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Positive periodic solutions of nonlinear functional difference equations *

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Abstract

In this paper, we apply a cone theoretic fixed point theorem to obtain sufficient conditions for the existence of multiple positive periodic solutions to the nonlinear functional difference equations

$$x(n+1) = a(n)x(n) \pm \lambda h(n)f(x(n-\tau(n))).$$

1 Introduction

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers and $\mathbb{R}^+$ the positive real numbers. Given $a < b$ in $\mathbb{Z}$, let $[a, b] = \{a, a+1, \dots, b\}$. In this paper, we investigate the existence of multiple positive periodic solutions for the nonlinear delay functional difference equation

$$x(n+1) = a(n)x(n) + \lambda h(n)f(x(n-\tau(n))) \tag{1.1}$$

where $a(n), h(n)$ and $\tau(n)$ are T-periodic for T is an integer with $T \geq 1$. We assume that λ, $a(n)$, $f(x)$ and $h(n)$ are nonnegative with $0 < a(n) < 1$ for all $n \in [0, T-1]$.

The existence of multiple positive periodic solutions of nonlinear functional differential equations have been studied extensively in recent years. We cite some appropriate references here [2] and [11]. We are particularly motivated by the work of Cheng and Zhang [2] on functional differential equations and the work of Eloe, Raffoul and others [5] on a boundary value problem involving functional difference equation. It is customary when working with boundary value problems, whether in differential or difference equations, to display the desired solution in terms of a suitable Green function and then apply cone theory [1, 3, 4, 5, 6, 7, 9]. Since our equation (1.1) is not of the type of boundary value we obtain a variation of parameters formula and then try to find a lower and upper estimates for the kernel inside the summation. Once those estimates are found we use Krasnoselskii's fixed point theorem to show the existence of multiple positive periodic solutions. In [10], the author studied the existence of

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periodic solutions of an equation similar to equation (1.1) using Schauder's Second fixed point theorem. Throughout this paper, we denote the product of $y(n)$ from $n = a$ to $n = b$ by $\prod_{n=a}^{b} y(n)$ with the understanding that $\prod_{n=a}^{b} y(n) = 1$ for all $a > b$,

2 Positive periodic solutions

We now state Krasnosel'skii fixed point theorem [8].

Theorem 2.1 (Krasnosel'skii) *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P}$ be a cone in $\mathcal{B}$. Suppose Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and suppose that*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

is a completely continuous operator such that

(i) $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$; *or*

(ii) $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Let $\mathcal{X}$ be the set of all real T-periodic sequences. This set endowed with the maximum norm $\|x\| = \max_{n\in[0,T-1]} |x(n)|$, $\mathcal{X}$ is a Banach space. The next Lemma is essential in obtaining our results.

Lemma 2.2 *$x(n) \in \mathcal{X}$ is a solution of equation* (1.1) *if and only if*

$$x(n) = \lambda \sum_{u=n}^{n+T-1} G(n,u)h(u)f(x(u - \tau(u))) \tag{2.1}$$

where

$$G(n,u) = \frac{\prod_{s=u+1}^{n+T-1} a(s)}{1 - \prod_{s=n}^{n+T-1} a(s)}, \quad u \in [n, n+T-1]. \tag{2.2}$$

Note that the denominator in $G(n,u)$ is not zero since $0 < a(n) < 1$ for $n \in [0, T-1]$. The proof of Lemma 2.1 is easily obtained by noting that (1.1) is equivalent to

$$\triangle\Big(\prod_{s=-\infty}^{n-1} a^{-1}(s)x(n)\Big) = \lambda h(n)f(x(n - \tau(n))) \prod_{s=-\infty}^{n} a^{-1}(s).$$

By summing the above equation from $u = n$ to $u = n + T - 1$ we obtain (2.1). Note that since $0 < a(n) < 1$ for all $n \in [0, T-1]$, we have

$$N \equiv G(n,n) \leq G(n,u) \leq G(n, n+T-1) = G(0, T-1) \equiv M$$

for $n \le u \le n+T-1$ and

$$1 \ge \frac{G(n,u)}{G(n,n+T-1)} \ge \frac{G(n,n)}{G(n,n+T-1)} = \frac{N}{M} > 0.$$

For each $x \in X$, define a cone by

$$\mathcal{P} = \big\{ y \in \mathcal{X} : y(n) \ge 0, n \in \mathbb{Z} \quad \text{and} \quad y(n) \ge \eta \|y\| \big\},$$

where $\eta = N/M$. Clearly, $\eta \in (0,1)$. Define a mapping $T : \mathcal{X} \to \mathcal{X}$ by

$$(Tx)(n) = \lambda \sum_{u=n}^{n+T-1} G(n,u)h(u)f(x(u-\tau(u))$$

where $G(n,u)$ is given by (2.2). By the nonnegativity of λ, f, a, h, and G, $Tx(n) \ge 0$ on $[0, T-1]$. It is clear that $(Tx)(n+T) = (Tx)(n)$ and T is completely continuous on bounded subset of $\mathcal{P}$. Also, for any $x \in \mathcal{P}$ we have

$$\begin{aligned} (Tx)(n) &= \lambda \sum_{u=n}^{n+T-1} G(n,u)h(u)f(x(u-\tau(u))) \\ &\le \lambda \sum_{u=0}^{T-1} G(0,T-1)h(u)f(x(u-\tau(u)). \end{aligned}$$

