $[t_1, t_2]$.

Proof. With (3.3) we get

$$\varphi(c) = (c - c^\natural)\exp\Big(\frac{1}{\varepsilon}\int_{t_1}^{t^*}(g(t,\tilde{y}(t,\varepsilon)) - g_0(t,0))dt\Big)$$

with $\tilde{y} : (t,c) \mapsto u^{\#}(t,c) - u^\natural(t)$, $g = \Delta_2 f(\cdot, u^\natural, \cdot)$ as before, and g_0 is the first term of the ε-expansion of g. Therefore,

$$\varphi(c) = (c - c^\natural)M\exp(J(c))$$

with

$$M := \exp\Big(\frac{1}{\varepsilon}\int_{t_1}^{t^*}(g(t,\tilde{y}(t,c)) - g(t,0))dt\Big)$$

independent of c, and

$$J(c) := \frac{1}{\varepsilon}\int_{t_1}^{t^*}(g(t,\tilde{y}(t,c)) - g(t,0))dt\,. \tag{3.9}$$

The function $\tilde{y}$ is the solution of the boundry-value problem $\varepsilon\dot{y} = g(t,y)y$, $y(t_1) = c - c^\natural$, hence is monotonous on $[t_1, t^*]$ and can be used as a change of variable. For convenience we use the notation $y^* := \tilde{y}(t^*, c)$ (it is an exponentially small number which depends on c) and $\tilde{t} = \tilde{t}(y,c)$ the “inverse” function of $\tilde{y}$. The change of variable $y = \tilde{y}(t,c)$ yields $dy = \frac{1}{\varepsilon}g(t,y)ydt$ and

$$J(c) = \int_{c-c^\natural}^{y^*}\psi(y)\,dy$$

with

$$\psi(y) := \frac{\Delta_2 g(\tilde{t}(y,c),0;y)}{g(\tilde{t}(y,c),y)}\,.$$

This function is not S-continuous, but is continuous and limited on $[c - c^\natural, y^*]$. Moreover, for $y \not\simeq 0$, one has

$$\psi(y) \simeq \psi_0(y) := \frac{\Delta_2 g_0(0,0;y)}{g_0(0,y)}\,.$$

Hence J is S-continuous and its shadow satisfies

$$^\circ J : c \mapsto \int_{c-c^\natural}^{0}\psi_0(y)dy\,.$$

We will show later that J has a regular expansion. This implies that

$$^\circ(J'(c)) = (^\circ J)'(c) = -\psi_0(c - c^\natural) = \frac{1}{c - c^\natural}\Big(\frac{g_0(0,0)}{g_0(0,c-c^\natural)} - 1\Big)\,.$$

Since we have $\frac{\varphi'}{\varphi}(c) = \frac{1}{c-c^\natural} + J'(c)$, we deduce that $(c - c^\natural)\frac{\varphi'}{\varphi}(c) \simeq \frac{g_0(0,0)}{g_0(0,c-c^\natural)}$ is appreciable. Using $M = @$ and $J(c) = \pounds$, this gives $\varphi'(c) = @$ which shows that φ is a S-diffeomorphism. Furthermore φ has an ε-expansion (by composition of the expansion (3.8) and the expansion of t^*). Hence by a well-known result on classical expansions, φ^{-1} has an ε-expansion, too. The consequence is clear, again by composition of the expansions of α and φ^{-1}.

It remains to prove that J has a regular expansion. For that purpose, it is better to use the change of variable $t = t_1 + \varepsilon x$ in (3.9). This gives, with $x^* := (t^* - t_1)/\varepsilon$ (independent of c) and $\hat{y}(x, c) := \tilde{y}(t_1 + \varepsilon x, c)$:

$$J(c) = \int_0^{x^*} G(x, c)dx \tag{3.10}$$

with

$$G(x, c) := (g(t_1 + \varepsilon x, \hat{y}(x, c)) - g(t_1 + \varepsilon x, 0))$$

At this scale, the function $\hat{y}$ has S-exponential decay in x and a regular ε-expansion $\hat{y}(x, c) \sim \sum_{n\geq 0} y_n(x, c)\varepsilon^n$ with respect to c (see the end of 2.3), where the y_n have exponential decay in x (see the end of 2.3). Hence the same holds for G, and J has a regular ε-expansion with respect to c.

