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EXPANSIONS OF SOLUTIONS OF HIGHER ORDER EVOLUTION EQUATIONS IN SERIES OF GENERALIZED HEAT POLYNOMIALS

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ABSTRACT. Upper bound estimates are established on generalized heat polynomials for higher order linear homogeneous evolution equations with coefficients depending on the time variable. These estimates are analogous to well known bounds of Rosenbloom and Widder on the heat polynomials. The bounds lead to further estimates on the width of the strip of convergence of series expansions in terms of these polynomial solutions. An application is given to a Cauchy problem, wherein the solution is expressed as the sum of a series of polynomial solutions.

1. INTRODUCTION

We consider a linear differential operator $\mathcal{L}$, represented by

$$\mathcal{L}u\,(x,t) = \partial_t u\,(x,t) - \sum_\alpha a_\alpha(t)\partial_x^\alpha u\,(x,t)\ . \tag{1.1}$$

Here $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and the coefficients $\{a_\alpha\}$, of which there are only a finite number, are real valued continuous functions of t on an interval $\mathbb{I}$ containing the origin. In an earlier paper [21] we developed explicit formulas for real valued solutions $\{p_\beta\}$ of the initial value problems

$$\mathcal{L}p_\beta(x,t) = 0\,, \quad p_\beta(x,0) = x^\beta\,. \tag{1.2}$$

Each p_β is for fixed t a polynomial in x of degree $|\beta|$, of the form

$$p_\beta(x,t) = \sum_{\nu\le\beta} c_\nu(t)x^\nu\,,$$

where each coefficient c_ν is a real valued function in $C^1(\mathbb{I})$. Moreover, when the coefficients $\{a_\alpha\}$ are constant and $\mathcal{L}$ has no zero order term, p_β is a polynomial in both x and t. The polynomials $\{p_\beta\}$ are direct analogs of the classical *heat polynomials*, appearing in Appell's work [1], and studied in depth by Rosenbloom and Widder [32, 35, 36, 37]. Indeed, when $\mathcal{L}$ is the heat operator $\mathcal{H} = \partial_t - \Delta$, these polynomials become the heat polynomials.

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As Rosenbloom and Widder demonstrated, heat polynomials serve as a basis for expansion of other solutions of the heat equation. In this paper we study series expansions of the form

$$u\,(x,t) = \sum_{\beta} \frac{c_\beta}{\beta!} p_\beta(x,t)\,, \tag{1.3}$$

investigating convergence questions in a horizontal strip $\{(x,t) : |t| < s\}$. We discuss also the Cauchy problem in such a strip,

$$\mathcal{L}u(x,t) = 0\,, \quad u\,(x,0) = f(x)\,, \tag{1.4}$$

and give conditions on f ensuring that a solution can be found as a series (1.3).

At the expense of some notational complexity, we have endeavored to make our operator $\mathcal{L}$ of (1.1) as general as possible, and we have presupposed an arbitrary number n of space dimensions. Allowing the coefficients of $\mathcal{L}$ to depend on t is significant, in that it allows for more applications to evolution operators in the physical sciences, where the convenience of constant coefficients is perhaps somewhat rare. Likewise, allowing more than only one space dimension permits a much wider range of applications. Although space dimensions higher than three do not appear in physical situations, we allow an arbitrary space dimension n because it permits treatment of the cases $n = 1, 2, 3$ all at once, and there is little additional burden in viewing n as an integer possibly larger than 3. We point out that allowing dependence of the coefficients of $\mathcal{L}$ on the space variable x would be unrealistic, as in that case polynomial solutions of the equation $\mathcal{L}u = 0$ would likely be scarce.

The main purpose of this paper is the derivation of pointwise upper bounds on the polynomials $\{p_\beta\}$, as such bounds are necessary for meaningful applications. For example, in [37] Widder showed, using bounds on the heat polynomials, that a temperature function can be expanded in a series of heat polynomials in the widest strip where it enjoys the Huygens property. Colton in [8] used these same bounds to demonstrate that the heat polynomials form a complete family of solutions of the heat equation in certain domains. In [20], Hile and Mawata used heat polynomial bounds to help describe the behaviour of the heat operator as a mapping between weighted Sobolev spaces in $\mathbb{R}^{n+1}$. We expect that perhaps the bounds we establish here will be crucial for similar applications involving more general evolution operators than the heat operator.

Rosenbloom and Widder established their bounds for the heat polynomials $\{v_\beta\}$ with use of the Poisson representation

$$v_\beta\,(x,t) = \int_{\mathbb{R}^n} k\,(x-y,t)\;\, y^\beta \,dy\,,$$

where $k\,(x,t)$ is the heat kernel. However, for our more general operators (1.1) – which we do not assume are even parabolic – we have no analogous representation. Instead we borrow liberally from techniques of H. Kemnitz [26], who studied polynomial expansions of solutions of the equation

$$\frac{\partial u\,(x,t)}{\partial t} = \frac{\partial^r u\,(x,t)}{\partial x^r}\,,$$

where r is an integer larger than 1. For this equation Kemnitz presented analogs of the heat polynomials,

$$v_{r,n}\left(x,t\right)=n!\sum_{k+r\ell=n}\frac{x^{k}t^{\ell}}{k!\ell!}=n!\sum_{\ell=0}^{[n/r]}\frac{x^{n-r\ell}t^{\ell}}{(n-r\ell)!\ell!}\,,$$

and bounded these polynomials directly with use of estimates involving factorials of integers. For our more general polynomials we strengthen and extend Kemnitz's estimates to factorials of general nonnegative numbers, and introduce a few additional factorial estimates of our own.

After deriving the polynomial estimates, we follow arguments of Kemnitz in analyzing series expansions in these polynomials. We give conditions on the coefficients $\{c_\beta\}$ in (1.3) that guarantee a minimum width s of a strip of convergence of the series.

Again following the example of Kemnitz [26], we solve the Cauchy problem (1.4) under the assumption that f has a power series expansion

$$f\left(x\right)=\sum_{\beta}\frac{c_\beta}{\beta!}x^{\beta}$$

valid in all of $\mathbb{R}^n$. We must specify conditions on the coefficients $\{c_\beta\}$ that guarantee convergence of the sum (1.3) to a solution u in a horizontal strip. These conditions at the same time mandate that f can be extended as an entire function to all of $\mathbb{C}^n$ obeying a certain exponential growth restriction at infinity. Indeed, these investigations lead us to a more general discussion on characterizations of the growth and type of an entire function of several complex variables by conditions on the coefficients in its power series expansion.

For further work regarding generalizations of the heat polynomials, see [4, 7, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 28].

There is an interesting body of work on the problem of determining all polynomial solutions of systems of partial differential equations with constant coefficients, as well as the dimensions of solution spaces of polynomials of specified degree. For important papers and further bibliographical references, see [23, 24, 25, 29, 30, 31, 33].

Another interesting application of the heat polynomials appears in the work of L. R. Bragg and J. W. Dettman concerning transmutation operators. (See [6] for a list of references.) For example, in [5] these authors demonstrate that various polynomial solutions of elliptic and hyperbolic equations can be obtained from the heat polynomials, and that these in turn can be used to represent solutions of problems involving these equations.

2. The Polynomials

We review briefly the description of the polynomial solutions $\{p_\beta\}$ of (1.2), as presented in [21]. First some explanation of the notation is required. Given $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n$ and a multi-index $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ in $\mathbb{R}^n$ we adopt the usual notation $\alpha!=\alpha_1!\alpha_2!\cdots\alpha_n!$, and

$$x^{\alpha}={x_1}^{\alpha_1}{x_2}^{\alpha_2}\cdots{x_n}^{\alpha_n}\,,\qquad \partial_x^{\alpha}=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}\,.$$

For multi-indices α and β of the same dimension, we write $\alpha \leq \beta$ whenever $\alpha_i \leq \beta_i$ for all i. As we assume each a_α appearing in (1.1) is continuous on an interval $\mathbb{I}$ containing the origin, for such a_α we may introduce the antiderivative

$$b_\alpha(t) = \int_0^t a_\alpha(s)\, ds\,,\ t \in \mathbb{I}\,. \tag{2.1}$$

We let K denote the number of indices α appearing in (1.1), and we label these indices as $\alpha^1,\ \alpha^2,\ \cdots,\ \alpha^K$, and the corresponding coefficients as $a_1,\ a_2,\ \cdots,\ a_K$. We further introduce vector functions

$$a = (a_1, a_2, \cdots, a_K)\ , \quad b = (b_1, b_2, \cdots, b_K)\ ,$$

as well as a *vector of multi-indices*

$$\overline{\alpha} = \left(\alpha^1, \alpha^2, \cdots, \alpha^K\right)\ .$$

Finally, given a multi-index $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_K)$ in $\mathbb{R}^K$, the "dot product" $\overline{\alpha} \cdot \sigma$ is the n-dimensional multi-index

$$\overline{\alpha} \cdot \sigma = \alpha^1\sigma_1 + \alpha^2\sigma_2 + \cdots + \alpha^K\sigma_K \quad .$$

At times it is convenient to write this sum also as

$$\overline{\alpha} \cdot \sigma = \sum_\alpha \alpha\sigma_\alpha\,,$$

where $\sigma_\alpha = \sigma_i$ if $\alpha = \alpha^i$. The polynomials $\{p_\beta\}$ are then defined on $\mathbb{R}^n \times \mathbb{I}$ according to

$$p_\beta(x,t) := \beta! \sum_{\gamma,\sigma:\gamma+\overline{\alpha}\cdot\sigma=\beta} \frac{x^\gamma b(t)^\sigma}{\gamma!\sigma!}\,, \tag{2.2}$$

where the summation is over all multi-indices γ in $\mathbb{R}^n$ and σ in $\mathbb{R}^K$ having the property that $\gamma + \overline{\alpha} \cdot \sigma = \beta$. An alternative formulation is

$$p_\beta(x,t) := \beta! \sum_{\overline{\alpha}\cdot\sigma\leq\beta} \frac{x^{\beta-\overline{\alpha}\cdot\sigma} b(t)^\sigma}{(\beta-\overline{\alpha}\cdot\sigma)!\sigma!}\,, \tag{2.3}$$

where now the sum is over all multi-indices σ in $\mathbb{R}^K$ such that $\overline{\alpha} \cdot \sigma \leq \beta$.

If the operator $\mathcal{L}$ of (1.1) has no zero order term then the summations (2.2) and (2.3) extend over only a finite number of terms, and trivially each p_β is as smooth as the individual terms in the sum. Since each b_α is of class C^1 on $\mathbb{I}$, and powers of x are C^∞ on $\mathbb{R}^n$, in this situation each p_β, along with all its partial derivatives involving at most one derivative with respect to t, is continuous in $\mathbb{R}^n \times \mathbb{I}$.

