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Existence of solutions for discontinuous functional equations and elliptic boundary-value problems *

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Abstract

We prove existence results for discontinuous functional equations in general L^p-spaces and apply these results to the solvability of implicit and explicit elliptic boundary-value problems involving discontinuous nonlinearities. The main tool in the proof is a fixed point result in lattice-ordered Banach spaces proved by the second author.

1 Introduction

In this paper we shall first prove existence results for the functional equations

$$h(x) = f(x, \phi(h(x)), h(x)) \quad \text{and} \quad h(x) = g(x, \phi(h(x)))$$

in the space $L^p(\Omega)$, $1 \leq p < \infty$, where Ω is a measure space, $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\phi : L^p(\Omega) \to L^p(\Omega)$. The proofs are based on a fixed point result in [4], which is derived by applying a recursion principle introduced in [5].

Then the existence results are applied to study the existence of weak solutions to boundary-value problems of elliptic differential equations of the form

$$\Lambda u(x) = f(x, u(x), \Lambda u(x)) \quad \text{and} \quad \Lambda u(x) = g(x, u(x)),$$

where

$$\Lambda u(x) := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\Big) + q(x, u(x)).$$

The functions f, g, and the mapping ϕ may be discontinuous in their arguments. Concrete and worked examples are provided to demonstrate the applicability of the results obtained.

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2 Existence results for functional equations

In this section we assume that $\Omega = (\Omega, \mathcal{A}, \mu)$ is a measure space, and that the space $L^p(\Omega)$, $1 \le p < \infty$, is ordered a.e. pointwise.

In the proof of our existence theorem for the functional equation

$$h(x) = f(x, \phi(h(x)), h(x)) \quad \text{a.e. in } \Omega, \tag{2.1}$$

we make use of the following fixed point result.

Lemma 2.1 *Assume that a mapping $G : L^p(\Omega) \to L^p(\Omega)$ is increasing, and that $\|Gh\|_p \le M + \psi(\|h\|_p)$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, and $M + \psi(R) \le R$ for some $R > 0$. Then G has a fixed point.*

Proof Choose an $R > 0$ such that $M + \psi(R) \le R$. Because ψ is increasing, then G maps the set $P = \{h \in L^p(\Omega) \mid \|h\|_p \le R\}$ into itself. Thus G has by [4, Corollary 5] a fixed point in P. $\diamondsuit$

For the functions $\phi : L^p(\Omega) \to L^p(\Omega)$ and $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ we have the following hypotheses:

(ϕ) ϕ is increasing, and $\|\phi \circ h\|_p \le m + \kappa\|h\|_p$ for some $m \ge 0$ and $\kappa > 0$.

(f1) f is sup-measurable, i.e., $x \mapsto f(x, u(x), v(x))$ is measurable in Ω whenever u, $v : \Omega \to \mathbb{R}$ are measurable.

(f2) $|f(x, y, z)| \le k(x) + c_1(x)|y|^\alpha + c_2(x)|z|^\beta$ for a.e. $x \in \Omega$ and for all y, $z \in \mathbb{R}$, where $k \in L^p(\Omega)$, and either

- (i) $0 < \alpha$, $\beta < 1$, $c_1 \in L^{\frac{p}{1-\alpha}}(\Omega)$, $c_2 \in L^{\frac{p}{1-\beta}}(\Omega)$, and $f(x, \cdot, \cdot)$ is increasing for a.e. $x \in \Omega$, or
- (ii) $\alpha = \beta = 1$, $\kappa\|c_1\|_\infty + \|c_2\|_\infty < 1$, where κ is the constant in (ϕ), and the function $(y, z) \mapsto f(x, y, z) + \lambda z$ is increasing for a.e. $x \in \Omega$ and for some $\lambda \ge 0$.

Our existence result for the functional equation (2.1) reads as follows.