(5) Using classical results of Gevrey analysis, it is possible to give an exact meaning to an expression like:

$$u^{\#}(t) \sim \sum_{n\geq 0} u_n(t)\varepsilon^n + \exp\big(\frac{A_0(t)}{\varepsilon}\big)(c_0^{\#} - c_0^{\natural}) \exp(A_1(t))\Big(1 + \sum_{n\geq 1} r_n(t)\varepsilon^n\Big) .$$

This expression may be considered as a start of *transasymptotic expansion.* The first expansion $\sum_{n\geq 0} u_n(t)\varepsilon^n$ is the classical expansion of slow curves, and the exponential term which follows arises from the boundary layer at t_1. Thanks to Gevrey analysis, — under analyticity hypothesis of f according to t and a simple geometrical hypothesis concerning the *relief* function $a : t \mapsto \frac{\partial f_0}{\partial u}(t, u_0(t))$ — it is possible to show that the first expansion $\sum_{n\geq 0} u_n(t)\varepsilon^n$ is defined up to exponentially small (of type strictly greater than $A_0(t)$). Therefore, a summation "to the least term" of both expansions accounts for the boundary layer.

Liouville's equation. We consider here Liouville's equation in its singularly perturbed form

$$\varepsilon\dot{u} = u^2 - t. \tag{3.11}$$

This equation has a repulsive river $u^\natural$ which is asymptotic to $\sqrt{t}$. Consider a solution $u^\flat$ that borders this river, on an appreciable interval, and join the attractive river. We will give an estimate of $u^\flat - u^\natural$ before the limit layer of $u^\flat$. The first two terms r_1 and r_2 will be explicit.

To set the ideas, we assume that $u^\flat$ links both rivers through 0 rather than infinity. Choosing adequately the parameter ε and doing linear change of variables of u and t, we can suppose that $u^\flat(1) = 0$. Given t fixed and standard in $]0, 1[$, we want to compute an ε-expansion of $u^\flat(t) - u^\natural(t)$. The first terms of the ε-expansion are

$$u^\natural(t) = \sqrt{t} + \frac{1}{4}t^{-1}\varepsilon - \frac{5}{32}t^{-5/2}\varepsilon^2 + \frac{15}{64}t^{-4}\varepsilon^3 + \pounds\varepsilon^5 .$$

The funtion $y(x) := u^\flat(1 + \varepsilon x) - u^\natural(1 + \varepsilon x)$, already introduced satisfies the differential equation

$$y' = (2u\natural(1 + \varepsilon x) + y)y ,$$

approximated by

$$y' = \left(2 + (x + \frac{1}{2})\varepsilon - (\frac{x^2}{4} + \frac{x}{2} + \frac{5}{16})\varepsilon^2 + (\frac{x^3}{8} + \frac{x^2}{2} + \frac{25x}{32} + \frac{15}{32})\varepsilon^3 + \pounds\varepsilon^4 - y\right) y$$

with the initial condition $y(0) = -u^\natural(1) = -1 - \frac{1}{4}\varepsilon + \frac{5}{32}\varepsilon^2 - \frac{15}{64}\varepsilon^3 + \pounds\varepsilon^4$.