If the operator $\mathcal{L}$ of (1.1) has a zero order term, then the summations (2.2) and (2.3) are over an infinite number of terms, as now $\alpha = (0, 0, \cdots, 0)$ appears as an index in (1.1) with corresponding nonzero a_α, resulting in an infinite number of multi-indices σ in $\mathbb{R}^K$ such that $\overline{\alpha} \cdot \sigma \leq \beta$. However it is shown in [21] that in this event the series (2.2) and (2.3) are uniformly and absolutely convergent on compact subsets of $\mathbb{R}^n \times \mathbb{I}$, as are the differentiated series $\partial_t p_\beta$ and $\partial_x^\tau p_\beta$ for all multi-indices τ in $\mathbb{R}^n$; consequently, each p_β and its derivatives $\partial_t p_\beta$ and $\partial_x^\tau p_\beta$ are continuous in $\mathbb{R}^n \times \mathbb{I}$. It is demonstrated in [21] also that

$$\partial_x^\alpha p_\beta(x,t) = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} p_{\beta-\alpha}(x,t)\ , & \text{if } \beta \geq \alpha\,, \\ 0\,, & \text{otherwise,} \end{cases} \tag{2.4}$$

$$\partial_t p_\beta(x,t) = \beta! \sum_{\alpha\le\beta} \frac{a_\alpha(t)}{(\beta-\alpha)!} p_{\beta-\alpha}(x,t)\,, \tag{2.5}$$

with the latter summation over only those α appearing in (1.1) for which $\alpha \le \beta$. From these formulas it follows further that all partial derivatives of p_β involving at most one derivative with respect to t and space derivatives of any order are continuous in $\mathbb{R}^n \times \mathbb{I}$.

Because there are only a finite number of powers x^γ with $\gamma \le \beta$, the functions p_β are polynomials in x for each fixed t.

Finally, if $\mathcal{L}$ has only constant coefficients $\{a_\alpha\}$, then from (2.1) we have $b_\alpha(t) = ta_\alpha$ for each α, and formulas (2.2) and (2.3) simplify to

$$p_\beta(x,t) = \beta! \sum_{\gamma,\sigma:\gamma+\overline{\alpha}\cdot\sigma=\beta} \frac{a^\sigma x^\gamma t^{|\sigma|}}{\gamma!\sigma!} = \beta! \sum_{\overline{\alpha}\cdot\sigma\le\beta} \frac{a^\sigma x^{\beta-\overline{\alpha}\cdot\sigma} t^{|\sigma|}}{(\beta-\overline{\alpha}\cdot\sigma)!\sigma!}\,. \tag{2.6}$$

Again, if furthermore $\mathcal{L}$ has no zero order term then the sums are over a finite number of terms, and in this event p_β is a polynomial in both variables (x,t), of degree $|\beta|$. (This is the situation in the case of the heat equation, for example.)

The reader is referred to [21] for further discussion of properties of the functions $\{p_\beta\}$.

3. Factorial Estimates

Recall that the factorial function, originally defined for nonnegative integers, can be extended as a C^∞ function to the interval $(-1,\infty)$ by the relation

$$x! = \Gamma(x+1)\,. \tag{3.1}$$

From properties of the gamma function it follows that $x!$ is positive on $(-1,\infty)$, and increases to $+\infty$ as x approaches -1 from the right. Formula (3.1) in fact defines the factorial function on all complex numbers except the negative integers, where $\Gamma(x+1)$ has poles, but we are interested mainly in the behaviour of $x!$ on the real interval $[0,\infty)$.

In this paper we must use two different norms on vectors in $\mathbb{R}^n$, but to conform with standard notation we are obliged to use the same notation for each. For general vectors $x = (x_1, x_2, \cdots, x_n)$ in $\mathbb{R}^n$ we use the Euclidean norm,

$$|x| = \left({x_1}^2 + {x_2}^2 + \cdots + {x_n}^2\right)^{1/2}\,, \tag{3.2}$$

while for vectors $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ in $\mathbb{R}^n$, all of whose components are nonnegative, we use also the ℓ^1 *norm* (or *multi-index norm* when each entry of α is a nonnegative integer),

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n\,. \tag{3.3}$$

We write $\alpha \ge 0$ to denote that $\alpha_i \ge 0$ for each i, and in this event say that α is a *nonnegative vector*. To avoid confusion between the two norms, we will use lower case Greek letters for nonnegative vectors on which we intend to use the multi-index norm, and lower case English letters for general vectors in $\mathbb{R}^n$ on which we use the Euclidean norm.

For $x, y \in \mathbb{R}^n$ we write $x \le y$, or $y \ge x$, whenever $x_i \le y_i$ for each i.

We will frequently encounter powers such as

$$x^\alpha = {x_1}^{\alpha_1} {x_2}^{\alpha_2} \cdots {x_n}^{\alpha_n}\,,$$

where $\alpha \geq 0$. If each α_i is an integer then x^α is defined on all x in $\mathbb{R}^n$, but for general nonnegative α it is required that also x be nonnegative.

Also for nonnegative vectors α in $\mathbb{R}^n$, we define $\alpha!$ as

$$\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n! = \Gamma(\alpha_1 + 1)\,\Gamma(\alpha_2 + 1) \cdots \Gamma(\alpha_n + 1)\ . \tag{3.4}$$

In this section we verify in the form of lemmas preliminary inequalities involving factorials. Perhaps a few of these might be of some interest independently of this paper.

Lemma 3.1. *For real numbers $x, y \geq 0$,*

$$x!y! \leq (x + y)!\,, \tag{3.5}$$

and for nonnegative vectors α in $\mathbb{R}^n$,

$$\alpha! \leq |\alpha|! = (\alpha_1 + \alpha_2 + \cdots + \alpha_n)! \leq n^{|\alpha|}\alpha!\,. \tag{3.6}$$

Proof. By (3.1), we verify (3.5) if we show that, for $x, y \geq 1$,

$$\log\Gamma(x) + \log\Gamma(y) \leq \log\Gamma(x + y - 1)\ .$$

As functions of x with y held fixed, the two sides of this inequality agree at $x = 1$. Thus it is sufficient to confirm the differentiated inequality,

$$\frac{\Gamma'(x)}{\Gamma(x)} \leq \frac{\Gamma'(x + y - 1)}{\Gamma(x + y - 1)}\,. \tag{3.7}$$

But as is well known ([2], chapter 2, or [34], section 12.1), the logarithmic derivative of the gamma function,

$$\frac{d}{dt}\log\Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}\,,$$

increases on $(0, \infty)$; thus (3.7) holds in particular if $x, y \geq 1$.

The left inequality of (3.6) is a trivial equality if $n = 1$, and for $n \geq 2$ it is an easy iteration of (3.5) from 2 factors to n factors.

To prove the right inequality of (3.6) we use a formula of Dirichlet ([34], section 12.5),

$$\begin{aligned}\Gamma(|\alpha|) &\int_P f(x_1 + x_2 + \cdots + x_n)\ {x_1}^{\alpha_1 - 1}{x_2}^{\alpha_2 - 1} \cdots {x_n}^{\alpha_n - 1}\,dx\\ &= \Gamma(\alpha_1)\,\Gamma(\alpha_2) \cdots \Gamma(\alpha_n) \int_0^1 f(t)\ t^{|\alpha| - 1}\,dt\,,\end{aligned}$$

where $\alpha_i > 0$ for each i, f is any continuous real valued function on $[0, 1]$, and P is the region in $\mathbb{R}^n$

$$P = \{x : x_i \geq 0 \text{ for each } i, \text{ and } x_1 + x_2 + \cdots + x_n \leq 1\}\ .$$

In Dirichlet's formula we take $f \equiv 1$, replace each α_i with $\alpha_i + 1$, perform the integration on the right, and use $\Gamma(t + 1) = t\Gamma(t)$ to obtain

$$(|\alpha| + n)! \int_P x^\alpha\,dx = \alpha!\,, \tag{3.8}$$

valid under our assumption that $\alpha_i \geq 0$ for each i.

For each $i = 1, 2, \cdots, n$ we define

$$P_i = \{x \in P : x_i \leq x_j\ , 1 \leq j \leq n\}\ ,$$

so that

$$\int_P x^\alpha \, dx = \sum_{i=1}^{n} \int_{P_i} x^\alpha \, dx \, . \tag{3.9}$$

(The region where two or more of the $P_i's$ overlap is a subset of a finite union of hyperplanes in $\mathbb{R}^n$ and thus has measure zero.) Looking at the first integral in this sum, we observe that

$$\int_{P_1} x^\alpha \, dx \geq \int_{P_1} {x_1}^{|\alpha|} \, dx = \int_0^{1/n} {x_1}^{|\alpha|} \, I\left(x_1\right) \, dx_1 \, ,$$

where

$$I\left(x_1\right) = \int_{x_2, x_3, \cdots, x_n \geq x_1 \, , \; x_2 + x_3 + \cdots + x_n \leq 1 - x_1} dx_2 \, dx_3 \cdots dx_n \, .$$

In this integral we substitute $y_i = x_i - x_1$ for $i = 2$ to n, and obtain

$$I\left(x_1\right) = \int_{y_2, y_3, \cdots, y_n \geq 0 \, , \; y_2 + y_3 + \cdots + y_n \leq 1 - n x_1} dy_2 \, dy_3 \cdots dy_n = \frac{\left(1 - n x_1\right)^{n-1}}{(n-1)!} \, .$$

Therefore,

$$\int_{P_1} x^\alpha \, dx \geq \frac{1}{(n-1)!} \int_0^{1/n} {x_1}^{|\alpha|} \, \left(1 - n x_1\right)^{n-1} \, dx_1 \, .$$

In the integral on the right we substitute $t = n x_1$ and find that

$$\int_{P_1} x^\alpha \, dx \geq \frac{n^{-|\alpha|}}{n!} \int_0^1 t^{|\alpha|} \left(1 - t\right)^{n-1} \, dt = \frac{n^{-|\alpha|}}{n!} B\left(|\alpha| + 1, n\right) \, ,$$

where the well-known *beta function* B (see [2], chapter 2, or [27], section 1.5) satisfies, for $p, q > 0$,

$$B\left(p, q\right) := \int_0^1 t^{p-1} \left(1 - t\right)^{q-1} \, dt = \frac{\Gamma\left(p\right) \Gamma\left(q\right)}{\Gamma\left(p + q\right)} \, .$$

Thus,

$$\int_{P_1} x^\alpha \, dx \geq \frac{n^{-|\alpha|}}{n!} \frac{\Gamma\left(|\alpha| + 1\right) \Gamma\left(n\right)}{\Gamma\left(|\alpha| + 1 + n\right)} = \frac{n^{-|\alpha|} \, |\alpha|!}{n \left(|\alpha| + n\right)!} \, .$$

By symmetry the same lower bound holds for each of the integrals in the summation of (3.9); thus

$$\int_P x^\alpha \, dx \geq \sum_{i=1}^{n} \frac{n^{-|\alpha|} \, |\alpha|!}{n \left(|\alpha| + n\right)!} = \frac{n^{-|\alpha|} \, |\alpha|!}{\left(|\alpha| + n\right)!} \, .$$

Finally, this inequality and (3.8) give the right inequality of (3.6). □

Lemma 3.2. *For real numbers $x \geq 0$,*

$$\frac{\sqrt{2\pi}}{e} \left(x + 1\right)^{x + 1/2} e^{-x} \leq x! \leq \left(x + 1\right)^{x + 1/2} e^{-x} \, , \tag{3.10}$$

$$x^x e^{-x} \leq x! \leq x^x e^{1-x} \sqrt{x + 1} \, , \tag{3.11}$$

and for $x, y \geq 0$,

$$\frac{y^x}{x!} \leq e^y \, . \tag{3.12}$$

(We interpret $0^0 = 1$ in these inequalities.)

