Electronic Journal of Differential Equations, Vol. 2002(2002), No. 59, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

POSITIVE SOLUTIONS AND NONLINEAR EIGENVALUE PROBLEMS FOR RETARDED SECOND ORDER DIFFERENTIAL EQUATIONS

G. L. KARAKOSTAS & P. CH. TSAMATOS

Abstract. We investigate the eigenvalues of a nonlocal boundary value problem for a second order retarded differential equation. We provide information on norm estimates, uniqueness, and continuity of solutions.

1. Introduction

We study the set of positive values λ for which second order nonlinear differential equations with retarded arguments admit a positive, nondecreasing, concave solution. Consider

$$(p(t)x'(t))' + \lambda \sum_{j=0}^{k} q_j(t) f_j(x(t), x(h_j(t))) = 0, \quad \text{a.a.} \quad t \in [0,1] \tag{1.1}$$

with the initial condition

$$x(0) = 0 \tag{1.2}$$

and the nonlocal boundary condition

$$x'(1) = \int_0^1 x'(s)dg(s), \tag{1.3}$$

where g is a nondecreasing function and the integral is meant in the Riemann-Stieljes sense. Boundary-value problems involving retarded and functional differential equations were recently studied by many authors using various methods. We especially refer to [1, 2, 4, 5, 7, 12, 13] and to [6, 8, 10] which were the motivation for this work. Our main results in this paper refer to the values of the positive real parameter λ for which the problem (1.1)-(1.3) has a solution. Note that the problem of finding eigenvalues, for which a second or a higher order differential equation with various boundary conditions has positive solutions, has been studied by several authors in the last decade. See for example the papers [2, 3, 6, 7, 10] and the references therein.

2000 *Mathematics Subject Classification.* 34K10.
Key words and phrases. Nonlocal boundary value problems, positive solutions, concave solutions, retarded second order differential equations .

Submitted March 27, 2002. Published June 21, 2002.

Problem of this type are usually transformed into operator equations of the form

$$Ax = \lambda^{-1}x, \tag{1.4}$$

where A is an appropriate completely continuous operator. (Obviously the form (1.4) justifies the term "eigenvalue problems" we use in the title of this article.)

Equation (1.4) is written as $x = Tx$, where $T := \lambda A$ and in a great number of works the following theorem is applied.

Theorem 1.1 (Krasnoselskii [11]). *Let $\mathcal{B}$ be a Banach space and let $\mathbb{K}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open bounded subsets of $\mathcal{B}$, with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let*

$$T : \mathbb{K} \cap (\overline{\Omega_2} \setminus \Omega_1) \to \mathbb{K}$$

be a completely continuous operator such that, either

$$\|Tu\| \le \|u\|, \quad u \in \mathbb{K} \cap \partial\Omega_1 \quad \textit{and} \quad \|Tu\| \ge \|u\|, \quad u \in \mathbb{K} \cap \partial\Omega_2,$$

or

$$\|Tu\| \ge \|u\|, \quad u \in \mathbb{K} \cap \partial\Omega_1 \quad \textit{and} \quad \|Tu\| \le \|u\|, \quad u \in \mathbb{K} \cap \partial\Omega_2.$$

Then T has a fixed point in $\mathbb{K} \cap (\overline{\Omega_2} \setminus \Omega_1)$.

In this paper we are interested in the existence of positive solutions and our approach is based on Theorem 1.1.

Note that, as the literature shows, in almost all the cases where Theorem 1.1 applies, concavity is the most significant property of te solutions. Indeed, the idea is to use concavity of the real valued functions x defined on the interval $[0, 1] =: I$ and which constitute the elements of a cone $\mathbb{K}$, the domain of the operator T. Then two elementary facts are the major steps in our proofs. The first fact read as follows:

Fact 1.2. *Let $x : I \to \mathbb{R}$ be a nonnegative, nondecreasing and concave function. Then, for any $\tau \in [0, 1]$ it holds*

$$x(t) \ge \tau\|x\|, \quad t \in [\tau, 1],$$

where $\|x\|$ is the sup-norm of x.