We then deduce the beginning of the ε-expansion $y(x) \sim \sum_{n\geq 0} y_n(x)\varepsilon^n$ of y:

$$y_0' = (2+y_0)y_0\,, \quad y_0(0) = -1,$$

hence $y_0(x) = -1 - \tanh x$,

$$y_1' = a(x)y_1 + (x+\frac{1}{2})y_0\,, \quad y_1(0) = -\frac{1}{4}$$

with $a(x) = 2 + 2y_0(x) = -2\tanh x$, hence $\exp\big(\int_u^x a\big) = \frac{\cosh^2 u}{\cosh^2 x}$; this gives

$$y_1(x) = \frac{-1}{4\cosh^2 x}\Big(1 + \int_0^x (2u+1)\left(e^{2u}+1\right)du\Big) = \frac{-1}{4\cosh^2 x}\left(1 + x + x^2 + x\,e^{2x}\right)\,.$$

Using the symbolic manipulation language Maple, we found It is possible to prove that y_n is of the form $y_n(x) = \frac{P_n(x,e^x)}{e^{2x}+1}$. We do not explicit the rather long expression y_2 because we need only the value of the integrals $I_n := \int_0^{-\infty} y_n(x)dx$. One finds for the first terms:

$$I_0 = \ln 2\,, \quad I_1 = \frac{1}{4}\,, \quad I_2 = \frac{1}{32}\,.$$

We remark that, despite the complexity of the functions y_n, their integrals on $\mathbb{R}^-$ are very simple. In particular, it is possible to show (using the associated linear Airy equation) that these numbers are rational for all $n \geq 1$.

Then, $u^\flat(t) - u^\natural(t) = \big(u^\flat(1) - u^\natural(1)\big)\exp(\frac{A(t)}{\varepsilon})$ with

$$u^\flat(1) - u^\natural(1) = -u^\natural(1) = -1 - \frac{\varepsilon}{4} + \frac{5}{32}\varepsilon^2 - \frac{15}{64}\varepsilon^3 + \pounds\varepsilon^4,$$

$$\begin{aligned}
A(t) &= \int_1^t \big(u^\natural(\tau) + u^\flat(\tau)\big)d\tau \\
&= 2\int_1^t u^\natural(\tau)d\tau + \varepsilon\int_0^{\frac{t-1}{\varepsilon}} y(x)dx \\
&= 2\sum_{n=0}^{3}\Big(\int_1^t u_n(\tau)d\tau\Big)\varepsilon^n + \sum_{n=0}^{2}\Big(\int_0^{-\infty} y_n(x)dx\Big)\varepsilon^{n+1} + \pounds\varepsilon^4 \\
&= \frac{4}{3}\Big(t^{3/2}-1\Big) + \Big(\frac{1}{2}\ln t\Big)\varepsilon + \frac{5}{24}\Big(t^{-3/2}-1\Big)\varepsilon^2 + \frac{5}{32}\left(1-t^{-3}\right)\varepsilon^3 \\
&\quad +\varepsilon\ln 2 + \frac{1}{4}\varepsilon^2 + \frac{1}{32}\varepsilon^3 + \pounds\varepsilon^4
\end{aligned}$$

Hence, we deduce $\exp\Big(\frac{A(t)}{\varepsilon}\Big)$ and obtain finally:

$$\begin{aligned}
u^\natural(t) - u^\flat(t) &= \exp\Big\{\frac{4}{3\varepsilon}\Big(t^{3/2}-1\Big)\Big\}\,2\sqrt{t}\;\Big[\,1 + \frac{1}{24}\Big(5t^{-3/2}+7\Big)\varepsilon \\
&\quad + \frac{1}{1152}\Big(49 + 70\,t^{-3/2} - 155\,t^{-3}\Big)\varepsilon^2 + \pounds\varepsilon^3\,\Big]\,.
\end{aligned}$$

Remark. This formula is valid only for t appreciable, but for Liouville equation, a formula can be given for t infinitely small, for example for $t=0$. Indeed, only the computation of the integral $\int_1^t u^\natural(\tau)d\tau$ of $A(t)$ poses a problem. But here, $u^\natural$ is linked to the logarithmic derivative of the Airy function. This yields an answer to the following natural question.