Theorem 2.2 *Under the assumptions (ϕ), (f1), and (f2), Equation* (2.1) *has a solution h in $L^p(\Omega)$.*

Proof The hypotheses (ϕ) and (f1) imply that for each $h \in L^p(\Omega)$ the relation

$$Gh := f(\cdot, \phi(h(\cdot)), h(\cdot)) \tag{2.2}$$

defines a measurable function $Gh : \Omega \to \mathbb{R}$. To show that (2.2) defines a mapping $G : L^p(\Omega) \to L^p(\Omega)$ we have to prove that $Gh \in L^p(\Omega)$. Applying the growth condition of (f2), the hypothesis (ϕ) and the Hölder inequality we obtain

$$\begin{aligned}
\|Gh\|_p &= \|f(\cdot,\phi(h(\cdot)),h(\cdot))\|_p \le \|k\|_p + \|c_1(\cdot)|\phi(h(\cdot))|^\alpha\|_p + \|c_2(\cdot)|h(\cdot)|^\beta\|_p \\
&\le \|k\|_p + (\|c_1(\cdot)^p\|_{\frac{1}{1-\alpha}}\| \,|\phi(h(\cdot)|^{p\alpha}\|_{\frac{1}{\alpha}})^{1/p} + (\|c_2(\cdot)^p\|_{\frac{1}{1-\beta}}\| \,|h(\cdot)|^{p\beta}\|_{\frac{1}{\beta}})^{1/p} \\
&= \|k\|_p + \|c_1\|_{\frac{p}{1-\alpha}}\|\phi\circ h\|_p^\alpha + \|c_2\|_{\frac{p}{1-\beta}}\|h\|_p^\beta \\
&\le \|k\|_p + \|c_1\|_{\frac{p}{1-\alpha}}(m+\kappa\|h\|_p)^\alpha + \|c_2\|_{\frac{p}{1-\beta}}\|h\|_p^\beta\,.
\end{aligned}$$

Thus $Gh \in L^p(\Omega)$, and

$$\|Gh\|_p \le M + \psi(\|h\|_p), \tag{2.3}$$

where $M = \|k\|_p$ and $\psi(r) := \|c_1\|_{\frac{p}{1-\alpha}}(m+\kappa r)^\alpha + \|c_2\|_{\frac{p}{1-\beta}} r^\beta$.

a) Assume first that the hypotheses (f2) (i) hold. If $h_1, h_2 \in L^p(\Omega)$, $h_1 \le h_2$, then $\phi(h_1) \le \phi(h_2)$ by (ϕ). Since $f(x,\cdot,\cdot)$ is increasing, then for a.e. $x \in \Omega$,

$$Gh_1(x) = f(x,\phi(h_1(x)),h_1(x)) \le f(x,\phi(h_2(x)),h_2(x)) = Gh_2(x)\,.$$

This proves that G is increasing. Since $0 < \alpha, \beta < 1$, then the mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined in (2.3) is increasing, and $r - \psi(r) \to \infty$ as $r \to \infty$. Thus $M + \psi(R) \le R$ when R is large enough, whence G has a fixed point by Lemma 2.1.

b) Assume next that the hypothesis (f2) (ii) holds with $\lambda = 0$. Then $f(x,\cdot,\cdot)$ is increasing, whence G is increasing by the above proof. Since $\alpha = \beta = 1$, then ψ given by (2.3) is of the form $\psi(r) = \|c_1\|_\infty(m+\kappa r) + \|c_2\|_\infty r$. If $\kappa\|c_1\|_\infty + \|c_2\|_\infty < 1$, then $M + \psi(R) \le R$ when R is sufficiently large. Thus G has a fixed point by Lemma 2.1

The above proof shows that in the cases a) and b) G has a fixed point $h \in L^p(\Omega)$. This implies by (2.2) that $h(x) = Gh(x) = f(x,\phi(h(x)),h(x))$ a.e. in Ω.

c) Assume finally that the hypotheses (f2) (ii) hold with $\lambda > 0$. Then a function $\tilde f : \Omega \times \mathbb{R} \times \mathbb{R}$, defined by

$$\tilde f(x,y,z) = \frac{f(x,y,z) + \lambda z}{1+\lambda}, \quad x \in \Omega,\ y,\ z \in \mathbb{R}, \tag{2.4}$$

is sup-measurable, $\tilde f(x,\cdot,\cdot)$ is increasing, and

$$|\tilde f(x,y,z)| \le \|\tilde k\|_2 + \tilde c_1(x)|y| + \tilde c_2(x)|z|,$$

where $\tilde k_2 = \frac{k_2}{1+\lambda}$, $\tilde c_1 = \frac{c_1}{1+\lambda}$, $\tilde c_2 = \frac{c_2+\lambda}{1+\lambda}$. Since $\kappa\|c_1\|_\infty + \|c_2\|_\infty < 1$, then

$$\kappa\|\tilde c_1\|_\infty + \|\tilde c_2\|_\infty = \frac{\kappa\|c_1\|_\infty + \|c_2\|_\infty + \lambda}{1+\lambda} < \frac{1+\lambda}{1+\lambda} = 1$$