Proof. From the concavity of x we have

$$x(t) \ge x(\tau) = x\left((1-\tau)0 + \tau 1\right) \ge (1-\tau)x(0) + \tau x(1) \ge \tau x(1) = \tau\|x\|,$$

for all $t \in [\tau, 1]$. □

The second fact is that the image Ax of a point x of the cone $\mathbb{K}$ is a concave function. And in case $p(t) = 1, t \in I$ this fact is obvious. (Indeed, one can show that the second derivative is nonnegative.) In the general case an additional assumption on p is needed. This step, which notice that, though it seems to be obvious, it should be added to the proofs of the main theorems in [8, 9], lies on the following elementary lemma:

Lemma 1.3. *Let a, b two real valued functions defined on I. If the product ab is a non-increasing function, then b is also non-increasing provided that, either*

(i) *a, b are nonnegative functions and a is nondecreasing, or*
(ii) *a is nonnegative and non-increasing and b is non-positive.*

Proof. For each $t_1, t_2 \in I$ with $t_1 \leq t_2$, it holds

$$\begin{aligned} a(t_1)[b(t_2) - b(t_1)] &= a(t_1)b(t_2) - a(t_1)b(t_1) \\ &\leq a(t_1)b(t_2) - a(t_2)b(t_2) = [a(t_1) - a(t_2)]b(t_2) \leq 0. \end{aligned}$$

Thus, in any case, we have $b(t_2) \leq b(t_1)$. □

From this lemma we get the following statement.

Fact 1.4. *If $y : I \to \mathbb{R}$ is a differentiable function with $y' \geq 0$ and $p : I \to \mathbb{R}$ is a positive and nondecreasing function such that $(p(t)y'(t))' \leq 0$, for all $t \in I$, then y is concave.*

Proof. We apply Lemma 1.3(i) with $a = p$, $b = y'$ and conclude that y' is nonincreasing. This implies that y is concave. □

Apart of positivity and concavity properties of the solutions which are guaranteed by applying Theorem 1.1 we know also monotonicity of them. Moreover we can have some information on the estimates of their sup-norm. Finally, some Lipschitz type conditions may provide uniqueness results as well as continuous dependence of the solutions under the corresponding eigenvalues.

2. Preliminaries and the assumptions

In the sequel we shall denote by $\mathbb{R}$ the real line and by I the interval $[0, 1]$. Then $C(I)$ will denote the space of all continuous functions $x : I \to \mathbb{R}$. This is a Banach space when it is furnished with the usual supremum norm $\|\cdot\|$.

Consider equation (1.1) associated with the conditions (1.2), (1.3). By a solution of the problem (1.1)-(1.3) we mean a function $x \in C(I)$, whose the first derivative x' is absolutely continuous on I and which satisfies equation (1.1) for almost all $t \in I$, as well as conditions (1.2), (1.3).

The basic assumptions on the functions involved are the following:

(H1) The function $p : I \to (0, +\infty)$ is continuous and nondecreasing.

(H2) The functions $q_j : I \to \mathbb{R}$, $j = 0, 1, \ldots, k$ are continuous and such that $q_j(t) \geq 0$, $t \in I$, $j = 0, \ldots, k$, as well as $q_0(1) > 0$.

(H3) The function $g\colon I \to \mathbb{R}$ is nondecreasing and such that

$$\int_0^1 \frac{1}{p(s)} dg(s) < \frac{1}{p(1)}.$$

(H4) The retardations $h_j : I \to I$ $(j = 0, \ldots, k)$ satisfy

$$0 \leq h_j(t) \leq h_0(t) \leq t, \quad t \in I, \quad j = 1, \ldots, k$$

and moreover h_0 is a nondecreasing function not identically zero.

(H5) The functions $f_j\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $0 = 1, \ldots, k$ are continuous and such that $f_j(u, v) \geq 0$, when $u \geq 0$ and $v \geq 0$, for all $j = 0, 1, \ldots, k$. Also, if for some $j_0 \in \{1, 2, \ldots, k\}$ there is a point $t \in I$ such that $h_{j_0}(t) < h_0(t)$, then we assume that the function $f_{j_0}(u, v)$ is nondecreasing with respect to v for all $u \geq 0$.