Thus,

$$\|Tx\| = \max_{n\in[0,T-1]} |Tx(n)| \le \lambda \sum_{u=0}^{T-1} G(0,T-1)h(u)f(x(u-\tau(u))).$$

Therefore,

$$\begin{aligned} Tx(n) &= \lambda \sum_{u=n}^{n+T-1} G(n,u)h(u)f(x(u-\tau(u))) \\ &\ge \lambda N \sum_{u=0}^{T-1} h(u)f(x(u-\tau(u))) \\ &= \lambda N \sum_{u=0}^{T-1} \frac{G(0,T-1)}{M} h(u)f(x(u-\tau(u))) \\ &\ge \eta \|Tx\|. \end{aligned}$$

That is, $T\mathcal{P}$ is contained in $\mathcal{P}$. ♢

In this paper we shall make the following assumptions.

(A1) the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous

(A2) $h(n) > 0$ for $n \in \mathbb{Z}$

(L1) $\lim_{x\to 0}\frac{f(x)}{x}=\infty$

(L2) $\lim_{x\to\infty}\frac{f(x)}{x}=\infty$

(L3) $\lim_{x\to 0}\frac{f(x)}{x}=0$

(L4) $\lim_{x\to\infty}\frac{f(x)}{x}=0$

(L5) $\lim_{x\to 0}\frac{f(x)}{x}=l$ with $0<l<\infty$

(L6) $lim_{x\to\infty}\frac{f(x)}{x}=L$ with $0<L<\infty$.

For the next theorem we let

$$A=\max_{0\leq n\leq T-1}\sum_{u=0}^{T-1}G(n,u)h(u) \tag{2.3}$$

and

$$B=\min_{0\leq n\leq T-1}\sum_{u=0}^{T-1}G(n,u)h(u). \tag{2.4}$$

Theorem 2.3 *Assume that (A1), (A2), (L5), and (L6) hold. Then, for each λ satisfying*

$$\frac{1}{\eta BL}<\lambda<\frac{1}{Al} \tag{2.5}$$

or

$$\frac{1}{\eta Bl}<\lambda<\frac{1}{AL} \tag{2.6}$$

equation (1.1) *has at least one positive periodic solution.*

Proof Suppose (2.5) hold. We construct the sets Ω_1 and Ω_2 in order to apply Theorem 2.1. Let $\epsilon>0$ be such that

$$\frac{1}{\eta B(L-\epsilon)}\leq\lambda\leq\frac{1}{A(l+\epsilon)}.$$

By condition (L5), there exists $H_1>0$ such that $f(y)\leq(l+\epsilon)y$ for $0<y\leq H_1$. Define

$$\Omega_1=\{x\in\mathcal{P}:\|x\|<H_1\}$$

Then, if $x\in\mathcal{P}\cap\partial\Omega_1$,

$$\begin{aligned}(Tx)(n)&\leq\lambda(l+\epsilon)\sum_{u=n}^{n+T-1}G(n,u)h(u)x(u-\tau(u)))\\&\leq\lambda(l+\epsilon)\|x\|\sum_{u=o}^{T-1}G(n,u)h(u)\\&\leq\lambda A(l+\epsilon)\|x\|\leq\|x\|.\end{aligned}$$

In particular, $\|Tx\| \leq \|x\|$, for all $x \in \mathcal{P} \cap \partial\Omega_1$.

Next we construct the set Ω_2. Apply condition $(L6)$ and find H such that $f(y) \geq (L-\epsilon)y$, for all $y \geq H$. Let $H_2 = \max\{2H_1, \eta H\}$. Define

$$\Omega_2 = \{x \in \mathcal{P} : \|x\| < H_2\}$$

Then, if $x \in \mathcal{P} \cap \partial\Omega_2$,

$$\begin{aligned} (Tx)(n) &\geq \lambda(L-\epsilon)\sum_{u=n}^{n+T-1} G(n,u)h(u)x(u-\tau(u))) \\ &\geq \lambda(L-\epsilon)\eta\|x\|\sum_{u=o}^{T-1} G(n,u)h(u) \\ &\geq \lambda(L-\epsilon)\eta B\|x\| \geq \|x\|. \end{aligned}$$

In particular, $\|Tx\| \geq \|x\|$, for all $x \in \mathcal{P} \cap \partial\Omega_2$. Apply condition (i) of Theorem 2.1, and this completes the proof. When condition (2.6) holds, the proof can be similarly obtained by invoking condition (ii) of Theorem 2.1.

Theorem 2.4 *Assume that (A1) and (A2) hold. Also, if either (L1) and (L4) hold, or, (L2) and (L3) hold, then* (1.1) *has at least one positive periodic solution for any* $\lambda > 0$.