Consider the Liouville equation

$$U' = U^2 - T \tag{3.12}$$

and its unique solution asymptotic to $\sqrt{T}$ at $+\infty$ given by $U^\natural(T) = -\mathrm{Ai}'(T)/\mathrm{Ai}(T)$. Consider another solution $U^\flat$ with the initial condition $U^\flat(0) = U^\natural(0) - \delta$, $0 < \delta \simeq 0$. This solution vanishes only at an infinitely large value ω. What is the asymptotic relation between δ and ω?

The change of variables $T = \omega t$, $U = \sqrt{\omega}\, u$, with $\varepsilon = \omega^{-3/2}$ leads to the previous equation (3.12). We deduce that $\exp\big(\frac{1}{\varepsilon}\int_0^1 u^\natural(\tau)d\tau\big) = \big(\frac{\mathrm{Ai}(\omega)}{\mathrm{Ai}(0)}\big)^2$ with $\omega = \varepsilon^{-2/3}$ (and $\mathrm{Ai}(0) = 3^{-2/3}/\Gamma(2/3)$). The asymptotic expansion of Ai implies:

$$\delta = \frac{1}{2\pi \mathrm{Ai}(0)^2}\exp\big(-\frac{4\omega^{3/2}}{3}\big)\big(1 + \alpha_1\omega^{-3/2} + \dots + \alpha_n\varepsilon^n + \pounds\omega^{-3(n+1)/2}\big)$$

with $\alpha_1 = \frac{7}{24}$ and $\alpha_2 = -\frac{49}{1152}$.

As mentioned above, concerning the associated Riccati equations to classical linear equations, it is more judicious to solve the problem directly with the linear equation. Nevertheless the method presented here can be applied to other types of equations. Liouville's equation is here only for illustration.

3.2. **The situation with a turning point.** Consider equation(2.10), $\varepsilon\dot{u} = f(t,u)$, with the hypotheses (H4) and (H5) below:

- H4 The function f is S^∞ and has a regular ε-expansion in an open subset U of $\mathbb{R}^2$ (*cf.* 2.2).
- H5 There is a C^∞slow curve $u = u_0(t)$ in U on a standard compact interval $T = [t_1, t_2]$, i.e.

$$\forall t \in [t_1, t_2]\ ,\ (t, u_0(t)) \in U \text{ and} f_0(t, u_0(t)) = 0\,. \tag{3.13}$$

Now, the slow curve $u = u_0(t)$ is assumed to be attractive for $t_1 \le t < t_0$ and repulsive for $t_0 < t \le t_2$, where t_0 is some standard point in $]t_1, t_2[$ called *turning point*:

- H6 The function $a(t,u) := \frac{\partial f_0}{\partial u}(t,u)$ satisfies

$$\forall t \in [t_1, t_0[\,,\ a(t, u_0(t)) < 0\,,\ \forall t \in]t_0, t_2]\,,\ a(t, u_0(t)) > 0\,. \tag{3.14}$$

Moreover, we assume that there is a canard solution $u^\natural$ that border the slow curve from t_1 to t_2. We are interested in the input-output relation about this canard solution.

Let t_e and t_s be the input and output instants respectively such that $t_1 < {}^\circ t_e < t_0 < {}^\circ t_s < t_2$. Let u_e and u_s be the input and output values which are appreciably different respectively from $u^\natural(t_e)$ and $u^\natural(t_s)$, then

$$\forall u \in [u_0(t_e), u_e],\ \ a(t_e, u) < 0 \quad \text{and} \quad \forall u \in [u_0(t_s), u_s],\ a(t_s, u) > 0\,. \tag{3.15}$$

We assume that there is a solution $u^\#$ of (2.10) with $u^\#(t_e) = u_e$, $u^\#(t_s) = u_s$. In this case, it is known [1] that the input-output relation is given by

$$\int_{{}^\circ t_e}^{{}^\circ t_s} a(t, u_0(t))dt = 0\,. \tag{3.16}$$

Proposition 3.3. *With the above notation and hypotheses (H4)–(H6), if three of the four quantities t_e, t_s, u_e, u_s have an ε-expansion, then the fourth one has, and this expansion can be computed modulo an inversion of diffeomorphism.*