The first step in our approach is to reformulate the problem (1.1)-(1.3) as an operator equation of the form (1.4) for an appropriate operator A, which does not depend on the parameter λ. Note that our requirement is $\lambda > 0$. To find such an operator A we integrate (1.1) from t to 1 and get

$$x'(t) = \frac{1}{p(t)}p(1)x'(1) + \frac{\lambda}{p(t)}\int_t^1 z(s)ds, \tag{2.1}$$

where

$$z(t) := \sum_{j=0}^{k} q_j(t)f_j(x(t), x(h_j(t))).$$

Taking into account condition (1.3) we obtain

$$x'(1) = \int_0^1 x'(s)dg(s) = p(1)x'(1)\int_0^1 \frac{1}{p(s)}dg(s) + \int_0^1 \frac{\lambda}{p(s)}\int_s^1 z(r)dr\, dg(s),$$

from which it follows that

$$p(1)x'(1) = \gamma\lambda\int_0^1 \frac{1}{p(s)}\int_s^1 z(r)drdg(s),$$

where the constant γ is

$$\gamma := \Big(\frac{1}{p(1)} - \int_0^1 \frac{1}{p(s)}dg(s)\Big)^{-1}.$$

Then, from (2.1) and (1.2), we derive

$$x(t) = \lambda\gamma\int_0^1 \frac{1}{p(s)}\int_s^1 z(r)dr\, dg(s)\int_0^t \frac{1}{p(s)}ds + \lambda\int_0^t \frac{1}{p(s)}\int_s^1 z(r)dr\, ds.$$

This fact shows that if x solves the boundary-value problem (1.1)-(1.3), then it solves the operator equation $\lambda Ax = x$, where A is the operator defined by

$$\begin{aligned} Ax(t) :=& \gamma P(t)\int_0^1 \frac{1}{p(s)}\int_s^1 \sum_{j=0}^{k} q_j(r)f_j(x(r), x(h_j(r)))dr\, dg(s) \\ &+ \int_0^t \frac{1}{p(s)}\int_s^1 \sum_{j=0}^{k} q_j(r)f_j(x(r), x(h_j(r)))dr\, ds. \end{aligned} \tag{2.2}$$

Here we have set

$$P(t) := \int_0^t \frac{1}{p(s)}\, ds, \quad t \in I.$$

Lemma 2.1. *A function $x \in C(I)$ is a solution of the boundary value problem (1.1)-(1.3) if and only if x solves the operator equation (1.4), where A is defined by (2.2). Also, any nonnegative solution of (1.4) is an increasing and concave function.*

Proof. The "only if" part was shown above. For the "if" part assume that x solves (1.4). Then, for every $t \in I$ we have

$$x(t) = \lambda Ax(t) = \lambda\gamma P(t)\int_0^1 \frac{1}{p(s)}\int_s^1 z(r)dr\, dg(s) + \lambda\int_0^t \frac{1}{p(s)}\int_s^1 z(r)dr\, ds.$$

Therefore

$$x'(t) = \lambda\gamma\frac{1}{p(t)}\int_0^1 \frac{1}{p(s)}\int_s^1 z(r)dr\, dg(s) + \lambda\frac{1}{p(t)}\int_t^1 z(r)dr\, ds\,.$$

and

$$(p(t)x'(t))' = -\lambda z(t) = -\lambda \sum_{j=0}^{k} q_j(t) f_j(x(t), x(h_j(t))).$$

