On the averaging method for differential equations with delay *

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Abstract

In this paper, we present averaging results for delay differential equations under weak conditions. The main result is formulated in both classical mathematics and nonstandard analysis. It is proved with internal set theory which is an axiomatic description of nonstandard analysis.

1 Introduction

Differential equations with delay frequently appear in the mathematical modelling of some evolution phenomena arising in various fields of science and technology. However, the investigation of these equations is rather difficult, hence there is a need for approximate methods of solution. Among such methods, we have the method of averaging which is a powerful tool that have been used for ordinary differential equations; see for example [1, 2, 7, 10, 15, 17, 25, 26].

This method of averaging has been extended to functional differential equations which include differential equations with delay and ordinary differential equations ($r = 0$), and to more general equations [6, 8, 9, 11, 13, 14, 16, 18, 19, 22, 28]. The most general results are given in [12], where the averaging is discussed for the functional differential equation

$$\dot{x}(t) = \varepsilon f(t, x_t). \tag{1.1}$$

The associated averaged equation is the ordinary differential equation

$$\dot{y}(t) = \varepsilon f^o(\tilde{y}), \quad \tilde{y}(\theta) = y, \quad \theta \in [-r, 0] \tag{1.2}$$

where

$$f^o(u) = \lim_{T \to \infty} \frac{1}{T} \int_s^{s+T} f(\tau, u) d\tau.$$

Under suitable conditions, one can compare solutions of (1.1) and the solutions of (1.2). The techniques used for this comparison follow different approaches. In [12], equation (1.1) is considered as a perturbation of

$$\dot{x}(t) = 0 \cdot x_t,$$

and the solution of (1.1) is decomposed as $x_t = \tilde{I}z(t) + w_t$ where $\tilde{I}(\theta) = I$, the identity, for $\theta \in [-r, 0]$. Then, conditions are derived such that w_t approaches zero faster than any exponential. By use of the invariant manifold theory, it is shown that the flow for (1.1) in any bounded set is equivalent to the flow defined by an ordinary differential equation

$$\dot{z}(t) = \varepsilon g(t, z, \varepsilon), \qquad g(t, z, 0) = f(t, \tilde{z}). \tag{1.3}$$

Also, classical averaging procedures for ordinary differential equations are applied to (1.3) obtaining the approximation of solutions of (1.1) by solutions of (1.2). In the particular case of differential equations with delay of the form

$$\dot{x}(t) = \varepsilon f(t, x(t), x(t - r)), \tag{1.4}$$

to relate the solutions of (1.4) and the averaged ordinary differential equation

$$\dot{y}(t) = \varepsilon f^o(y(t), y(t))$$

where

$$f^o(x_1, x_2) = \lim_{R \to \infty} \frac{1}{R} \int_s^{s+R} f(\tau, x_1, x_2) d\tau,$$

authors such as Halanay [8], Medvedev [22] and Volosov [28] propose near identity change of coordinates, similar to the ones proposed in the case of ordinary differential equations. Then it is shown that as $\varepsilon \to 0$, all "delay" terms become negligible. Averaging procedures of ordinary differential equations can then be applied.

Note that, if we let $t \mapsto t/\varepsilon$ and $x(t/\varepsilon) = z(t)$, equation (1.1) becomes

$$\dot{z}(t) = f\left(\frac{t}{\varepsilon}, z_{t,\varepsilon}\right)$$

with $z_{t,\varepsilon}(\theta) = z(t + \varepsilon\theta)$, $\theta \in [-r, 0]$, which is an equation with a small delay.

There are a few works dedicated to the use of the method of averaging to functional differential equations in the general case

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x_t\right). \tag{1.5}$$

In [12], the authors introduce an extension of the method of averaging to abstract evolutionary equations in Banach spaces. In particular, they rewrite equation (1.5) as an ordinary differential equation in an infinite dimensional Banach space and proceed formally from there.

The approach in [19] differs from [12] since all the analysis is kept in the associated natural phase space. The first result given there generalizes the corresponding one of [16] where the differential equation with delay of the particular form

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x(t - r)\right)$$

is considered.

The purpose of this paper is to consider differential equations with delay of the form

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x(t), x(t-r)\right) \tag{1.6}$$

and to show that the averaging approach works under much more general conditions (which are, in particular, less restrictive than those ones in the existing literature).