Proof. The quantities t_e, t_s, u_e, u_s are linked by the relation

$$u_s - u^\natural(t_s) = (u_e - u^\natural(t_e)) \exp\left\{\frac{1}{\varepsilon}\int_{t_e}^{t_s} \Delta_2 f\left(t, u^\natural(t); u^\#(t) - u^\natural(t)\right) dt\right\}.$$

The solution $u^\#$ presents two limit layers. First, we split this integral in two parts, for example at t_0. For each limit layer, we use the combined asymptotic expansions, more especially proposition 2.6.

When the three quantities t_e, t_s and u_e have an ε-expansion, proposition 2.6 shows that $u^\#(t_0) - u^\natural(t_0)$ is of the form $\exp\left(\frac{A_0}{\varepsilon} + b\right)$ where A_0 is standard and b has an ε-expansion. Hence, by corollary 3.2, u_s has an ε-expansion. The case where the three given quantities are t_e, t_s and u_s is similar.

If the three quantities are t_e, u_e and u_s, then the previous case is used as follows: we set the change of variable $t_s = {}^\circ t_s + \varepsilon x_s$, where ${}^\circ t_s$ is given by the input-output relation (3.16). When t_e and u_e are fixed, the previous case shows that u_s is of the form $u_s = \varphi(t_s)$ where φ is S^∞ and has an ε-expansion. Moreover, we have

$$\varphi'(x_s) = \frac{d}{dx} u^\#({}^\circ t_s + \varepsilon x_s) = f({}^\circ t_s + \varepsilon x_s, u^\#({}^\circ t_s + \varepsilon x_s)) = f(t_s, u_s)\,,$$

which is appreciable thanks to (3.15), to $f(t_s, u^\natural(t_s)) \simeq 0$ and to $u_s \not\simeq u^\natural(t_s)$. Thus, φ is a S-diffeomorphism (i.e. its shadow is a diffeomorphism) and its inverse is S^∞ and has an ε-expansion. We conclude that $t_s = \varphi^{-1}(x_s)$ has an ε-expansion. □

3.3. **Equations with parameter.** Here, we prove theorem 1.1. It is classical [2] to consider the associated Riccati equation by setting $v = \frac{\varepsilon\dot\psi}{\psi}$. This leads to

$$\varepsilon\dot v = U(t) - E - v^2\,. \tag{3.17}$$

Recall that the solutions of a Riccati equation are naturally considered on the cylinder $\mathbb{R}\times(\mathbb{R}\cup\{\infty\})$, whose variable v is one chart among others. In particular, it is natural to consider solutions with poles: the passage through infinity is perfectly regular.

The equation (3.17) is of slow-fast type and usual techniques are applied. For any infinitesimal value of E, we have two repulsive rivers; the first one is close to $-\varphi(t)$ for $t \to -\infty$ and the second one is close to $\varphi(t)$ for $t \to +\infty$ (here, repulsiveness does not take into account the direction of t). For each river, there is a unique solution, called "exceptional" which borders it and all other solutions join the attractive river ($\varphi(t)$ for $t \to -\infty$, $-\varphi(t)$ for $t \to +\infty$). Moreover, the formula $\psi = \exp(\int v/\varepsilon)$ shows that to each solution bordering an attractive river corresponds a subspace of solutions of the corresponding linear equation with exponential growth. The subspace associated to an exceptional solution is constituted by solutions with exponential decay.

For E fixed, the symmetry of the equation shows that if v is a solution of (3.17), then $\tilde v : t \mapsto -v(-t)$ is. Hence, a solution bordering both repulsive rivers is necessarily an odd function. The posed problem has two solutions, denoted by $(E^\#, v^\#)$ and $(E^\flat, v^\flat)$ with $v^\#(0) = \infty$, $v^\flat(0) = 0$, which are canard solutions. It is more convenient to work in another chart. We therefore turn the Riccati cylinder by a quarter of turn.