Hence, if $x = \lambda Ax$, then x satisfies (1.1) and, moreover, since $x(0) = \lambda Ax(0) = 0$, it follows that x satisfies (1.2). Also, for every $t \in I$ we have

$$\begin{aligned}
\int_0^1 x'(t)dg(t) =& \lambda\gamma \int_0^1 \frac{1}{p(t)} dg(t) \cdot \int_0^1 \frac{1}{p(s)} \int_s^1 z(r)dr\, dg(s) \\
&+ \lambda \int_0^1 \frac{1}{p(t)} \int_t^1 z(r)dr\, dg(t) \\
=& \lambda\big[\gamma \int_0^1 \frac{1}{p(t)} dg(t) + 1\big] \int_0^1 \frac{1}{p(t)} \int_t^1 z(r)dr\, dg(t) \\
=& \lambda\Big[\frac{\int_0^1 \frac{1}{p(t)} dg(t)}{\frac{1}{p(1)} - \int_0^1 \frac{1}{p(t)} dg(t)} + 1\Big] \int_0^1 \frac{1}{p(t)} \int_t^1 z(r)dr\, dg(t) \\
=& \frac{\lambda\gamma}{p(1)} \int_0^1 \frac{1}{p(t)} \int_t^1 z(r)dr\, dg(t) = x'(1).
\end{aligned}$$

Thus x satisfies (1.3). The additional properties, which the lemma claims that any $x \geq 0$ with $x = \lambda Ax$ has, are implied from the fact that $x' \geq 0$, $(p(t)x'(t))' \leq 0$ and Fact 1.4. We keep in mind that $\lambda > 0$. $\square$

By using the continuity of the functions f_j, q_j and p it is not hard to show that A is a completely continuous operator.

Now consider the set

$$\mathbb{K} := \{x \in C(I) : x(0) = 0, \quad x \geq 0, \quad x' \geq 0 \quad \text{and } x \text{ concave}\},$$

which, obviously, is a cone in $C(I)$. We show that the operator λA maps the cone $\mathbb{K}$ into itself. Indeed we have the following statement.

Lemma 2.2. *Consider functions p, g, f_j, q_j, h_j, $(j = 0, 1, \dots, k)$, satisfying the assumptions (H1)-(H5). Then*

$$\lambda A(\mathbb{K}) \subset \mathbb{K}.$$

Proof. Let $x \in \mathbb{K}$ be fixed. Then we observe that $Ax(0) = 0$, $Ax \geq 0$ and $(Ax)' \geq 0$. Moreover, since, obviously, $\big(p(t)(Ax)'(t)\big)' \leq 0$ for all $t \in I$, by Fact 1.4, we know that the function $y = \lambda Ax$ is concave and the proof is complete. $\square$

3. Existence Results

Let x be a function in the cone $\mathbb{K}$. Then x is nondecreasing and nonnegative, hence $\|x\| = x(1)$. Also, from Lemma 2.1 we have $\lambda Ax \in \mathbb{K}$, thus $\|Ax\| = Ax(1)$.

But then we have

$$\begin{aligned}\|Ax\| = Ax(1) =&\gamma P(1)\int_0^1 \frac{1}{p(s)}\int_s^1 \sum_{j=0}^k q_j(r)f_j(x(r),x(h_j(r)))dr\,dg(s)\\ &+\int_0^1 \frac{1}{p(s)}\int_s^1 \sum_{j=0}^k q_j(r)f_j(x(r),x(h_j(r)))drds\\ =&\int_0^1 \frac{1}{p(s)}\int_s^1 \sum_{j=0}^k q_j(r)f_j(x(r),x(h_j(r)))drdk(s),\end{aligned}$$

where $k(s) := s + \gamma P(1)g(s)$, $s \in I$. Applying Fubini's Theorem we get

$$\|Ax\| = \int_0^1 \sum_{j=0}^k q_j(s)f_j(x(s),x(h_j(s)))R(s)ds, \tag{3.1}$$

where

$$R(s) := \int_0^s \frac{1}{p(r)}\,dk(r).$$

Next, let $0 < K < S < M < +\infty$ be fixed and define the functions

$$\begin{aligned}\Phi(u,v) &:= sup\{f_0(u',v') : 0 \le u' \le u, \quad 0 \le v' \le v\}, \quad u,v \in [0,K]\\ \phi(u,v) &:= inf\{f_0(u',v') : u \le u' \le M, \quad v \le v' \le M\}, \quad u,v \in [S,M].\end{aligned}$$