Proof. The inequalities (3.10) are minor variations on Stirling's formula; we include the brief proofs because a statement of Stirling's formula in this exact form is not easily found. In chapter 1 of [27] and section 12.31 of [34] we find two equations of Binet for $\log\Gamma(x)$, $x>0$ – namely,

$$\log\Gamma(x)=\left(x-\frac{1}{2}\right)\log x-x+1+\int_0^\infty f(t)\left[e^{-tx}-e^{-t}\right]dt\,, \tag{3.13}$$

$$\log\Gamma(x)=\left(x-\frac{1}{2}\right)\log x-x+\frac{1}{2}\log 2\pi+\int_0^\infty f(t)\,e^{-tx}\,dt\,, \tag{3.14}$$

where f is given for $t\geq 0$ as

$$f(t)=\left[\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right]\frac{1}{t}=2\sum_{k=1}^{\infty}\frac{1}{t^2+4\pi^2k^2}\,.$$

Moreover, f is continuous, positive, and decreasing on $[0,\infty)$, with $f(0^+)=1/12$. If $x\geq 1$ then the integrand in (3.13) is nonpositive, and we get

$$\log\Gamma(x)\leq\left(x-\frac{1}{2}\right)\log x-x+1\,,$$

and then the right inequality of (3.10) by exponentiation. On the other hand, if $x>0$ then the integrand of (3.14) is positive, and thus

$$\log\Gamma(x)\geq\left(x-\frac{1}{2}\right)\log x-x+\frac{1}{2}\log 2\pi\,,$$

which yields the left side of (3.10).

The right side of (3.11) follows from the right side of (3.10), and the observation

$$(x+1)^{x+1/2}=\left(\frac{x+1}{x}\right)^x x^x\sqrt{x+1}\leq ex^x\sqrt{x+1}\,.$$

To verify the left side of (3.11) we show that the function

$$h(x)=x!x^{-x}e^x$$

is increasing on $(0,\infty)$; then it follows that $h(x)\geq h(0^+)=1$. We replace x by $x+1$ in (3.14) and differentiate with respect to x to find that

$$\frac{(x!)'}{x!}=\log(x+1)-\frac{1}{2(x+1)}-\int_0^\infty tf(t)e^{-t(x+1)}\,dt\,.$$

Using $0<f(t)\leq 1/12$, we deduce that

$$\begin{aligned}\frac{(x!)'}{x!}&\geq\log(x+1)-\frac{1}{2(x+1)}-\frac{1}{12}\int_0^\infty te^{-t(x+1)}\,dt\\&=\log(x+1)-\frac{1}{2(x+1)}-\frac{1}{12(x+1)^2}\,.\end{aligned}$$

If $x>0$ then $\log(x+1)\geq\log x+1/(x+1)$, and consequently

$$\frac{(x!)'}{x!}\geq\log x+\frac{1}{2(x+1)}-\frac{1}{12(x+1)^2}>\log x\,.$$

This inequality implies that, for $x>0$,

$$\frac{d}{dx}\log h(x)=\frac{d}{dx}\left[\log x!-x\log x+x\right]=\frac{(x!)'}{x!}-\log x>0\,,$$

and consequently $h(x)$ increases with x.

Finally, we write (3.12) as $y^x e^{-y} \le x!$, and observe that $y^x e^{-y}$ is maximized as a function of y on $[0,\infty)$ by taking $y = x$. Thus it is sufficient to verify the one variable inequality $x^x e^{-x} \le x!$, which is contained in (3.11). $\square$

Lemma 3.3. *Let $x, \alpha \in \mathbb{R}^n$, $n \ge 1$, with $\alpha \ge 0$ and $1 < \ell < \infty$. If either (a) each component α_i of α is an integer, or (b) $x \ge 0$, then*

$$\frac{|x^\alpha|}{\alpha!} \le \frac{1}{(|\alpha|/\ell)!} \exp\left[(n|x|)^{\ell/(\ell-1)}\right], \tag{3.15}$$

where for x we use the Euclidean norm (3.2) and for α the ℓ^1 norm (3.3).

Proof. Under either condition (a) or (b), x^α is defined and

$$|x^\alpha| = |{x_1}^{\alpha_1} {x_2}^{\alpha_2} \cdots {x_n}^{\alpha_n}| \le |x|^{\alpha_1} |x|^{\alpha_2} \cdots |x|^{\alpha_n} = |x|^{|\alpha|}.$$

We use this inequality, along with (3.6), (3.5), and (3.12), to derive

$$\begin{aligned} |x^\alpha| &\le \frac{|\alpha|!\,|x|^{|\alpha|}}{\left[\frac{|\alpha|}{\ell} + \left(|\alpha| - \frac{|\alpha|}{\ell}\right)\right]!} \le \frac{n^{|\alpha|}\alpha!\,|x|^{|\alpha|}}{\left(\frac{|\alpha|}{\ell}\right)!\left(|\alpha| - \frac{|\alpha|}{\ell}\right)!} \\ &= \frac{\alpha!}{\left(\frac{|\alpha|}{\ell}\right)!} \frac{\left[(n|x|)^{\ell/(\ell-1)}\right]^{\left(\frac{\ell-1}{\ell}\right)|\alpha|}}{\left(\frac{\ell-1}{\ell}|\alpha|\right)!} \le \frac{\alpha!}{\left(\frac{|\alpha|}{\ell}\right)!} \exp\left[(n|x|)^{\ell/(\ell-1)}\right]. \end{aligned}$$

The last inequality implies (3.15). $\square$

Lemma 3.4. *If x, y, p are nonnegative real numbers, then*

$$x \le y \Longrightarrow \frac{(x+p)!}{x!} \le \frac{(y+p)!}{y!}.$$

Proof. It is sufficient to confirm that the function

$$f(x) = \frac{(x+p)!}{x!} = \frac{\Gamma(x+p+1)}{\Gamma(x+1)}$$

increases with x on $[0,\infty)$. But

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\log f(x) = \frac{\Gamma'(x+p+1)}{\Gamma(x+p+1)} - \frac{\Gamma'(x+1)}{\Gamma(x+1)}.$$

Again recalling that Γ'/Γ increases on $(0,\infty)$, we infer that $f'(x) \ge 0$. $\square$

The next three lemmas yield estimates of certain sums involving factorials.

Lemma 3.5. *For $y = (y_1, y_2, \cdots, y_n)$ with all $y_i \ge 0$, and for real numbers $s, r \ge 0$,*

$$\sum_{\sigma \in \mathbb{R}^n,\ |\sigma| \le s} \frac{|y^\sigma|\, r^{s-|\sigma|}}{\sigma!\,(s-|\sigma|)!} \le \frac{(|y_1| + |y_2| + \cdots + |y_n| + r)^s}{s!}, \tag{3.16}$$

where the summation is over all multi-indices $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ in $\mathbb{R}^n$ such that $|\sigma| \le s$. (We interpret $0^0 = 1$.)

Proof. Since $|y^\sigma| \le (|y_1|, |y_2|, \cdots, |y_n|)^\sigma$, we may assume that each $y_i \ge 0$.

First we consider the case $n = 1$, when y and σ are scalars. When $r = 0$ the right side of (3.16) is $y^s/s!$, while the left side is the same if s is an integer, and 0 if s is not an integer. Thus we may assume $r > 0$. When $y = 0$ both sides reduce to $r^s/s!$, so likewise we may assume $y > 0$. We set

$$f(y) = (y + r)^s .$$

Let ℓ be the largest integer such that $\ell \le s$. By Taylor's formula with remainder, there exists x between 0 and y such that

$$\begin{aligned} f(y) &= \sum_{\sigma=0}^{\ell} \frac{f^{(\sigma)}(0)}{\sigma!} y^\sigma + \frac{f^{(\ell+1)}(x)}{(\ell+1)!} y^{\ell+1} \\ &= \sum_{\sigma=0}^{\ell} \frac{f^{(\sigma)}(0)}{\sigma!} y^\sigma + \frac{s(s-1)\cdots(s-\ell)(x+r)^{s-\ell-1}}{(\ell+1)!} y^{\ell+1} \\ &\ge \sum_{\sigma=0}^{\ell} \frac{f^{(\sigma)}(0)}{\sigma!} y^\sigma = s! \sum_{\sigma \le s} \frac{y^\sigma r^{s-\sigma}}{\sigma!\,(s-\sigma)!} . \end{aligned}$$

This inequality gives (3.16) for $n = 1$.

Next we assume the lemma holds for $n - 1$, and consider the case of n. We use the induction hypotheses for $n - 1$ and the established result for $n = 1$ to derive

$$\begin{aligned} &\sum_{|\sigma| \le s} \frac{y^\sigma r^{s-|\sigma|}}{\sigma!\,(s-|\sigma|)!} \\ &= \sum_{\sigma_n \le s} \frac{y_n^{\sigma_n}}{\sigma_n!} \sum_{\sigma_1+\cdots+\sigma_{n-1} \le s-\sigma_n} \frac{(y_1, \cdots, y_{n-1})^{(\sigma_1,\cdots,\sigma_{n-1})}\, r^{s-\sigma_n-\sigma_1-\cdots-\sigma_{n-1}}}{\sigma_1! \cdots \sigma_{n-1}!\,(s-\sigma_n-\sigma_1-\cdots-\sigma_{n-1})!} \\ &\le \sum_{\sigma_n \le s} \frac{y_n^{\sigma_n}}{\sigma_n!} \frac{(y_1 + y_2 + \cdots + y_{n-1} + r)^{s-\sigma_n}}{(s-\sigma_n)!} \le \frac{(y_1 + y_2 + \cdots + y_n + r)^s}{s!} , \end{aligned}$$

which is (3.16). □

Lemma 3.6. *For real numbers $r, s \ge 0$ and for any integers $m, n \ge 1$,*

$$J(r,s,m,n) := \sum_{\sigma \in \mathbb{R}^n,\ |\sigma| \le s} \frac{r^{(s-|\sigma|)/m}}{\left(\frac{\sigma}{m}\right)! \left(\frac{s-|\sigma|}{m}\right)!} \le m^n \frac{(n+r)!}{r!} \frac{(n+r)^{s/m}}{\left(\frac{s}{m}\right)!} ,$$

where the summation is over all multi-indices σ in $\mathbb{R}^n$ such that $|\sigma| \le s$.