The change of the unknown $u := \frac{v-1}{v+1}$ leads to

$$\varepsilon\dot u = \frac{1}{2}(U(t) - E - 1)(1 + u^2) - (U(t) - E + 1)u,$$

with $u^{\#}(0) = 1,\ u^{\flat}(0) = -1$ (and respectively with the values $E^{\#}$ and $E^{\flat}$ for E).

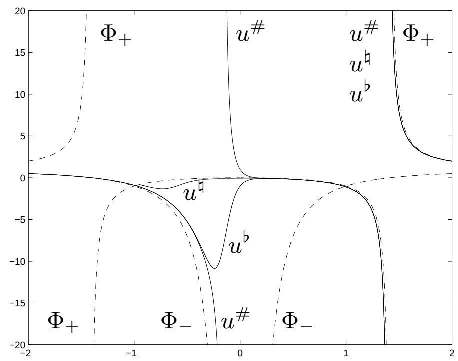

FIGURE 3. The solutions $u^{\#}$, $u^{\flat}$ and $u^{\natural}$ for the potential (1.2) and $\varepsilon = \frac{1}{10}$. $\Phi_+ := \frac{\varphi-1}{\varphi+1}$, $\Phi_- := \frac{-\varphi-1}{-\varphi+1}$.

The difference $y := u^{\#} - u^{\flat}$ satisfies the equation

$$\varepsilon\dot{y} = A(t)y - B(t)(E^{\#} - E^{\flat})\ , \quad y(+\infty) = 0$$

with

$$A(t) = \frac{1}{2}(U(t) - E^{\flat} - 1)(u^{\#} + u^{\flat}) - (U(t) - E^{\flat} + 1), \quad B(t) = \frac{1}{2}(1 - u^{\#})^2\,.$$

Here occurs a technical difficulty which does not appear in theory if ones uses the Riccati cylinder, but in practice the cylindrical coordinates are difficult to use. The difficulty comes form A and B possibly having poles with $u^{\#}$ and $u^{\flat}$, for some values of t near t^* that satisfies $\varphi(t^*) = -1$. Therefore, we consider some standard $t_1 > t_0$ close enough to t_0 such that A and B are limited on $[0, t_1]$, and we apply the variation of constant formula between 0 and t_1, in the form

$$y(0) = y(t_1)\exp\Big(\frac{1}{\varepsilon}\int_{t_1}^{0} A\Big) - \frac{E^{\#} - E^{\flat}}{\varepsilon}\int_{t_1}^{0} B(t)\exp\Big(\frac{1}{\varepsilon}\int_{t}^{0} A\Big)\,dt\,. \tag{3.18}$$

Notice that, if $t \in]]0, t_1]$, then $A(t) \simeq \frac{1}{2}\left(\varphi(t)^2 - 1\right) 2\frac{\varphi(t)-1}{\varphi(t)+1} - \left(\varphi(t)^2 + 1\right) = -2\varphi(t)$. Furthermore, $u^{\#}$ and $u^{\flat}$ admit combined expansions and $E^{\flat}$ has an ε-expansion which is the ε-expansion of every canard value. Therefore A, B and the primitive of A have also combined expansions. and the ε-expansion of $\int_{t_0}^{0} A$ begins with a given by (1.5). Notice also that $y(t_1) = \pounds\left(E^{\#} - E^{\flat}\right)$ by item 2 of proposition 2.3 in section 2.3. Since we already know that $E^{\#} - E^{\flat}$ is exponentially small of order a, and since $\int_{t_1}^{0} A \ll \int_{t_0}^{0} A \simeq a$, the first term of (3.18), namely $y(t_1)\exp\left(\frac{1}{\varepsilon}\int_{t_1}^{0} A\right)$, is exponentially small (of order $\int_{t_0}^{t_1}(-2\varphi)$). With $y(0) = 2$, formula (3.18) then yields:

$$E^{\#} - E^{\flat} = 2\varepsilon\Big(\int_0^{t_1} B(t)\exp\Big(-\frac{1}{\varepsilon}\int_0^t A\Big)dt\Big)^{-1}\left(1 + \pounds e^{-@/\varepsilon}\right)\,,$$

which can be rewritten as

$$\begin{aligned} E^{\#} - E^{\flat} =& 2\varepsilon \exp\big(-\frac{a}{\varepsilon}\big) \exp\Big(\frac{1}{\varepsilon}\int_0^{t_0} (A - 2\varphi)\Big) \\ & \times \Big(\int_0^{t_1} B(t) \exp\big(-\frac{1}{\varepsilon}\int_{t_0}^{t} A\big) dt\Big)^{-1} \big(1 + \pounds e^{-@/\varepsilon}\big) . \end{aligned}$$

Now Laplace method (proposition 2.3 in section 2.3) shows that the integral

$$\int_0^{t_1} B(t) \exp\Big(-\frac{1}{\varepsilon}\int_{t_0}^{t} A\Big) dt$$

has an expansion in powers of $\sqrt{\varepsilon}$ with a non-zero first term, i.e. is of order $\sqrt{\varepsilon}$ since $B(t_0) \simeq \frac{1}{2}$. This completes the proof of theorem 1.1.

3.4. **Symbolic and numerical results.** Using the Maple package at http://www.univ-lr.fr/Labo/MATH/DAC, we studied the expansion (1.6) for several symmetric potentials. For instance, the potential $U(t) = (2\cosh(t) - 5/2)^2$ yields

$$\begin{aligned} E^{\#} - E^{\flat} =& \frac{3^{5/2}}{\sqrt{2\pi}} \varepsilon^{1/2} \exp\left(-\frac{1}{\varepsilon}(5\ \ln 2 - 3)\right) \\ & \times \left(1 - \frac{793}{324}\varepsilon - \frac{534959}{209952}\varepsilon^2 - \frac{2490060889}{204073344}\varepsilon^3 + O\left(\varepsilon^4\right)\right) . \end{aligned}$$

The polynomial potential with rational coefficients $U(t) = (1 - t^4)^2$ yields an ε-expansion with irrational coefficients:

$$\begin{aligned} E^{\#} - E^{\flat} =& \frac{64 e^{\pi/2}}{\sqrt{\pi}} \sqrt{\varepsilon} e^{-8/5\varepsilon} \Big(1 + \big(-\frac{131}{64} - \frac{3}{32}\pi\big)\epsilon \\ & + \big(-\frac{5923}{8192} - \frac{57}{2048}\pi + \frac{9}{2048}\pi^2\big)\epsilon^2 + O(\varepsilon^3)\Big). \end{aligned}$$

Concerning the potential (1.2), numerical simulations give the following results:

- $a^{\#}$ is given by the relation $E^{\#} - E^{\flat} = \frac{16\sqrt{2\varepsilon}}{\sqrt{\pi}} \exp\big(\frac{-4}{3\varepsilon}\big) a^{\#}$,
- $\tilde{a}^{\#} = 1 - \frac{71}{96}\varepsilon - \frac{6299}{18432}\varepsilon^2 - \frac{2691107}{5308416}\varepsilon^3$ is the same parameter value obtained by symbolic manipulations with three terms in the asymptotic expansion,
- $\Delta E = \frac{E^{\#} + E^{\flat}}{2} - E^{\natural}$ is expected to be of order $\exp(-\frac{8}{3\varepsilon})$.