It is clear that both the functions Φ and ϕ are nondecreasing with respect to their variables and they satisfy

$$f_0(u,v) \le \Phi(u,v), \quad u,v \in [0,K] \tag{3.2}$$

$$f_0(u,v) \ge \phi(u,v), \quad u,v \in [S,M]. \tag{3.3}$$

Also we make the following assumption

(H6) The following quantities are finite numbers:

$$L_j := \sup_{0<v\le u<K} \frac{f_j(u,v)}{f_0(u,v)}, \quad j = 1,2,\dots,k,$$

Now we define the set

$$E(S,M) := \big\{\eta \in I : S < h_0(\eta)M\big\},$$

which maybe empty. Also we define the following real numbers:

$$\begin{aligned}\zeta &:= \sup\Big\{h_0(\eta)\int_\eta^1 q_0(s)R(s)ds : \eta \in E(S,M)\Big\},\\ \xi &:= \int_0^1 \big(q_0 + \sum_{j=1}^k L_jq_j(s)\big)R(s)ds,\\ b(S,M) &:= \frac{1}{\zeta}\sup_{u\in[S,M]} \frac{u}{\phi(u,u)},\\ B(K) &:= \frac{1}{\xi}\frac{K}{\Phi(K,K)}.\end{aligned}$$

Theorem 3.1. *Assume that p, g, f_j, q_j, h_j $(j = 0, \dots, k)$ are functions which satisfy assumptions (H1)-(H6). Then for every λ such that $b(S, M) < \lambda < B(K)$, the boundary value problem (1.1)-(1.3) admits at least one positive, nondecreasing and concave solution x such that $K < \|x\| < M$.*

Proof. We set $T := \lambda A$. First we shall prove that

$$\text{if} \quad x \in \mathbb{K} \quad \text{and} \quad \|x\| = M, \quad \text{then} \quad \|Tx\| > M. \tag{3.4}$$

Indeed, assume on the contrary, that there is a x in $\mathbb{K}$ such that

$$\|x\| = M \quad \text{and} \quad \|Tx\| \le M \tag{3.5}$$

and consider any $\eta \in E(S, M)$ fixed. Then $\eta > 0$ and $h_0(\eta)M > S$. Also for all $s \in [\eta, 1]$ we have $h_0(s) \ge h_0(\eta) := \tau$. From the concavity and monotonicity of x and Fact 1.2 we have

$$M = \|x\| \ge x(s) \ge x(h_0(s)) \ge \tau\|x\| = h_0(\eta)M.$$

Now taking into account (3.1) and (3.3) from (3.5) we get

$$\begin{aligned}
M &\ge \lambda \int_0^1 q_0(s) f_0(x(s), x(h_0(s)))R(s)ds \\
&\ge \lambda \int_\eta^1 q_0(s) f_0(x(s), x(h_0(s)))R(s)ds \\
&\ge \lambda \phi\big(h_0(\eta)M, h_0(\eta)M\big) \int_\eta^1 q_0(s)R(s)ds.
\end{aligned}$$

So

$$\frac{h_0(\eta)M}{\phi\big(h_0(\eta)M, h_0(\eta)M\big)} \ge \lambda h_0(\eta) \int_\eta^1 q_0(s)R(s)ds. \tag{3.6}$$

This implies that

$$\sup_{u \in [S,M]} \frac{u}{\phi(u, u)} \ge \lambda\zeta, \tag{3.7}$$

which contradicts to the fact that $\lambda > b(S, M)$. Thus (3.4) holds.