Proof. First we consider the case $n = 1$, when the summation is over nonnegative integers σ such that $0 \le \sigma \le s$. We may write any such σ in a unique way as

$$\sigma = m\tau - \rho , \tag{3.17}$$

where τ and ρ are nonnegative integers and $0 \le \rho < m$. (If $\sigma = 0$ we must take $\tau = \rho = 0$; otherwise, $\tau \ge 1$.) Note that $0 \le \sigma \le s$ implies

$$\frac{\rho}{m} \le \tau \le \frac{s+\rho}{m} . \tag{3.18}$$

Conversely, if ρ is an integer with $0 \le \rho < m$ and τ is a nonnegative integer satisfying (3.18), then $\sigma = m\tau - \rho$ satisfies $0 \le \sigma \le s$. Thus there is a one-to-one correspondence, as determined by (3.17), between nonnegative integers σ with

$\sigma \le s$ and pairs of integers (τ,ρ) such that $0 \le \rho < m$ and (3.18) holds. It follows that we can write the summation $J = J(r,s,m,1)$ as

$$J(r,s,m,1) = \sum_{\rho=0}^{m-1} \sum_{\rho/m \le \tau \le (s+\rho)/m} \frac{r^{(s+\rho)/m-\tau}}{\left(\tau - \frac{\rho}{m}\right)! \left(\frac{s+\rho}{m} - \tau\right)!} .$$

By Lemma 3.4, if $\rho/m \le \tau \le (s+\rho)/m$ then

$$\frac{1}{\left(\tau - \frac{\rho}{m}\right)!} = \frac{1}{\tau!} \frac{\tau!}{\left(\tau - \frac{\rho}{m}\right)!} \le \frac{1}{\tau!} \frac{\left(\frac{s+\rho}{m}\right)!}{\left(\frac{s}{m}\right)!} \quad ;$$

therefore

$$J(r,s,m,1) \le \frac{1}{\left(\frac{s}{m}\right)!} \sum_{\rho=0}^{m-1} \left(\frac{s+\rho}{m}\right)! \sum_{\tau \le (s+\rho)/m} \frac{1^\tau r^{(s+\rho)/m-\tau}}{\tau! \left(\frac{s+\rho}{m} - \tau\right)!} \quad .$$

Applying Lemma 3.5 with $n = 1$ to the interior sum gives

$$\begin{aligned} J(r,s,m,1) &\le \frac{1}{\left(\frac{s}{m}\right)!} \sum_{\rho=0}^{m-1} (1+r)^{(s+\rho)/m} \le \frac{1}{\left(\frac{s}{m}\right)!} \sum_{\rho=0}^{m-1} (1+r)^{(s+m)/m} \\ &= \frac{m\,(1+r)^{(s+m)/m}}{\left(\frac{s}{m}\right)!} = m^1 \frac{(1+r)!}{r!} \frac{(1+r)^{s/m}}{\left(\frac{s}{m}\right)!} . \end{aligned}$$

Having established the Lemma for $n = 1$, next we assume it is valid also for $n-1$. Then

$$\begin{aligned} J(r,s,m,n) &= \sum_{\sigma \in \mathbb{R}^n,\ |\sigma| \le s} \frac{r^{(s-|\sigma|)/m}}{\left(\frac{\sigma}{m}\right)! \left(\frac{s-|\sigma|}{m}\right)!} \\ &= \sum_{\sigma_n \le s} \frac{1}{\left(\frac{\sigma_n}{m}\right)!} \sum_{\sigma_1 + \dots + \sigma_{n-1} \le s - \sigma_n} \frac{r^{(s-\sigma_1-\dots-\sigma_n)/m}}{\left(\frac{\sigma_1}{m}\right)! \dots \left(\frac{\sigma_{n-1}}{m}\right)! \left(\frac{s-\sigma_1-\dots-\sigma_n}{m}\right)!} \\ &= \sum_{\sigma_n \le s} \frac{1}{\left(\frac{\sigma_n}{m}\right)!} J(r, s-\sigma_n, m, n-1) \\ &\le \sum_{\sigma_n \le s} \frac{1}{\left(\frac{\sigma_n}{m}\right)!} m^{n-1} \frac{(n-1+r)!}{r!} \frac{(n-1+r)^{(s-\sigma_n)/m}}{\left(\frac{s-\sigma_n}{m}\right)!} \\ &= m^{n-1} \frac{(n-1+r)!}{r!} J(n-1+r, s, m, 1) . \end{aligned}$$

Applying the lemma to $J(n-1+r,s,m,1)$ now gives the desired inequality for $J(r,s,m,n)$. □

Lemma 3.7. *Let $\mathcal{M}$ denote a nonempty collection of M nonzero multi-indices α in $\mathbb{R}^n$, and let m be a positive integer and s a nonnegative real number. Then*

$$\sum_{|\overline{\alpha}\cdot\sigma| \le s} \frac{1}{\left(\frac{s-|\overline{\alpha}\cdot\sigma|}{m}\right)! \prod_{\alpha \in \mathcal{M}} \left(\frac{|\alpha|}{m}\sigma_\alpha\right)!} \le m^M (M+1)! \frac{(M+1)^{s/m}}{\left(\frac{s}{m}\right)!} , \tag{3.19}$$

where the summation is taken over all multi-indices σ in R^M such that

$$|\overline{\alpha}\cdot\sigma| = \left| \sum_{\alpha \in \mathcal{M}} \alpha \sigma_\alpha \right| = \sum_{\alpha \in \mathcal{M}} |\alpha|\, \sigma_\alpha \le s . \tag{3.20}$$

Proof. Let ℓ denote the maximum magnitude of any multi-index in $\mathcal{M}$, so that $|\alpha| \le \ell$ for all $\alpha \in \mathcal{M}$. First we consider the special case when $|\alpha| = \ell$ for all $\alpha \in \mathcal{M}$. Then (3.20) shows that $|\overline{\alpha} \cdot \sigma| = \ell\,|\sigma| = |\ell\sigma|$. Denoting the sum on the left of (3.19) as S, we have

$$S = \sum_{\sigma\in\mathbb{R}^M\,,\ |\ell\sigma|\le s} \frac{1}{\left(\frac{s-|\ell\sigma|}{m}\right)!\,\left(\frac{\ell\sigma}{m}\right)!}\,.$$

We can view this sum as taken over all multi-indices $\nu \in \mathbb{R}^M$ such that $|\nu| \le s$ and ν has the special form $\nu = \ell\sigma$ for some multi-index $\sigma \in \mathbb{R}^M$. The sum will increase if we remove the last restriction on ν, allowing us to use Lemma 3.6 to deduce that

$$S \le \sum_{\nu\in\mathbb{R}^M\,,\ |\nu|\le s} \frac{1}{\left(\frac{s-|\nu|}{m}\right)!\,\left(\frac{\nu}{m}\right)!} = J(1,s,m,M) \le m^M (M+1)!\frac{(M+1)^{s/m}}{\left(\frac{s}{m}\right)!}\,.$$

We now proceed by induction on ℓ. When $\ell = 1$ we have $|\alpha| = 1 = \ell$ for all $\alpha \in \mathcal{M}$, and the result has been established. Now we assume the lemma is valid for $1,\ 2,\ \cdots,\ \ell-1$, and consider the case of ℓ. We let $\mathcal{M}_\ell$ consist of all multi-indices α in $\mathcal{M}$ with $|\alpha| = \ell$, and we let $\mathcal{M}_0$ consist of all remaining multi-indices in $\mathcal{M}$. We denote by L and K the number of multi-indices in $\mathcal{M}_\ell$ and $\mathcal{M}_0$, respectively, so that $M = L + K$. We have $L > 0$ by assumption, and we may assume $K > 0$ as otherwise the result is already proved. Given any multi-index σ in $\mathbb{R}^M$ we may write

$$\sigma = \tau + \lambda\,,$$

where τ and λ are multi-indices also in $\mathbb{R}^M$, defined according to

$$\tau_\alpha = \begin{cases} \sigma_\alpha\ , & \text{if } |\alpha| = \ell\,, \\ 0\ , & \text{if } 0 < |\alpha| < \ell\,, \end{cases} \tag{3.21}$$

$$\lambda_\alpha = \begin{cases} 0\ , & \text{if } |\alpha| = \ell\,, \\ \sigma_\alpha\ , & \text{if } 0 < |\alpha| < \ell\,. \end{cases} \tag{3.22}$$

Then

$$|\overline{\alpha}\cdot\sigma| = |\overline{\alpha}\cdot\tau| + |\overline{\alpha}\cdot\lambda|\ , \qquad |\overline{\alpha}\cdot\tau| = \ell\,|\tau| = |\ell\tau|\ ,$$
$$\prod_{\alpha\in\mathcal{M}}\left(\frac{|\alpha|}{m}\sigma_\alpha\right)! = \prod_{\alpha\in\mathcal{M}_\ell}\left(\frac{\ell}{m}\tau_\alpha\right)!\prod_{\alpha\in\mathcal{M}_0}\left(\frac{|\alpha|}{m}\lambda_\alpha\right)!\,,$$

and

$$S = \sum_{|\overline{\alpha}\cdot\tau|+|\overline{\alpha}\cdot\lambda|\le s} \frac{1}{\left(\frac{s-|\overline{\alpha}\cdot\tau|-|\overline{\alpha}\cdot\lambda|}{m}\right)!\,\left(\frac{\ell\tau}{m}\right)!\prod_{\alpha\in\mathcal{M}_0}\left(\frac{|\alpha|}{m}\lambda_\alpha\right)!} = \sum_{|\ell\tau|\le s}\frac{1}{\left(\frac{\ell\tau}{m}\right)!}\cdot S_0\,,$$

where

$$S_0 := \sum_{|\overline{\alpha}\cdot\lambda|\le s-|\ell\tau|} \frac{1}{\left(\frac{s-|\ell\tau|-|\overline{\alpha}\cdot\lambda|}{m}\right)!\prod_{\alpha\in\mathcal{M}_0}\left(\frac{|\alpha|}{m}\lambda_\alpha\right)!}\,.$$