Thus $\tilde f$ satisfies the hypotheses (f1) and (f2) (ii) with $\lambda = 0$. The proof of the case b) above implies an existence of a $h \in L^p(\Omega)$ such that $h(x) = \tilde f(x,\phi(h(x)),h(x))$, or equivalently, by (2.4), $h(x) = f(x,\phi(h(x)),h(x))$ a.e. in Ω. This concludes the proof. $\diamondsuit$

As a consequence of Theorem 2.2 we obtain an existence result for the equation

$$h(x) = g(x, \phi(h(x))) \quad \text{a.e. in } \Omega. \tag{2.5}$$

For the next proposition we assume the following hypotheses:

(g1) g is sup-measurable, and $g(x, \cdot)$ is increasing for a.e. $x \in \Omega$.

(g2) $|g(x, y)| \le k(x) + c_1(x)|y|^\alpha$ for a.e. $x \in \Omega$ and for all $y \in \mathbb{R}$, where $k \in L^p(\Omega)$, and either $0 < \alpha < 1$ and $c_1 \in L^{\frac{p}{1-\alpha}}(\Omega)$, or $\alpha = 1$ and $\kappa \|c_1\|_\infty < 1$.

Proposition 2.3 *Assume that $\phi : L^p(\Omega) \to L^p(\Omega)$ satisfies the hypothesis (ϕ) and that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (g1) and (g2). Then* (2.5) *has a solution h in $L^p(\Omega)$.*

Remark 2.4 The hypotheses of Theorem 2.2 and Proposition 2.2 allow the functions f and g to be discontinuous in all their arguments. Even the mapping ϕ may be discontinuous.

3 Applications to elliptic boundary-value problems

Let $\Omega \subset \mathbb{R}^N$, $N \ge 3$, be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this section we study the existence of weak solutions of the implicit elliptic BVP

$$\begin{gathered} \Lambda u(x) = f(x, u(x), \Lambda u(x)) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where

$$\Lambda u(x) := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\Big) + q(x, u(x)).$$

Theorem 2.2 will be the main tool in our investigations. We assume that the coefficients $a_{ij} \in L^\infty(\Omega)$ satisfy the ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge \gamma \sum_{i=1}^{N} \xi_i^2 \tag{3.2}$$

for a.e. $x \in \Omega$, all $\xi_1, \dots, \xi_N \in \mathbb{R}$, and some $\gamma > 0$.

Let $W_0^{1,2}(\Omega)$ denote the usual Sobolev space of square integrable functions having generalized homogeneous boundary values, and denote its dual space by $W^{-1,2}(\Omega)$. We are going to introduce conditions which ensure that (3.1) has a weak solution in the following sense.

Definition A function $u \in W_0^{1,2}(\Omega)$ is called a *weak solution* of the BVP (3.1) if there exists a function $h \in L^2(\Omega)$ such that

$$h(x) = f(x, u(x), h(x)) \quad \text{for a.e. } x \in \Omega, \tag{3.3}$$

and u is a weak solution of the semilinear BVP

$$\begin{aligned} \Lambda u(x) &= h(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

We shall first prove an existence, uniqueness and comparison result for the BVP (3.4) assuming that $W_0^{1,2}(\Omega)$ and $L^2(\Omega)$ are equipped with the natural partial ordering of functions defined by the order cone $L_+^2(\Omega)$ of all nonnegative functions of $L^2(\Omega)$. On the function q we assume the following hypotheses:

(q1) q is a Carathéodory function and $q(x, \cdot)$ is increasing for a.e. $x \in \Omega$.

(q2) $|q(x, y)| \le k_0(x) + c_0(x)\,|s|^{p_0-1}$ for a.e. $x \in \Omega$ and for all $x \in \mathbb{R}$, where $k_0 \in L^{\frac{p_0}{p_0-1}}(\Omega)$, $c_0 \in L_+^\infty(\Omega)$ and $1 < p_0 \le 2^* := \frac{2N}{N-2}$ (critical exponent).