Next we claim that

$$\text{if} \quad x \in \mathbb{K} \quad \text{and} \quad \|x\| = K, \quad \text{then} \quad \|Tx\| < K. \tag{3.8}$$

Indeed if not, then assume that for some $x \in \mathbb{K}$ with $\|x\| = K$ we have $\|Tx\| \ge K$. The first one implies that $0 \le x(s) \le K$, $s \in I$ and so, taking into account (3.1), (3.2), assumptions (H5), (H6) and the fact that x is nondecreasing we get in any case that

$$\begin{aligned}
K &\le \lambda \int_0^1 \sum_{j=0}^k q_j(s) f_j(x(s), x(h_j(s)))R(s)ds \\
&\le \lambda \int_0^1 \sum_{j=0}^k q_j(s) f_j(x(s), x(h_0(s)))R(s)ds \qquad (3.9) \\
&\le \lambda \int_0^1 \Big(q_0(s) + \sum_{j=1}^k L_j q_j(s)\Big) f_0(x(s), x(h_0(s)))R(s)ds.
\end{aligned}$$

Now, from (3.2) it follows that

$$K \leq \lambda\Phi(K,K)\int_0^1 \Big(q_0(s) + \sum_{j=1}^{k} L_j q_j(s)\Big)R(s)ds = \lambda\Phi(K,K)\xi. \tag{3.10}$$

Hence it holds

$$\frac{K}{\Phi(K,K)} < \lambda\xi,$$

which contradicts to the fact that $\lambda < B(K)$ and our claim is proved.

Finally, we set $\Omega_1 := \{x \in C(I) : \|x\| < K\}$ and $\Omega_2 := \{x \in C(I) : \|x\| < M\}$. Note that $K < M$. Taking into account that T is a completely continuous operator and Lemma 2.2, from Theorem 1.1 we conclude that there exists a solution x of the boundary value problem $(1.1)-(1.3)$ such that $K \leq \|x\| \leq M$. From (3.4) and (3.8) we see that equalities $\|x\| = K$ and $\|x\| = M$ cannot hold. □

One of the main questions, on the existence problem solved above, is whether we may enlarge the set of eigenvalues. Partial answers to this question are given in the following theorems.

Theorem 3.2. *Assume that p, g, f_j, q_j, h_j, $(j = 0, \dots, k)$ satisfy assumptions (H1)-(H6) and moreover that*

$$f_0(u,v) = 0 \quad \text{and} \quad u \geq v \geq 0 \quad \text{imply} \quad v = 0.$$

Also let h_0 be a continuous function, with $h_0(1) > 0$. If $b(S,M) \leq \lambda < B(K))$, then there is a positive, nondecreasing and concave solution x of the boundary value problem (1.1)-(1.3) such that $K < \|x\| < M$.

Proof. As in Theorem 3.1 we have that $x \in \mathbb{K}$ and $\|x\| = K$ imply $\|Ax\| < K$ and we will show that $x \in \mathbb{K}$ and $\|x\| = M$ imply $\|Ax\| > M$. To do this we proceed as in Theorem 3.1 and obtain (3.6).

Now, if for some $\eta' \in I$ equality holds, we must have $h_0(\eta) > \frac{S}{M}$ and

$$\int_0^\eta q_0(s)f_0(x(s), x(h_0(s)))R(s)ds = 0$$

for all $\eta \in [\eta', 1]$. Since $q_0(s) \geq 0$ and $q_0(1) > 0$ it follows that for all s close to 1 it holds

$$f_0(x(s), x(h_0(s))) = 0$$

and so, by our hypothesis we have $x(h_0(s)) = 0$ for all $s > 0$ close to 1. This gives $h_0(s) = 0$, because of Fact 1.2, hence, by continuity, $h_0(1) = 0$, a contradiction.

Therefore in (3.6) we have the strict inequality. Since both sides are continuous functions of η, it follows that (3.7) holds as a strict inequality, which contradicts to $b(S,M) \leq \lambda$. Now the result follows as in Theorem 3.1. □

Theorem 3.3. *Assume that p, g, f_j, q_j, h_j, $(j = 0, \dots, k)$ satisfy (H1)-(H6) and moreover assume that there is an index $j_1 \in \{1, \dots, k\}$ such that* $\operatorname{meas}\{s \in I : q_{j_1}(s) \neq 0\} > 0$ *and for all $u \in (0, K]$ and $v \in (0, u]$ it holds*

$$f_{j_1}(u,v) < L_{j_1} f_0(u,v). \tag{3.11}$$

If $b(S,M) < \lambda \leq B(K)$, then there is a positive, nondecreasing and concave solution x of the boundary-value problem (1.1)-(1.3) such that $K < \|x\| < M$.