Since

$$\overline{\alpha}\cdot\lambda = \sum_{\alpha\in\mathcal{M}_0}\alpha\lambda_\alpha\,,$$

and because the zero entries of λ play no role in the calculations, we may disregard the zero entries and regard λ as a multi-index in $\mathbb{R}^K$; then by the induction hypotheses,

$$S_0 \le m^K (K+1)! \frac{(K+1)^{(s-|\ell\tau|)/m}}{\left(\frac{s-|\ell\tau|}{m}\right)!} ,$$

and consequently

$$S \le m^K (K+1)! \sum_{|\ell\tau|\le s} \frac{(K+1)^{(s-|\ell\tau|)/m}}{\left(\frac{\ell\tau}{m}\right)! \left(\frac{s-|\ell\tau|}{m}\right)!} .$$

As the zero entries of τ play no role in the calculations, we may ignore these entries as well and regard τ as a multi-index in $\mathbb{R}^L$; then we use Lemma 3.6 once again to derive

$$\begin{aligned} S &\le m^K (K+1)! \sum_{\nu\in\mathbb{R}^L\,,\,|\nu|\le s} \frac{(K+1)^{(s-|\nu|)/m}}{\left(\frac{\nu}{m}\right)! \left(\frac{s-|\nu|}{m}\right)!} \\ &\le m^K (K+1)! m^L \frac{(K+L+1)!}{(K+1)!} \frac{(K+L+1)^{s/m}}{(s/m)!} , \end{aligned}$$

which with the substitution $M = K + L$ gives (3.19). □

4. Polynomial Bounds

We now derive bounds on the polynomials $\{p_\beta\}$, as specified by (2.2) and (2.3), for the operator (1.1). We assume that the highest order of any space derivative is m, so that the operator may be written as

$$\mathcal{L}u(x,t) = \partial_t u(x,t) - \sum_{\alpha\in\mathcal{A}} a_\alpha(t)\partial_x^\alpha u(x,t) , \tag{4.1}$$

where $\mathcal{A}$ is a finite collection of multi-indices α in $\mathbb{R}^n$ with $|\alpha| \le m$ for each $\alpha \in \mathcal{A}$, and with $|\alpha| = m$ for at least one such α. We suppose also that there are finite bounds $\{\Lambda_\alpha\}$ and Λ such that

$$\Lambda_\alpha = \sup_{t\in\mathbb{I}} |a_\alpha(t)| \quad (\alpha\in\mathcal{A}) , \qquad \Lambda = \sup_{t\in\mathbb{I}} \sum_{\alpha\in\mathcal{A}\,,\,|\alpha|=m} |a_\alpha(t)| . \tag{4.2}$$

The utility of Theorem 4.1 below will arise mainly from the nature of the dependence of the bounds on β. Series in the polynomials $\{p_\beta\}$ will, under rather general circumstances, converge in strips in $\mathbb{R}^{n+1}$ of the form $|t| < s$, and convergence will be uniform on compact subsets. Our goal in producing inequality (4.3) is to obtain the most efficient bounds possible in terms of the β – dependent factors, so as to get the best estimate on s, while being less concerned with factors independent of β. In later results on strips of convergence we will see that the β – independent bounds, as magnanimous as they seem, turn out to be easily manageable as they are trivially bounded on compact sets in $\mathbb{R}^{n+1}$. While for any single polynomial p_β a bound of the type (4.3) below might seem obvious and overly generous, the point is that the inequality is valid for all $\{p_\beta\}$ simultaneously. Note that, in our assumptions (4.2) on the coefficients of $\mathcal{L}$, the bound Λ on the highest order coefficients is singled out separately. As we will see in later results, it is this bound that is relevant for the width of the strip of convergence, while the bounds on the lower order coefficients are less significant.

Theorem 4.1. *Assume $m \geq 2$, where m is the order of (4.1), and that the coefficents $\{a_\alpha\}$ are continuous with the bounds (4.2) in an interval $\mathbb{I}$ containing the origin. Then for $x \in \mathbb{R}^n$ and $t \in \mathbb{I}$, and for any $\delta > 0$,*

$$\begin{aligned}\frac{|p_\beta(x,t)|}{\beta!} \leq \frac{m^M\,(M+1)!}{(|\beta|/m)!}\,(\Lambda\,|t|+\delta)^{|\beta|/m} \\ \cdot \exp\Bigg[\left(\frac{M+1}{\delta}\right)^{1/(m-1)}(n\,|x|)^{m/(m-1)} \\ + \sum_{\alpha\in\mathcal{A}\ ,\ |\alpha|<m}\left(\frac{M+1}{\delta}\right)^{|\alpha|/(m-|\alpha|)}|\Lambda_\alpha t|^{m/(m-|\alpha|)}\Bigg]\,.\end{aligned} \tag{4.3}$$

where M is the number of multi-indices $\alpha \in \mathcal{A}$ such that $0 < |\alpha| < m$.

Proof. We initially assume (4.1) contains no zero order term; that is, there is no term in the sum corresponding to $\alpha = (0, \cdots, 0)$. Let K denote the number of space derivative terms in this sum; then by (2.2),

$$\frac{p_\beta(x,t)}{\beta!} = \sum_{\gamma+\overline{\alpha}\cdot\sigma=\beta}\frac{x^\gamma b(t)^\sigma}{\gamma!\sigma!}\,, \tag{4.4}$$

where the sum is over all multi-indices γ in $\mathbb{R}^n$ and σ in $\mathbb{R}^K$ satisfying $\gamma+\overline{\alpha}\cdot\sigma = \beta$. Given $\delta > 0$, Lemma 3.3 (with $\ell = m$) yields

$$\begin{aligned}\frac{|x^\gamma|}{\gamma!} &= \frac{\delta^{|\gamma|/m}}{\gamma!}\left|\left(\frac{x_1}{\delta^{1/m}},\frac{x_2}{\delta^{1/m}},\cdots,\frac{x_n}{\delta^{1/m}}\right)^\gamma\right| \\ &\leq \frac{\delta^{|\gamma|/m}}{(|\gamma|/m)!}\exp\left(\frac{n\,|x|}{\delta^{1/m}}\right)^{m/(m-1)}\,.\end{aligned}$$

This estimate, used in (4.4), leads to the bound

$$\frac{|p_\beta(x,t)|}{\beta!} \leq \exp\left(\frac{n\,|x|}{\delta^{1/m}}\right)^{m/(m-1)}\cdot S\,, \tag{4.5}$$

where

$$S = \sum_{\gamma+\overline{\alpha}\cdot\sigma=\beta}\frac{\delta^{|\gamma|/m}}{(|\gamma|/m)!}\frac{|b^\sigma|}{\sigma!}\,. \tag{4.6}$$

(Henceforth, for brevity we suppress the arguments of functions of t.)

We let N denote the number of terms in the summation of (4.1) corresponding to $|\alpha| = m$, and M the number with $0 < |\alpha| < m$, so that $K = N+M$. We number the multi-indices appearing in this sum as

$$\alpha^1\,,\ \alpha^2\,,\ \cdots\,,\ \alpha^N\,,\ \alpha^{N+1}\,,\ \alpha^{N+2}\,,\ \cdots\,,\ \alpha^{N+M}\,,$$

where $\alpha^1, \cdots, \alpha^N$ are the highest order multi-indices with $|\alpha| = m$, and $\alpha^{N+1}, \cdots, \alpha^{N+M}$ are of lower order. (Thus $N \geq 1$, as $\mathcal{L}$ has order m). We write a general multi-index σ in $\mathbb{R}^K$ as

$$\sigma = (\nu_1, \cdots, \nu_N, \lambda_1, \cdots, \lambda_M) = \nu + \lambda\,,$$

where the multi-indices ν and λ, each also in $\mathbb{R}^K$, are defined as

$$\nu = (\nu_1, \cdots, \nu_N, 0, \cdots, 0) \,, \qquad \lambda = (0, \cdots, 0, \lambda_1, \cdots, \lambda_M) \,, \tag{4.7}$$

and

$$\overline{\alpha} \cdot \sigma = \sum \alpha \sigma_\alpha = \sum_{i=1}^{N} \alpha^i \nu_i + \sum_{i=1}^{M} \alpha^{N+i} \lambda_i \,.$$

(We admit the possibility $M = 0$, in which case $\lambda = (0, \cdots, 0)$ and $\sigma = \nu$.) Then

$$\overline{\alpha} \cdot \sigma = \overline{\alpha} \cdot \nu + \overline{\alpha} \cdot \lambda \,, \quad \sigma! = \nu! \lambda! \,,$$

and (4.6) can be written as

$$\begin{aligned} S &= \sum_{\gamma + \overline{\alpha} \cdot \nu + \overline{\alpha} \cdot \lambda = \beta} \frac{\delta^{|\gamma|/m}}{(|\gamma|/m)!} \frac{|b^\nu|}{\nu!} \frac{|b^\lambda|}{\lambda!} \\ &= \sum_{\overline{\alpha} \cdot \nu + \overline{\alpha} \cdot \lambda \le \beta} \frac{\delta^{|\beta - \overline{\alpha} \cdot \nu - \overline{\alpha} \cdot \lambda|/m}}{(|\beta - \overline{\alpha} \cdot \nu - \overline{\alpha} \cdot \lambda|/m)!} \frac{|b^\nu|}{\nu!} \frac{|b^\lambda|}{\lambda!} \\ &= \sum_{\overline{\alpha} \cdot \nu \le \beta} \delta^{|\beta - \overline{\alpha} \cdot \nu|/m} \frac{|b^\nu|}{\nu!} \cdot T \,, \end{aligned} \tag{4.8}$$

where

$$T = \frac{1}{(|\beta - \overline{\alpha} \cdot \nu|/m)!} \,, \text{ if } M = 0 \quad , \tag{4.9}$$

$$T = \sum_{\overline{\alpha} \cdot \lambda \le \beta - \overline{\alpha} \cdot \nu} \frac{\delta^{-|\overline{\alpha} \cdot \lambda|/m}}{(|\beta - \overline{\alpha} \cdot \nu - \overline{\alpha} \cdot \lambda|/m)!} \frac{|b^\lambda|}{\lambda!} \,, \text{ if } M > 0 \,. \tag{4.10}$$