Lemma 3.1 *Assume that $q : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (q1) and (q2). Then (3.4) has a unique weak solution u for each $h \in L^2(\Omega)$. Moreover, u is increasing with respect to h and there exist constants $m \ge 0$ and $\kappa > 0$ such that*

$$\|u\|_{1,2} \le m + \kappa \|h\|_2. \tag{3.5}$$

Proof It is well-known that

$$a(u, v) := \int_\Omega \sum_{i,j=1}^N a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j}, \quad u,\, v \in W_0^{1,2}(\Omega) \tag{3.6}$$

defines a bounded bilinear form $a : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$. The assumptions (q1) and (q2) imply that the mapping $A : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ given by

$$\langle Au, v \rangle := a(u, v) + \int_\Omega q(x, u(x))\, v(x)\, dx, \quad u,\, v \in W_0^{1,2}(\Omega), \tag{3.7}$$

is well-defined, continuous and strongly monotone, and hence bijective by [7, Theorem 26.A].

To each $h \in L^2(\Omega)$ there corresponds a unique functional $\tilde{h} \in W^{-1,2}(\Omega)$ given by

$$\langle \tilde{h}, v \rangle = \int_\Omega h(x) v(x)\, dx, \quad v \in W_0^{1,2}(\Omega). \tag{3.8}$$

Denoting $u = A^{-1}\tilde{h}$, we then have $Au = \tilde{h}$, which by (3.7) and (3.8) is equivalent to

$$a(u, v) + \int_\Omega q(x, u(x))\, v(x)\, dx = \int_\Omega h(x) v(x)\, dx, \; v \in W_0^{1,2}(\Omega). \tag{3.9}$$

Thus u is, by definition, a weak solution of (3.4). To prove that u is increasing with respect to h, let $h_1,\ h_2 \in L^2(\Omega)$ satisfy $h_1 \le h_2$. Denoting by u_i the weak solutions of (3.4) with $h = h_i$, $i = 1, 2$, it follows from (3.9) that

$$\begin{aligned} a(u_1 - u_2, v) &+ \int_\Omega (q(x, u_1(x)) - q(x, u_2(x)))v(x)\,dx \\ =& \int_\Omega (h_1(x) - h_2(x))v(x)\,dz \le 0 \end{aligned} \tag{3.10}$$

for all $v \in (W_0^{1,2}(\Omega))_+ := W_0^{1,2}(\Omega) \cap L^2_+(\Omega)$. Choosing in (3.10) $v = (u_1 - u_2)^+$, and noticing that due to the monotonicity of $q(x,\cdot)$ the inequality

$$\int_\Omega (q(x, u_1(x)) - q(x, u_2(x)))(u_1 - u_2)^+(x)\,dx \ge 0$$

holds, and that a is coercive and $a((u_1 - u_2)^-, (u_1 - u_2)^+) = 0$, we obtain from (3.10)

$$c\|(u_1 - u_2)^+\|_{1,2}^2 \le a((u_1 - u_2)^+, (u_1 - u_2)^+) = a(u_1 - u_2, (u_1 - u_2)^+) \le 0.$$

This result implies that $(u_1 - u_2)^+ = 0$, i.e. $u_1 \le u_2$, and hence proves that the weak solution u of (3.4) is increasing with respect to h.

To prove estimate (3.5), let $h \in L^2(\Omega)$ be given, and let $u \in W_0^{1,2}(\Omega)$ be the weak solution of (3.4). The monotonicity of $q(x,\cdot)$ along with the continuous embedding $W_0^{1,2}(\Omega) \subset L^{p_0}(\Omega)$ and the coercivity of a yield the following estimate:

$$\begin{aligned} c\|u\|_{1,2}^2 &\le a(u,u) \le a(u,u) + \int_\Omega (q(x,u(x)) - q(x,0))u(x)\,dx \\ &= \int_\Omega h(x)u(x)\,dx - \int_\Omega q(x,0)u(x)\,dx \\ &\le \|h\|_2\|u\|_2 + \|k_0\|_{\frac{p_0}{p_0-1}}\|u\|_{p_0} \le (b\,\|k_0\|_{\frac{p_0}{p_0-1}} + \|h\|_2)\|u\|_{1,2}, \end{aligned}$$

for some positive constant b. Thus (3.5) holds with $m = \frac{b}{c}\|k_0\|_{\frac{p_0}{p_0-1}}$ and $\kappa = 1/c$. $\diamondsuit$

As an application of Theorem 2.2 and Lemma 3.1, we shall prove the following existence result for (3.1).