Proof. As in Theorem 3.1 we can show that $x \in \mathbb{K}$ and $\|x\| = M$ imply $\|Ax\| > M$. It remains to show that if $x \in \mathbb{K}$ and $\|x\| = K$, then $\|Ax\| < K$. To do this we obtain (3.10) and we will show that equality cannot hold. Indeed if it is so, then taking into account (3.9) we conclude that for all $j = 1, \dots, k$ it holds

$$q_j(s)f_j(x(s), x(h_j(s))) = q_j(s)L_j f_0(x(s), x(h_j(s))),$$

where all these quantities are nonnegative. But for $j = j_1$, (3.11) cannot be true, thus in (3.10) we have the strict inequality, a contradiction. $\square$

4. Uniqueness and continuous dependence Results

Here we give results on the uniqueness and the continuous dependence of the solutions on the eigenvalues. For this we make the following condition.

(H7) For every $j = 0, \dots, k$ there exist real nonnegative constants ρ_j, σ_j such that

$$|f_j(u, v) - f_j(u', v')| \le \rho_j|u - v| + \sigma_j|u' - v'|$$

for all $u, v, u', v' \in [0, +\infty)$.

Theorem 4.1. *Assume that $p, g, f_j, q_j, h_j, \rho_j, \sigma_j$, $(j = 0, \dots, k)$ satisfy (H1)-(H7) and*

$$c := B(K)\sum_{j=0}^{k}(\rho_j + \sigma_j)\int_0^1 q_j(s)R(s)ds < 1. \tag{4.1}$$

Then for every $\lambda \in (b(S, M), B(K))$ there exists exactly one positive (nondecreasing and concave) solution x_λ of the boundary value problem (1.1) − (1.3)*. Also the function $\lambda \to x_\lambda$ is uniformly continuous.*

Proof. It is clear that conditions (H7) and (4.1) imply that the operator $T = \lambda A$ is a contraction. Hence, by the Contraction Principle, for every $\lambda \in (b(S, M), B(K))$ the solution x_λ, say, obtained by Theorem 3.1 is unique.

Now, consider the solutions $x_{\lambda_1}, x_{\lambda_2}$, where $\lambda_1, \lambda_2 \in (b(S, M), B(K))$. Then, for any $t \in I$ we have

$$\begin{aligned}|x_{\lambda_1}(t) - x_{\lambda_2}(t)| &\le |\lambda_1 - \lambda_2||Ax_{\lambda_1}(t)| + B(K)|Ax_{\lambda_1}(t)| - Ax_{\lambda_2}(t)| \\ &\le \frac{|\lambda_1 - \lambda_2|}{b(S, M)}\|x_{\lambda_1}\| + c\|x_{\lambda_1} - x_{\lambda_2}\|.\end{aligned}$$

Therefore,

$$\|x_{\lambda_1} - x_{\lambda_2}\| \le \frac{1}{b(S, M)(1 - c)}|\lambda_1 - \lambda_2|$$

which completes our proof. $\square$

5. Some Applications

(a) Consider the equation

$$x''(t) + \lambda\Big[x^\mu(t)x^\nu(h(t)) + \theta\big|sin[x(t)x(h(t))]\big|x^{\mu-1}(t)x^{\nu-1}(h(t))\Big] = 0, \quad t \in I, \tag{5.1}$$

associated with the conditions (1.2) and (1.3). The function h is any retardation, $\theta \ge 0$ and $\mu, \nu > 0$ with $\mu + \nu > 1$. Also consider the constants ξ and ζ as in Theorem 3.1. Here we have

$$\begin{aligned}f_0(u, v) = \phi(u, v) = \Phi(u, v) &:= u^\mu v^\nu, \\ f_1(u, v) &:= |sin(uv)|u^{\mu-1}v^{\nu-1}.\end{aligned}$$