In the case $M > 0$ we let $\mathcal{M}$ denote the collection of multi-indices $|\alpha|$ appearing in (4.1) such that $0 < |\alpha| < m$, and apply Lemma 3.3 but with $\ell = m/|\alpha|$ and $n = 1$ to estimate

$$\begin{aligned} \frac{|b^\lambda|}{\lambda!} &= \prod_{\alpha \in \mathcal{M}} \frac{|b_\alpha|^{\lambda_\alpha}}{\lambda_\alpha!} = \prod_{\alpha \in \mathcal{M}} \left\{ \delta^{|\alpha|\lambda_\alpha/m} \frac{1}{\lambda_\alpha!} \left(\frac{|b_\alpha|}{\delta^{|\alpha|/m}} \right)^{\lambda_\alpha} \right\} \\ &\le \prod_{\alpha \in \mathcal{M}} \left\{ \delta^{|\alpha|\lambda_\alpha/m} \frac{1}{\left(\frac{|\alpha|\lambda_\alpha}{m} \right)!} \exp \left[\left(\frac{|b_\alpha|}{\delta^{|\alpha|/m}} \right)^{m/(m-|\alpha|)} \right] \right\} \\ &= \delta^{|\overline{\alpha} \cdot \lambda|/m} \exp \left[\sum_{\alpha \in \mathcal{M}} \left(\frac{|b_\alpha|}{\delta^{|\alpha|/m}} \right)^{m/(m-|\alpha|)} \right] \prod_{\alpha \in \mathcal{M}} \frac{1}{\left(\frac{|\alpha|\lambda_\alpha}{m} \right)!} \,. \end{aligned}$$

We substitute this inequality into (4.10), noting that $|\overline{\alpha} \cdot \lambda| \le |\beta - \overline{\alpha} \cdot \nu|$ in this sum, to obtain

$$\begin{aligned} T \le& \exp \left[\sum_{\alpha \in \mathcal{M}} \left(\frac{|b_\alpha|}{\delta^{|\alpha|/m}} \right)^{m/(m-|\alpha|)} \right] \\ &\cdot \sum_{|\overline{\alpha} \cdot \lambda| \le |\beta - \overline{\alpha} \cdot \nu|} \frac{1}{\left(\frac{|\beta - \overline{\alpha} \cdot \nu| - |\overline{\alpha} \cdot \lambda|}{m} \right)! \prod_{\alpha \in \mathcal{M}} \left(\frac{|\alpha|}{m} \lambda_\alpha \right)!} \,. \end{aligned}$$

Applying Lemma 3.7 to estimate the latter sum, we arrive at

$$T \le m^M \,(M+1)! \frac{(M+1)^{|\beta-\overline{\alpha}\cdot\nu|/m}}{\left(\frac{|\beta-\overline{\alpha}\cdot\nu|}{m}\right)!} \exp\left[\sum_{\alpha\in\mathcal{M}} \left(\frac{|b_\alpha|}{\delta^{|\alpha|/m}}\right)^{m/(m-|\alpha|)}\right] .$$

Note that this estimate reduces to (4.9) in the case $M = 0$, when $\mathcal{M}$ is empty, so that in fact it is valid for both $M > 0$ and $M = 0$. We insert this estimate into (4.8), and then (4.8) into (4.5), to obtain

$$\begin{aligned} \frac{|p_\beta(x,t)|}{\beta!} &\le m^M \,(M+1)! \cdot U \\ &\cdot \exp\left[\left(\frac{n\,|x|}{\delta^{1/m}}\right)^{m/(m-1)} + \sum_{\alpha\in\mathcal{M}}\left(\frac{|b_\alpha|}{\delta^{|\alpha|/m}}\right)^{m/(m-|\alpha|)}\right] , \end{aligned} \tag{4.11}$$

where

$$U = \sum_{\overline{\alpha}\cdot\nu\le\beta} \frac{|b^\nu|}{\nu!} \frac{\delta^{|\beta-\overline{\alpha}\cdot\nu|/m}\,(M+1)^{|\beta-\overline{\alpha}\cdot\nu|/m}}{(|\beta-\overline{\alpha}\cdot\nu|\,/m)!} . \tag{4.12}$$

Now, in (4.12) we have $|\overline{\alpha}\cdot\nu| = m\,|\nu| \le |\beta|$, which allows us to use Lemma 3.5 to estimate

$$\begin{aligned} U &\le \sum_{|\nu|\le|\beta|/m} \frac{|b^\nu|}{\nu!} \frac{[\delta\,(M+1)]^{|\beta|/m-|\nu|}}{(|\beta|\,/m-|\nu|)!} \\ &\le \frac{1}{(|\beta|\,/m)!}\left[\sum_{|\alpha|=m}|b_\alpha(t)| + \delta\,(M+1)\right]^{|\beta|/m} . \end{aligned} \tag{4.13}$$

We substitute this inequality into (4.11) and replace δ by $\delta/(M+1)$ to obtain

$$\begin{aligned} \frac{|p_\beta(x,t)|}{\beta!} &\le \frac{m^M\,(M+1)!}{(|\beta|\,/m)!}\left[\sum_{|\alpha|=m}|b_\alpha(t)|+\delta\right]^{|\beta|/m} \\ &\cdot\exp\left[\left(\frac{M+1}{\delta}\right)^{1/(m-1)}(n\,|x|)^{m/(m-1)}\right. \\ &\left.+\sum_{\alpha\in\mathcal{A}\ ,\ |\alpha|<m}\left(\frac{M+1}{\delta}\right)^{|\alpha|/(m-|\alpha|)}|b_\alpha|^{m/(m-|\alpha|)}\right] . \end{aligned} \tag{4.14}$$

From (2.1) and (4.2) we deduce that

$$|b_\alpha(t)| \le \Lambda_\alpha\,|t| \ , \qquad \sum_{|\alpha|=m}|b_\alpha(t)| \le \Lambda\,|t| \ , \tag{4.15}$$

and we insert these estimates into (4.14) to arrive at (4.3).

It remains now only to verify the theorem when the operator (4.1) has a zero order term, corresponding to $\alpha_0 := (0,\cdots,0)$. To analyze this situation, we assume the operator $\mathcal{L}$ of (4.1) has no zero order term, and we subtract such a term from

$\mathcal{L}$ to produce a modified operator

$$\begin{aligned}\mathcal{L}_0 u(x,t) &= \partial_t u(x,t) - \sum_{\alpha\in\mathcal{A}} a_\alpha(t)\partial_x^\alpha u(x,t) - a_0(t)u(x,t) \\ &= \partial_t u(x,t) - \sum_{\alpha\in\mathcal{B}} a_\alpha(t)\partial_x^\alpha u(x,t) \ ,\end{aligned}$$

where $\mathcal{B} = \mathcal{A}\cup\{\alpha_0\} = \mathcal{A}\cup\{(0,\cdots,0)\}$. We let $\{q_\beta\}$ denote the polynomials (2.2) for $\mathcal{L}_0$, and $\{p_\beta\}$ the corresponding polynomials for $\mathcal{L}$. The analogue of (4.4) for q_β is

$$\frac{q_\beta(x,t)}{\beta!} = \sum_{\gamma+\overline{\alpha}\cdot\sigma=\beta} \frac{x^\gamma b(t)^\sigma}{\gamma!\sigma!}\,, \tag{4.16}$$

where here

$$\overline{\alpha}\cdot\sigma = \sum_{\alpha\in\mathcal{B}} \alpha\sigma_\alpha = \sum_{\alpha\in\mathcal{A}} \alpha\sigma_\alpha + \alpha_0\sigma_0 = \sum_{\alpha\in\mathcal{A}} \alpha\sigma_\alpha\,.$$

Thus the condition $\gamma+\overline{\alpha}\cdot\sigma=\beta$ places no restriction on the component σ_0 of σ corresponding to α_0; that is, for fixed acceptable choices of the other components $\{\sigma_\alpha\}$ of σ, we may allow σ_0 to range over all nonnegative integers. If we write

$$\sigma = (\xi,\sigma_0) = (\xi_1,\cdots,\xi_K,\sigma_0)\ , \quad b_0(t) = \int_0^t a_0(s)\,ds\,,$$

then (4.16) may be written as

$$\frac{q_\beta(x,t)}{\beta!} = \sum_{\gamma+\overline{\alpha}\cdot\xi=\beta} \frac{x^\gamma b(t)^\xi}{\gamma!\xi!} \sum_{\sigma_0=0}^{\infty} \frac{b_0(t)^{\sigma_0}}{(\sigma_0)!}\,, \tag{4.17}$$

where here we interpret

$$\overline{\alpha}\cdot\xi = \sum_{\alpha\in\mathcal{A}} \alpha\xi_\alpha\,, \quad b(t)^\xi = \prod_{\alpha\in\mathcal{A}} b_\alpha(t)^{\xi_\alpha}\,.$$

Thus equation (4.17) confirms the formula

$$q_\beta(x,t) = p_\beta(x,t)\ e^{b_0(t)}\,.$$

But $b_0(t)\le\Lambda_0\,|t|$ where $|a_0(t)|\le\Lambda_0$, giving

$$|q_\beta(x,t)| \le e^{\Lambda_0|t|}\,|p_\beta(x,t)|\ .$$

As p_β has the bound (4.3), the identical bound will hold for q_β because the multiplicative factor $e^{\Lambda_0|t|}$ only adds another term to the sum in the expression

$$\exp\left[\sum_{\alpha\in\mathcal{A}\ ,\ |\alpha|<m} \left(\frac{M+1}{\delta}\right)^{|\alpha|/(m-|\alpha|)} |\Lambda_\alpha t|^{m/(m-|\alpha|)}\right]\,,$$

so that $\alpha\in\mathcal{A}$ changes to $\alpha\in\mathcal{B}$. The number M of multi-indices α satisfying $0<|\alpha|<m$ is the same for $\mathcal{A}$ and $\mathcal{B}$. These remarks complete the proof of the theorem. □

When the operator (4.1) has no lower order terms, assuming the form

$$\mathcal{L}u(x,t) = \partial_t u(x,t) - \sum_{|\alpha|=m} a_\alpha(t)\partial_x^\alpha u(x,t)\ , \tag{4.18}$$

the bound on the polynomials $\{p_\beta\}$ is simplified; we have $M = 0$, while sums over lower order multi-indices are 0. Then in Theorem 4.1 the bound (4.3) reduces to

$$\frac{|p_\beta(x,t)|}{\beta!} \le \frac{(\Lambda\,|t| + \delta)^{|\beta|/m}}{(|\beta|\,/m)!} \exp\left[\frac{(n\,|x|)^{m/(m-1)}}{\delta^{1/(m-1)}}\right]. \tag{4.19}$$

5. Series Expansions

We consider polynomial series of the form

$$u(x,t) = \sum_\beta \frac{c_\beta}{\beta!} p_\beta\,(x,t)\;, \tag{5.1}$$

where again the polynomials $\{p_\beta\}$ are those associated with the operator (4.1). Borrowing techniques from [26, 32, 35, 36, 37], we prescribe conditions on the coefficients $\{c_\beta\}$ that guarantee convergence of the series in a strip

$$S_s = \{(x,t) : x \in \mathbb{R}^n,\ t \in \mathbb{I},\ |t| < s\} \quad . \tag{5.2}$$

Theorem 5.1 below is our main tool for solving the Cauchy problem for the operator $\mathcal{L}$ of (4.1), as described later in Theorem 6.3. The analogous Cauchy problem was solved for the heat equation by Rosenbloom and Widder, and subsequently by Kemnitz and Haimo and Markett for other one dimensional equations, as mentioned earlier.