Theorem 3.2 *Assume that $q : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses (q1) and (q2), and that $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses (f1) and (f2) with $p = 2$. Then the BVP* (3.1) *possesses a weak solution.*

Proof If follows from Lemma 3.1 that the mapping $\phi : L^2(\Omega) \to L^2(\Omega)$, which assigns to each $h \in L^2(\Omega)$ the weak solution $u := \phi(h) \in W_0^{1,2}(\Omega) \subset L^2(\Omega)$ of the BVP (3.4), is increasing. Moreover, the inequality (3.5) holds, whence

$$\|\phi \circ h\|_2 = \|u\|_2 \le \|u\|_{1,2} \le m + \kappa\|h\|_2, \quad h \in L^2(\Omega).$$

This proves that ϕ satisfies the hypothesis (ϕ). Thus the hypotheses of Theorem 2.2 hold when $p = 2$, whence there exists a function $h \in L^2(\Omega)$ such that

$$h(x) = f(x, \phi(h(x)), h(x)) = f(x, u(x), h(x)) \text{ a.e. in } \Omega,$$

and u is the weak solution of (3.4). This implies by Definition 3.1 that u is a weak solution of (3.1). ♢

As a consequence of Theorem 3.2, we obtain an existence result for the (explicit) BVP

$$\begin{gathered} \Lambda u(x) = g(x, u(x)) \quad \text{ a.e. in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.11}$$

where

$$\Lambda u(x) := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\Big) + q(x, u(x)).$$

Proposition 3.3 *Assume that $q : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses (q1) and (q2), and that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses (g2) and (g2) with $p = 2$. Then the BVP* (3.11) *has a weak solution.*

Remark 3.4 (i) The hypotheses of Theorem 3.2 and Proposition 3.3 allow both functions f and g to be discontinuous in all their arguments.
(ii) Theorem 3.2 and Proposition 3.3 also apply to problems in domains Ω of dimensions $N = 1$ and $N = 2$, since in these cases the critical exponent $2^* = \infty$ and Lemma 3.1 is valid with an exponent p_0 satisfying $1 < p_0 < \infty$.
(iii) If the coefficients a_{ij} are uniformly Lipschitz continuous, it follows by the regularity result [3, Theorem 8.8] that the weak solutions of problems (3.1) and (3.11) satisfy their differential equation a.e. pointwise. This holds, in particular, when $a_{ij} = \delta_{ij}$ which is the case in the following examples, where $[z]$ denotes the greatest integer $\leq z \in \mathbb{R}$.

Example 3.5 Assume that $\mathbb{R}^4$ is equipped with the Euclidean norm $|\cdot|$. Choose $\Omega = \{x \in \mathbb{R}^4 : \frac{1}{2} < |x| < 1\}$, and consider the BVP

$$\begin{gathered} \Lambda u(x) = 5 + [6|x|] + 7[10^9 u(x)]^{\frac{1}{3}} + 8[10^{10}\Lambda u(x)]^{\frac{1}{5}}, \quad \text{a.e. in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.12}$$

where $\Lambda u(x) := -\Delta u(x) + u(x)^3$, for $x \in \Omega$. The BVP (3.12) is of the form (3.1), where

$$a_{ij}(x) \equiv \delta_{ij}, \; q(x, y) = y^3 \; \text{ and } \; f(x, y, z) = 5 + [6|x|] + 7[10^9 y]^{1/3} + 8[10^{10} z]^{1/5}.$$

The critical exponent here is $2^* = 4$ and it is easy to see that the hypotheses (q), (f1) and (f2) with $p = 2$ hold, whence the BVP (3.12) has by Theorem 3.2 a weak solution.