Take any constants ϵ, Θ such that $0 < \epsilon\zeta < \Theta\xi < +\infty$ and consider constants $K, S(> 0)$ so that

$$K^{\mu+\nu-1} < (\Theta\xi)^{-1} < (\epsilon\zeta)^{-1} < S^{\mu+\nu-1}.$$

Fix any $M > S$. Then observe that $L_1 = 1$, as well as

$$b(S,M) = \frac{1}{\zeta}\sup_{u\in[S,M]}\frac{u}{\Phi(u,u)} = \frac{1}{\zeta}\sup_{u\in[S,M]}\frac{1}{u^{\mu+\nu-1}} = \frac{1}{\zeta}\frac{1}{S^{\mu+\nu-1}} < \epsilon$$

and

$$B(K) = \frac{1}{\xi}\frac{1}{K^{\mu+\nu-1}} > \Theta.$$

Since ϵ, Θ are arbitrary, we have the following statement.

Corollary 5.1. *Assume that $g, h : I \to \mathbb{R}$ are nondecreasing functions (with h not identically zero) and such that $g(1) - g(0) < 1$ and $0 \le h(t) \le t$, $t \in I$. Then for every $\lambda > 0$ the boundary value problem (5.1),(1.2),(1.3) admits at least one positive, nondecreasing and concave solution x such that $K < \|x\| < M$.*

(b) Consider the retarded differential equation

$$x''(t) + \lambda[x^{m+1}(t) + \rho x^{n+1}(t^2)] = 0, \quad t \in [0,1], \tag{5.2}$$

with initial condition (1.2), i.e. $x(0) = 0$ and the boundary condition

$$x'(1) = \delta x(1) \tag{5.3}$$

where $m, n, \lambda, \rho, \delta$ are real positive numbers with $\delta < 1$. To apply Theorem 3.1 we write this problem in the form (1.1)-(1.3) by setting

$$p(t) := 1, \quad q_0(t) := 1, \quad g(t) := \delta t, \quad h_0(t) := t^2, \quad t \in I,$$

and

$$f_0(u,v) := \phi(u,v) = \Phi(u,v) = u^{m+1} + \rho v^{n+1}, \quad f_j(u,v) := 0, \quad (j = 1,\dots,k).$$

Then we obtain

$$\gamma = \frac{1}{1-\delta} \quad \text{and} \quad R(t) = \frac{t}{1-\delta}, \quad t \in I.$$

Choose $S := 1$, $M := 2$ and observe that $E(S,M) = (2^{-1/2}, 1]$. Thus, we have

$$\zeta = \sup\left\{\frac{1}{2(1-\delta)}\eta^2(1-\eta^2) : \quad \eta \in (2^{-1/2}, 1]\right\} = \frac{1}{8(1-\delta)},$$

$$\xi = \frac{1}{2(1-\delta)}, \qquad \sup_{u\in[1,2]}\frac{u}{\phi(u,u)} = \frac{1}{1+\rho} \quad \frac{K}{\phi(K,K)} = \frac{1}{K^{m+1} + \rho K^{n+1}}.$$

Therefore, Theorem 3.2 applies and the following result follows.

Corollary 5.2. *Assume that $m, n, \lambda, \rho, \delta$ are real positive numbers with $\delta < 1$. Then for any $K > 0$, with*

$$K^{m+1} + \rho K^{n+1} < \frac{1}{4}(1+\rho)$$

and any λ such that

$$\frac{8(1-\delta)}{1+\rho} \le \lambda < \frac{2(1-\delta)}{K^{m+1} + \rho K^{n+1}},$$

the boundary value problem (5.2), (1.2), (5.3) admits at least one solution x which is a positive, nondecreasing and concave function such that $K < \|x\| < 2$.

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G. L. Karakostas

Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

E-mail address: `gkarako@cc.uoi.gr`

P. Ch. Tsamatos

Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

E-mail address: `ptsamato@cc.uoi.gr`