Proof: Apply (L1) and choose $H_1 > 0$ such that if $0 < y < H_1$, then

$$f(y) \geq \frac{y}{\lambda\eta B}.$$

Define $\Omega_1 = \{x \in \mathcal{P} : \|x\| < H_1\}$. If $x \in \mathcal{P} \cap \partial\Omega_1$, then

$$\begin{aligned} (Tx)(n) &\geq \lambda\frac{1}{\lambda\eta B}\sum_{u=0}^{T-1} G(n,u)h(u)x(u-\tau(u))) \\ &\geq \frac{1}{\lambda\eta B}\lambda\|x\|\sum_{u=o}^{T-1} G(n,u)h(u) \\ &\geq \|x\|. \end{aligned}$$

In particular, $\|Tx\| \geq \|x\|$, for all $x \in \mathcal{P} \cap \partial\Omega_1$. In order to construct Ω_2, we consider two cases, f bounded and f unbounded. The case where f is bounded is straight forward. If $f(y)$ is bounded by $Q > 0$, set

$$H_2 = \max\{2H_1, \lambda QA\}.$$

Then if $x \in \mathcal{P}$ and $\|x\| = H_2$,

$$\begin{aligned} Tx(n) &\leq \lambda N\sum_{u=0}^{T-1} G(n,u)h(u) \\ &\leq \lambda QA \leq H_2. \end{aligned}$$

Now assume f is unbounded. Apply condition $(L4)$ and set $\epsilon_1 > 0$ such that if $x > \epsilon_1$, then
$$f(y) < \frac{y}{\lambda A}.$$
Set $H_2 = \max\{2H_1, \epsilon_1\}$ and define $\Omega_2 = \{x \in \mathcal{P} : \|x\| < H_2\}$. If $x \in \mathcal{P} \cap \partial\Omega_2$, then

$$\begin{aligned}(Tx)(n) &\leq \lambda \frac{1}{\lambda A} \sum_{u=0}^{T-1} G(n,u)h(u)x(u-\tau(u)) \\ &\leq \frac{1}{\lambda A} \lambda H_2 \sum_{u=o}^{T-1} G(n,u)h(u) \\ &\leq H_2 = \|x\|.\end{aligned}$$

In particular, $\|Tx\| \leq \|x\|$, for all $x \in \mathcal{P} \cap \partial\Omega_2$. Apply condition (ii) of Theorem 2.1, and this completes the proof. The proof of the other part, follows similarly by invoking condition (ii) of Theorem 2.1. The next two corollaries are consequence of the previous two theorems.

Corollary 2.5 *Assume that* (A1) *and (A2) hold. Also, if either (L1) and (L6) hold, or, (L2) and (L5) hold, then* (1.1) *has at least one positive periodic solution if* λ *satisfies either* $0 < \lambda < 1/(AL)$*, or,* $0 < \lambda < 1/(Al)$.

Corollary 2.6 *Assume that (A1) and (A2) hold. Also, if either (L3) and (L6) hold, or, (L4) and (L5) hold, then* (1.1) *has at least one positive periodic solution if* λ *satisfies either* $1/(\eta BL) < \lambda < \infty$*, or,* $1/(\eta Bl) < \lambda < \infty$.

Next we turn our attention to the equation
$$x(n+1) = a(n)x(n) - \lambda h(n)f(x(n-\tau(n))) \tag{2.7}$$
where $\lambda, a(n), f(x)$ and $h(n)$ satisfy the same assumptions stated for (1.1) except that $a(n) > 1$ for all $n \in [0, T-1]$. In view of (2.7) we have that
$$x(n) = \lambda \sum_{u=n}^{n+T-1} K(n,u)h(u)f(x(u-\tau(u))) \tag{2.8}$$
where
$$K(n,u) = \frac{\prod_{s=u+1}^{n+T-1} a(s)}{\prod_{s=n}^{n+T-1} a(s) - 1}, \qquad u \in [n, n+T-1]. \tag{2.9}$$
Note that the denominator in $G(n,u)$ is not zero since $a(n) > 1$ for $n \in [0, T-1]$. Also, it is easily seen that since $a(n) > 1$ for all $n \in [0, T-1]$, we have
$$M \equiv K(n,n) \geq K(n,u) \geq K(n, n+T-1) = K(0, T-1) \equiv N$$
for $n \leq u \leq n+T-1$ and
$$1 \geq \frac{K(n,u)}{K(n,n)} \geq \frac{K(n,n+T-1)}{K(n,n)} = \frac{N}{M} > 0.$$

Finally, by defining

$$A^1 = \max_{0 \le n \le T-1} \sum_{u=0}^{T-1} K(n,u)h(u)$$

and

$$B^1 = \min_{0 \le n \le T-1} \sum_{u=0}^{T-1} K(n,u)h(u)$$

similar theorems and corollaries can be easily stated and proven regarding equation (2.7).

We conclude this paper with the following open problems. Assume that (A1) and (A2) hold. In view of this paper, what can be said about equations (1.1) and (2.7) when:

1. The conditions ($L1$) and ($L2$) hold?
2. The conditions ($L3$) and ($L4$) hold?
3. $0 < a(n) < 1$ in (2.7) and $a(n) > 1$ in (1.1) for all $n \in [0, T-1]$?

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