Given a real valued function f defined on multi-indices β in $\mathbb{R}^n$, we define

$$\limsup_{\beta\to\infty}\ f\,(\beta) := \lim_{k\to\infty}\ \sup_{|\beta|\ge k}\ f\,(\beta)\ .$$

Theorem 5.1. *Under the hypotheses of Theorem 4.1, assume that*

$$\limsup_{\beta\to\infty} \frac{|c_\beta|^{m/|\beta|}\, me\Lambda}{|\beta|} = \frac{1}{s}\,, \tag{5.3}$$

where $0 < s \le \infty$.

(a) Then the series (5.1) converges absolutely and uniformly on compact subsets of the strip S_s to a function u.

(b) Both u and $\partial u/\partial t$ are continuous in this strip, as are their space derivatives $\partial_x^\nu u$ and $\partial_x^\nu u_t$ of all orders. All these derivatives can be obtained from (5.1) by termwise differentiation, with convergence of the differentiated series likewise absolute and uniform on compact subsets of S_s. Finally, u solves in S_s the equation $\mathcal{L}u = 0$, where $\mathcal{L}$ is the operator (4.1), while for each β,

$$c_\beta = \partial_x^\beta u(0,0)\,. \tag{5.4}$$

Proof. Given X, T with $0 < X < \infty$, $0 < T < s$, consider the region $R(X,T)$ containing all points (x,t) such that

$$(x,t) \in \mathbb{R}^n \times \mathbb{I}\,, \quad |x| \le X\,, \quad |t| \le T\,.$$

Given $\delta > 0$, inequality (4.3) shows that, under the hypotheses of Theorem 4.1, there is a constant $C = C\,(X,T,\delta)$, independent of β, such that for all $(x,t) \in R\,(X,T)$,

$$\frac{|p_\beta\,(x,t)|}{\beta!} \le C\frac{(\Lambda T + \delta)^{|\beta|/m}}{(|\beta|\,/m)!}\,. \tag{5.5}$$

By (3.11), in (5.5) we have

$$\frac{1}{(|\beta|/m)!} \le \frac{e^{|\beta|/m}}{(|\beta|/m)^{|\beta|/m}} = \left(\frac{em}{|\beta|}\right)^{|\beta|/m} . \tag{5.6}$$

Let T_1 satisfy $T < T_1 < s$. Under assumption (5.3) there exists $k > 0$ such that $|\beta| \ge k$ implies

$$\frac{|c_\beta|^{m/|\beta|}\, me\Lambda}{|\beta|} \le \frac{1}{T_1} . \tag{5.7}$$

We combine (5.5), (5.6), and (5.7) to conclude that, for $|\beta| \ge k$ and under the hypotheses of Theorem 4.1,

$$|c_\beta| \frac{|p_\beta(x,t)|}{\beta!} \le C \left(\frac{\Lambda T + \delta}{\Lambda T_1}\right)^{|\beta|/m} . \tag{5.8}$$

As Λ is positive and $T_1 > T$, in (5.8) we can choose δ sufficiently small that the quanitity in parentheses on the right is less than some positive constant r, with $r < 1$; then we obtain

$$\sum_{|\beta|\ge k} \left|\frac{c_\beta}{\beta!} p_\beta(x,t)\right| \le C \sum_{|\beta|\ge k} r^{|\beta|/m} < \infty .$$

Thus (5.1) converges absolutely and uniformly in the region $R(X,T)$ – and likewise on compact subsets of S_s because X can be arbitrarily large and T arbitrarily close to s.

Keeping in mind the identity (2.4), we consider a formal space derivative of the series (5.1),

$$\begin{aligned} \partial_x^\nu u(x,t) &= \partial_x^\nu \sum_\beta \frac{c_\beta}{\beta!} p_\beta(x,t) = \sum_\beta \frac{c_\beta}{\beta!} \partial_x^\nu p_\beta(x,t) \\ &= \sum_{\beta\ge\nu} \frac{c_\beta}{(\beta-\nu)!} p_{\beta-\nu}(x,t) = \sum_\beta \frac{c_{\beta+\nu}}{\beta!} p_\beta(x,t) . \end{aligned} \tag{5.9}$$

Under assumption (5.3), for the coefficients in the derived series we have

$$\begin{aligned} &\limsup_{\beta\to\infty} \frac{|c_{\beta+\nu}|^{m/|\beta|}\, me\Lambda}{|\beta|} \\ &= \limsup_{\beta\to\infty} \left[\frac{|c_{\beta+\nu}|^{m/|\beta+\nu|}\, me\Lambda}{|\beta+\nu|}\right]^{|\beta+\nu|/|\beta|} \frac{|\beta+\nu|}{|\beta|} \frac{|\beta+\nu|^{|\nu|/|\beta|}}{(me\Lambda)^{|\nu|/|\beta|}} \\ &= \left(\frac{1}{s}\right)^1 \cdot 1 \cdot \frac{1}{1} = \frac{1}{s} . \end{aligned}$$

We may now apply the result of (a) to deduce that each derived series (5.9) converges absolutely and uniformly on compact subsets of S_s. This conclusion legitimizes our formal termwise differentiations of the series, while also confirming the continuity of the sums.

Next, in view of (2.5), we compute the formal derivative of (5.1) with respect to t as

$$\partial_t \left(\sum_\beta \frac{c_\beta}{\beta!} p_\beta(x,t)\right) = \sum_\beta \sum_{\alpha\le\beta} c_\beta a_\alpha(t) \frac{p_{\beta-\alpha}(x,t)}{(\beta-\alpha)!} , \tag{5.10}$$

where

$$\sum_{\beta}\sum_{\alpha\le\beta}\left|c_{\beta}a_{\alpha}(t)\frac{p_{\beta-\alpha}(x,t)}{(\beta-\alpha)!}\right| = \sum_{\alpha}|a_{\alpha}(t)|\sum_{\beta}|c_{\beta+\alpha}|\,\frac{|p_{\beta}\,(x,t)|}{\beta!}\,.$$

As above, each of the series

$$\sum_{\beta}|c_{\beta+\alpha}|\,\frac{|p_{\beta}(x,t)|}{\beta!}$$

converges uniformly and absolutely on compact subsets of S_s; as the coefficients $\{a_{\alpha}(t)\}$ are bounded, the same is true for the series (5.10). Thus termwise differentiation is justified in (5.10), and the sum of the series is continuous in S_s.

Applying again (2.4) and (2.5), we compute also formal mixed derivatives of (5.1),

$$\partial_t\partial_x^{\nu}\sum_{\beta}\frac{c_{\beta}}{\beta!}p_{\beta}\,(x,t) = \partial_x^{\nu}\partial_t\sum_{\beta}\frac{c_{\beta}}{\beta!}p_{\beta}\,(x,t) = \sum_{\beta}\sum_{\alpha\le\beta}c_{\beta+\nu}a_{\alpha}\,(t)\,\frac{p_{\beta-\alpha}\,(x,t)}{(\beta-\alpha)!}\,,$$

where

$$\sum_{\beta}\sum_{\alpha\le\beta}\left|c_{\beta+\nu}a_{\alpha}\,(t)\,\frac{p_{\beta-\alpha}\,(x,t)}{(\beta-\alpha)!}\right| = \sum_{\alpha}|a_{\alpha}(t)|\sum_{\beta}|c_{\beta+\alpha+\nu}|\,\frac{|p_{\beta}\,(x,t)|}{\beta!}\,.$$

By arguments above, these series likewise converge absolutely and uniformly on compact subsets of S_s, and termwise differentiation and continuity of the sums are justified.

The formula $\mathcal{L}u = 0$ follows by termwise differentiation and the fact that $\mathcal{L}p_{\beta} = 0$ for each β. Finally, setting $(x,t) = (0,0)$ in (5.9) and using $p_{\beta}\,(x,0) = x^{\beta}$ gives $\partial_x^{\nu}u\,(0,0) = c_{\nu}$, and thereby (5.4). $\square$

It is noteworthy in formula (5.3) that bounds on lower order terms do not appear – only highest order bounds affect the estimate on the width of the strip of convergence.

For the special case of the Kemnitz operator in one space dimension,

$$\partial_t u\,(x,t) - \partial_x^m u\,(x,t)\ ,$$

we have $\Lambda = 1$ and (5.3) becomes the estimate of Kemnitz [26].

6. Cauchy Problems

Again we let $\mathcal{L}$ denote the operator (4.1). We consider Cauchy problems

$$\begin{cases} \mathcal{L}u(x,t) = 0\,, \\ u\,(x,0) = f\,(x)\ . \end{cases} \tag{6.1}$$

We shall assume f is a real valued function on $\mathbb{R}^n$, with a power series expansion in $\mathbb{R}^n$ which we write in the two ways

$$f\,(x) = \sum_{\beta}a_{\beta}x^{\beta} = \sum_{\beta}\frac{c_{\beta}}{\beta!}x^{\beta}\,. \tag{6.2}$$

We introduce some terminology regarding growth conditions on the function f. We state the conditions for complex valued analytic functions defined in $\mathbb{C}^n$, as they are more natural in that setting. Whereas for entire functions of a single complex variable there is general agreement on definitions of order and type (see [3], for example), for functions of several complex variables there is no standard terminology, as there are several possible definitions depending on one's choice of norms in $\mathbb{C}^n$ and on other considerations. (See [10], Chapter 1, for a discussion of this topic.) Although it appears that all definitions give the same order of an entire function, the type varies with the definition. The definitions below are most suitable for our purposes, as they allow the natural extension to higher space dimensions of one-dimensional results of Rosenbloom and Widder, and later of Kemnitz, regarding the Cauchy problem (6.1). We explain in Theorem 6.2 how the prescribed conditions on the coefficients of the power series expansion of an entire function are equivalent to growth limits at infinity on the function.