Example 3.6 For $\Omega = (0,1)$, consider the boundary-value problem

$$\begin{gathered} -u''(x) = 2 + 2[2-2x] + 2\,[(2u(x)-2x)^{\frac{1}{3}}] + [(-u''(x)-1)^{\frac{1}{3}}] \quad \text{a.e. in } (0,1) \\ u(0) = u(1) = 0\,. \end{gathered} \tag{3.13}$$

Problem (3.13) is of the form (3.1), with

$$q(x,y) \equiv -1, \quad \text{and} \quad f(x,y,z) = 1 + 2[2-2x] + 2\,[(2y-2x)^{\frac{1}{3}}] + [(z-1)^{\frac{1}{3}}]. \tag{3.14}$$

By elementary calculations one can show that for each $h \in L^2(\Omega)$ the function

$$\begin{aligned} u(x) = \phi(h(x)) &= (1-x)\int_0^x t(1+h(t))dt + x\int_x^1 (1-t)(1+h(t))dt \\ &= \frac{x-x^2}{2} + (1-x)\int_0^x th(t)dt + x\int_x^1 (1-t)h(t)dt, \quad x \in [0,1] \end{aligned} \tag{3.15}$$

is a unique solution of the BVP

$$\begin{gathered} \Lambda u(x) := -u''(x) - 1 = h(x) \quad \text{in} \quad (0,1), \\ u(0) = u(1) = 0 \end{gathered}$$

in $W_0^{1,2}(0,1)$, and that

$$\|u\| = \|\phi \circ h\|_2 \le \frac{1}{2\sqrt{30}}(1 + 2\|h\|_2).$$

Thus the hypothesis (ϕ) holds. Obviously, f is sup-measurable, i.e., (f1) is fulfilled. Since

$$|f(x,y,z)| \le 15 + 4|y|^{1/3} + |z|^{1/3},$$

then the hypothesis (f2) (i) is satisfied. It then follows from Theorem 3.1 that the BVP (3.13) has a solution.

Example 3.7 For $\Omega = (0,1)$, consider the boundary-value problem

$$\begin{gathered} -u''(x) = 2 + 2\,[u(x) - 2x + 1] + \frac{[-u''(x) - 2]}{2} \quad \text{a.e. in } (0,1), \\ u(0) = u(1) = 0. \end{gathered} \tag{3.16}$$

Problem (3.16) is of the form (3.1), where

$$q(x,y) \equiv -2, \quad \text{and} \quad f(x,y,z) = 2\,[y - 2x + 1] + \frac{[z-2]}{2}. \tag{3.17}$$

For each $h \in L^2(\Omega)$ the function

$$\begin{aligned} u(x) = \phi(h(x)) &= (1-x)\int_0^x t(2+h(t))dt + x\int_x^1 (1-t)(2+h(t))dt \\ &= x - x^2 + (1-x)\int_0^x th(t)dt + x\int_x^1 (1-t)h(t)dt, \quad x \in [0,1] \end{aligned} \tag{3.18}$$

is a unique solution of the BVP

$$\Lambda u(x) := -u''(x) - 2 = h(x) \quad \text{in } (0,1)$$
$$u(0) = u(1) = 0$$

in $W_0^{1,2}(0,1)$. Moreover,

$$\|\phi \circ h\|_2 \le \frac{1}{\sqrt{30}}(1 + \|h\|_2).$$

Thus the hypothesis (ϕ) holds with $m = \kappa = 1/\sqrt{30}$. Obviously, f is sup-measurable, i.e. the hypothesis (f1) is satisfied. Since

$$|f(x,y,z)| \le 6 + 4|x| + 2|y| + \frac{1}{2}|z|,$$

and since $(2/\sqrt{30}) + (1/2) < 1$, then also the hypothesis (f2) (ii) is fulfilled. Thus it follows from Theorem 3.2 that the BVP (3.16) possesses a solution.

Remark 3.8 (i) Based on the method of proof of the abstract fixed point result obtained in [4, Corollary 5] an algorithm has been developed to calculate approximations for Examples 3.6 and 3.7, which can be used to infer the exact solutions. Computational results will be given in a forthcoming paper.
(ii) We have restricted to homogeneous boundary value problems only for the sake of simplicity. Nonhomogeneous Dirichlet boundary conditions as well as Neumann or Robin type boundary conditions involving even discontinuous nonlinearities can be treated.
(iii) As for existence results for discontinuous explicit and implicit elliptic BVP's different from those presented in this paper, see e.g., [1, 2, 5, 6] and the references therein. The existence of extremal solutions is considered in [1, 2, 5].

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