$1/\varepsilon$	$E^{\#}$	$a^{\#}$	$\tilde{a}^{\#}$	$a^{\#} - \tilde{a}^{\#}$
2	1.04614053857975	0.55129173	0.48140379	6.98879402e-02
4	0.47412601902887	0.77628568	0.78582414	-9.5384546e-03
6	0.31849701709428	0.86337190	0.864896269	-1.52436242e-03
8	0.24157410341774	0.90083691	0.90122221	-3.85297223e-04
10	0.19467902372582	0.92197555	0.92211728	-1.41734313e-04
12	0.16301207073454	0.93563721	0.93570146	-6.42491620e-05
14	0.14019322986821	0.94521099	0.94524428	-3.32946274e-05
16	0.12297222653846	0.95229836	0.95231734	-1.91678736e-05
18	0.10951599390792	0.95775964	0.95777035	-1.57096142e-05
20	0.09871245541601	0.96211977	0.96210310	-1.11318467e-05
22	0.08984800332140	0.96677395	0.96562888	6.41511616e-04
24	0.08244380009277	0.98149952	0.96855405	1.59136097e-03

$1/\varepsilon$	$E^\natural$	$(E^\# + E^\flat)/2$	ΔE	$-(3\varepsilon/8)\ln\{\Delta E\}$
2	0.72722933436087	0.87324733697904	1.46018002e-01	3.60754754e-01
4	0.46065652094628	0.46216455343306	1.50803248e-03	6.09089012e-01
6	0.31773469668930	0.31774227863839	7.58194908e-06	7.36858765e-01
8	0.24152668022043	0.24152671668585	3.64654158e-08	8.02823509e-01
10	0.19467600946049	0.19467600963480	1.74309428e-10	8.42632090e-01
12	0.16301187671876	0.16301187671959	8.31751334e-13	8.69226339e-01
14	0.14019321725968	0.14019321725968	4.38538094e-15	8.85549104e-01
16	0.12297222571282	0.12297222571282	-8.88178419e-16	8.12281852e-01
18	0.10951599385352	0.10951599385352	-9.71445146e-17	7.68131915e-01
20	0.09871245541241	0.09871245541241	2.77555756e-17	7.14808029e-01
22	0.08984800332116	0.08984800332116	1.24900090e-16	6.24187798e-01
24	0.08244380009275	0.08244380009275	-1.66533453e-16	5.67677116e-01

We can observe that $a^\# - \tilde{a}^\#$ has approximatively the same order as ε^4, except for $\varepsilon \leq 1/20$. The results of the second table are in accordance with our third conjecture: The last column seems to tend to 1 (as we ask an exponential order $2\,a$, which is twice than for $E^\# - E^\flat$, only the upper half of the table is relevant).

To determine $E^\flat$, we initialize $E := 2\varepsilon$ and compare the values of the solutions v_+ and v_-, at $t = 1$, of the following Cauchy's problems

$$\begin{aligned} \varepsilon\dot{v}_+ &= (t^2-1)^2 - E - v_+^2 \\ v_+(0) &= 0 \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \varepsilon\dot{v}_- &= (t^2-1)^2 - E - v_-^2 \\ v_-(2) &= -(2^2-1) = -3. \end{aligned} \tag{3.20}$$

For this purpose, we used an adaptative fourth Runge-Kutta method such that the computations are valid for $\varepsilon \geq 1/20$.

If we denote by $\zeta(E) = v_+(1) - v_-(1)$, the computation of $E^\flat$ consists in finding the zero of ζ. This computation was performed with the secant method. To compute $E^\natural$, we consider the problems (3.19) and (3.20) respectively with the initial conditions $v_+(-0.95) = \varphi(-0.95)$ and $v_-(2) = \varphi(2)$. Concerning the computation of $E^\#$, the initial condition $v(0) = \infty$ is replaced by $w(0) = 0$, where $w = 1/v$. This change of chart produces a pole (for w) in the neighborhood of $t = 1$, thus we return to the first chart at, for example, $t = 1/2$, with the initial condition $v_+(1/2) = 1/w(1/2)$. We compare finally the values of $v_+(1)$ to $v_-(1)$ where v_- is the solution of (3.20) with the initial condition $v_-(2) = \varphi(2)$.

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