Definition 6.1. Let $f = f(z) = f(z_1, z_2, \cdots, z_n)$ be an entire function f in $\mathbb{C}^n$, with the power series expansion (with complex coefficients)

$$f(z) = \sum_\beta a_\beta z^\beta = \sum_\beta \frac{c_\beta}{\beta!} z^\beta \,. \tag{6.3}$$

Given ρ, τ with $0 < \rho, \tau < \infty$, we say that f has growth $\{\rho, \tau\}$ provided that

$$\limsup_{\beta\to\infty} |\beta|^{1-\rho} \left(\beta^{\beta/|\beta|}\right)^\rho |a_\beta|^{\rho/|\beta|} \le e\tau\rho \,. \tag{6.4}$$

Theorem 6.2. *Assume $0 < \rho, \tau < \infty$, and that $a_\beta = c_\beta/\beta!$ for each β. Then condition (6.4) is equivalent to*

$$\limsup_{\beta\to\infty} \left(\frac{e}{|\beta|}\right)^{\rho-1} |c_\beta|^{\rho/|\beta|} \le \tau\rho \,, \tag{6.5}$$

and under these conditions on the coefficients the series (6.3) converges to an entire function f in $\mathbb{C}^n$ satisfying the growth condition

$$|f(z)| = \mathcal{O}\left(\exp \tau' \left[\sum_{i=1}^n |z_i|\right]^\rho\right) \quad \text{as } z \to \infty, \text{ for all } \tau' > \tau \,. \tag{6.6}$$

Conversely, if f is an entire function in $\mathbb{C}^n$ with growth condition (6.6) and the expansion (6.3), then (6.4) and (6.5) must hold.

Before proving Theorem 6.2 we demonstrate how the coefficient conditions of that theorem pertain to the Cauchy problem (6.1). A real valued function f with an expansion (6.2) in $\mathbb{R}^n$ can be viewed as the restriction to $\mathbb{R}^n$ of an analytic function in $\mathbb{C}^n$ with the expansion (6.3). Accordingly, we may apply Definition 6.1 just as well to real valued functions with expansions (6.2) valid in $\mathbb{R}^n$.

Theorem 6.3. *Under the hypotheses of Theorem 4.1 assume that the function (6.2) has growth $\{m/(m-1), \tau\}$ for some τ, $0 < \tau < \infty$. Then the series*

$$u(x,t) = \sum_\beta \frac{c_\beta}{\beta!} p_\beta(x,t) \tag{6.7}$$

solves the Cauchy problem (6.1) in the strip

$$S_s = \{(x,t) : x \in \mathbb{R}^n \,,\ t \in \mathbb{I} \,,\ |t| < s\} \,,$$

where

$$s = \frac{1}{m\Lambda} \left(\frac{m-1}{m\tau}\right)^{m-1} \,. \tag{6.8}$$

Moreover, the series (6.7) and its sum $u(x,t)$ have the convergence and regularity properties described in Theorem 5.1.

Proof. Under the hypotheses of Theorem 4.1, the growth condition (6.5) on f, with $\rho = (m-1)/m$, implies

$$\limsup_{\beta\to\infty} \frac{|c_\beta|^{m/|\beta|}\, me\Lambda}{|\beta|} \le \frac{1}{s}\,,$$

where s is prescribed by (6.8). Thus Theorem 5.1 applies, confirming the stated convergence properties of the series (6.7) and the regularity properties of its sum $u(x,t)$, and affirming as well the equation $\mathcal{L}u = 0$ in S_s. The initial value of u is

$$u(x,0) = \sum_\beta \frac{c_\beta}{\beta!} p_\beta(x,0) = \sum_\beta \frac{c_\beta}{\beta!} x^\beta = f(x)\,.$$

□

The proof of Theorem 6.2 requires a technical lemma.

Lemma 6.4. *If* $0 < r, \rho < \infty$, *then*

$$\sum_{k=0}^{\infty} \frac{r^k}{k^{k/\rho}} \le (2^\rho r^\rho + 2)\exp\left(\frac{r^\rho}{e\rho}\right)\,. \tag{6.9}$$

Proof. The function $\psi(k) = r^k k^{-k/\rho}$ attains a maximum on $[0,\infty)$ at $k = r^\rho/e$; thus

$$r^k k^{-k/\rho} \le \psi(r^\rho/e) = \exp\left(\frac{r^\rho}{e\rho}\right)\,. \tag{6.10}$$

The sum on the left of (6.9) may be written as $S_1 + S_2$, where

$$S_1 = \sum_{k<2^\rho r^\rho} \frac{r^k}{k^{k/\rho}}\,, \qquad S_2 = \sum_{k\ge 2^\rho r^\rho} \frac{r^k}{k^{k/\rho}}\,.$$

In S_1 we bound each term by (6.10) to obtain

$$S_1 \le \sum_{k<2^\rho r^\rho} \exp\left(\frac{r^\rho}{e\rho}\right) \le (2^\rho r^\rho + 1)\exp\left(\frac{r^\rho}{e\rho}\right)\,.$$

In S_2 we have $r \le k^{1/\rho}/2$, and thus

$$S_2 \le \sum_{k\ge 2^\rho r^\rho} \frac{\left(k^{1/\rho}/2\right)^k}{k^{k/\rho}} \le \sum_{k=1}^{\infty} 2^{-k} = 1 \le \exp\left(\frac{r^\rho}{e\rho}\right)\,.$$

Combining the estimates for S_1 and S_2 gives (6.9). □

Now we prove Theorem 6.2. We use techniques adapted from similar arguments in [3, 10, 35].

First, from (3.11) it follows that for any multi-index β in $\mathbb{R}^n$,

$$\beta^\beta e^{-|\beta|} \le \beta! \le \beta^\beta e^{-|\beta|} e^n \prod_{i=1}^{n} \sqrt{\beta_i + 1}\,.$$

From this inequality and the observation

$$\lim_{|\beta|\to\infty} \left(e^n \prod_{i=1}^{n} \sqrt{\beta_i + 1}\right)^{\rho/|\beta|} = 1\,,$$

it is readily checked that (6.4) is equivalent to (6.5).

Now we assume condition (6.5) holds on the coefficients of (6.3), and verify the series converges to an entire function f in $\mathbb{C}^n$ satisfying (6.6). Let δ be fixed, $\delta > 0$. Then (6.5) guarantees that for $|\beta|$ sufficiently large,

$$|c_\beta| \le [(\tau+\delta)\,\rho]^{|\beta|/\rho} \left(|\beta|/e\right)^{|\beta|-|\beta|/\rho} .$$

But the right hand side of this inequality is bounded below on any finite collection of $\beta' s$ (as we interpret $0^0 = 1$), so for some constant M we have for all β that

$$|c_\beta| \le M\,[(\tau+\delta)\,\rho]^{|\beta|/\rho} \left(|\beta|/e\right)^{|\beta|-|\beta|/\rho} .$$

Then for the series (6.3) we have the bound

$$\begin{aligned}
|f(z)| &\le M \sum_\beta [(\tau+\delta)\,\rho]^{|\beta|/\rho} \left(|\beta|/e\right)^{|\beta|-|\beta|/\rho} \frac{\left|z^\beta\right|}{\beta!} \\
&= M \sum_{k=0}^\infty \frac{[e\rho(\tau+\delta)]^{k/\rho}\, k^{k-k/\rho}}{e^k} \sum_{|\beta|=k} \frac{\left|z^\beta\right|}{\beta!} .
\end{aligned}$$

Writing $\|z\|_1 := \sum_{i=1}^n |z_i|$, we have

$$\left(\|z\|_1\right)^k = \sum_{|\beta|=k} \frac{k!}{\beta!} \prod_{i=1}^n |z_i|^{\beta_i} = k! \sum_{|\beta|=k} \frac{\left|z^\beta\right|}{\beta!} ,$$

and thereby

$$|f(z)| \le M \sum_{k=0}^\infty \frac{[e\rho(\tau+\delta)]^{k/\rho}}{k^{k/\rho}} \frac{k^k \left(\|z\|_1\right)^k}{e^k k!} .$$

By (3.11), $e^k k! \ge k^k$, and consequently

$$|f(z)| \le M \sum_{k=0}^\infty \frac{\left\{[e\rho(\tau+\delta)]^{1/\rho} \|z\|_1\right\}^k}{k^{k/\rho}} .$$

Finally, by Lemma 6.4 the series on the right converges uniformly on compact subsets of $\mathbb{C}^n$, and

$$\begin{aligned}
|f(z)| &\le M\left[2^\rho e\rho(\tau+\delta)\left(\|z\|_1\right)^\rho + 2\right] \exp\left[(\tau+\delta)\left(\|z\|_1\right)^\rho\right] \\
&= \mathcal{O}\left(\exp\left[(\tau+2\delta)\left(\|z\|_1\right)^\rho\right]\right) .
\end{aligned}$$

Since δ can be arbitrarily small, (6.6) follows. Clearly f, the sum in $\mathbb{C}^n$ of a power series, is entire.

Next suppose f is entire in $\mathbb{C}^n$ with the expansion (6.3), and that (6.6) holds. We verify (6.4). Given $\tau' > \tau$, (6.6) implies there is a positive constant M such that

$$|f(z)| \le M \exp\left[\tau' \left(\|z\|_1\right)^\rho\right] .$$

By the generalized Cauchy inequality for analytic functions in $\mathbb{C}^n$ (see [22], Theorem 2.2.7), for any multi-index β and positive vector $r = (r_1, \cdots, r_n)$,

$$\begin{aligned}
|c_\beta| = \left|\partial^\beta f(0)\right| &\le \frac{\beta!}{r^\beta} \sup\{|f(z)| : |z_i| \le r_i\,,\ 1 \le i \le n\} \qquad (6.11) \\
&\le \frac{M\beta!}{r^\beta} \exp\left[\tau' \left(\sum_{i=1}^n r_i\right)^\rho\right] .
\end{aligned}$$

The gradient with respect to r of the expression on the right vanishes when

$$r_i = \frac{\beta_i}{\rho^{1/\rho}\left(\tau'\right)^{1/\rho}|\beta|^{(\rho-1)/\rho}}\,,\ 1 \le i \le n\ ; \tag{6.12}$$

we choose these values of r_i and obtain

$$|a_\beta| = |c_\beta|\,/\beta! \le \frac{M\left(e\tau'\rho\right)^{|\beta|/\rho}|\beta|^{|\beta|-|\beta|/\rho}}{\beta^\beta}\,. \tag{6.13}$$

(In the event that $\beta_i = 0$ for some i, we replace β_i with $\beta_i + \varepsilon_i$ in the numerator on the right side of (6.12), as we require $r_i > 0$; but then (6.13) follows by letting $\varepsilon_i \to 0^+$.) From (6.13) it follows that

$$\limsup_{\beta\to\infty} |\beta|^{1-\rho}\left(\beta^{\beta/|\beta|}\right)^\rho |a_\beta|^{\rho/|\beta|} \le \limsup_{\beta\to\infty} M^{\rho/|\beta|}e\tau'\rho = e\tau'\rho\,.$$

As this inequality holds for all $\tau' > \tau$, we have verified (6.4).

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