The organization of this paper is as follows. Notation and hypotheses required to state and prove our main result as well as the main result itself are presented in Section 2. The proof of this result is given in Subsection 4.2. To avoid complicating the proof unnecessarily several subsidiary lemmas have been placed in Subsection 4.1.

The main result is formulated in both classical mathematics and nonstandard analysis. For its proof we use *Internal Set Theory* (IST) which is an axiomatic approach, given by Nelson [23], of the *Nonstandard Analysis* (NSA) of Robinson [24]. For that, Appendix A in the end of the paper is devoted to a short description of IST. In Section 3 we present the nonstandard translate (Theorem 3.1) in the language of IST of our main result (Theorem 2.2). As IST is a conservative extension of ordinary mathematics, that is, every classical statement which is proved in IST is a theorem of ordinary mathematics, we do not give a classical proof.

2 Notation, Hypotheses, and Main Result

In this section we introduce notation and hypotheses that are used throughout this paper. We also state our main result.

Let $r \geq 0$ be given. By $\mathcal{C}_o = \mathcal{C}([-r, 0], \mathbb{R}^d)$ we denote the Banach space of all continuous functions from $[-r, 0]$ into $\mathbb{R}^d$ with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\},$$

where $|.|$ is any convenient norm in $\mathbb{R}^d$. Let $t_0 \in \mathbb{R}$ and $L \geq 0$. For any continuous function $x : [t_0 - r, t_0 + L] \to \mathbb{R}^d$ and any $t \in [t_0, t_0 + L]$, we denote by x_t the element of $\mathcal{C}_o$ defined by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-r, 0].$$

Here $x_t(.)$ represents the history of the state from time $t - r$ up to the present time t.

We list the following hypotheses:

(H1) The function f in (1.6) is continuous on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$.

(H2) The continuity of $f = f(\tau, x_1, x_2)$ in x_1, x_2 is uniform with respect to τ.

(H3) For all $x_1, x_2 \in \mathbb{R}^d$ there exists a limit

$$f^o(x_1, x_2) := \lim_{R \to \infty} \frac{1}{R} \int_s^{s+R} f(\tau, x_1, x_2) d\tau$$

which is uniform with respect to $s \in \mathbb{R}_+$.

(H4) The averaged equation

$$\dot{y}(t) = f^o(y(t), y(t-r)) \qquad (2.1)$$

has the uniqueness of the solutions with the prescribed initial conditions.

Remark 2.1 Using hypotheses (H1)-(H3) we will prove in Lemma 4.1 below that the function $f^o : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous so that the existence of solutions of (2.1) is guaranteed.

The main result of this paper reads as follows.

Theorem 2.2 *Assume that hypotheses (H1)-(H4) hold. Let $\phi \in \mathcal{C}_0$. Let y be the solution of (2.1) with $y_0 = \phi$, and let J be its maximal interval of definition. For any $L > 0$, $L \in J$, and any $\delta > 0$ there exists an $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$, any solution x of (1.6) with $x_0 = \phi$, which is defined on $[0, L]$, satisfies the inequality $|x(t) - y(t)| < \delta$ on $t \in [0, L]$.*

3   Nonstandard Main Result

Hereafter we give the nonstandard formulation of Theorem 2.2. Then, by use of the reduction algorithm, we show that the reduction of Theorem 3.1 bellow is Theorem 2.2.

Theorem 3.1 *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be standard. Assume that hypotheses (H1)-(H4) hold. Let $\phi \in \mathcal{C}_o$ be standard. Let y be the solution of (2.1) with $y_0 = \phi$, and let J be its maximal interval of definition. Let $\varepsilon > 0$ be infinitesimal. Then for any standard $L > 0$, $L \in J$, any solution x of (1.6) with $x_0 = \phi$, which is defined on $[0, L]$, is such that $x(t) \simeq y(t)$ for all $t \in [0, L]$.*

The proof of Theorem 3.1 is postponed to Section 4. Theorem 3.1 is an external statement. We show that the reduction of Theorem 3.1 is Theorem 2.2.

Reduction of Theorem 3.1.   The characterization of the conclusion of Theorem 3.1 is

$$\forall \varepsilon : \ \varepsilon \simeq 0 \implies \begin{array}{l} \text{any solution } x \text{ of (1.6) with } x_0 = \phi, \text{ which is defined} \\ \text{on } [0, L], \text{ is such that } x(t) \simeq y(t) \ \text{ for all } t \in [0, L]. \end{array} \qquad (3.1)$$

Let B be the formula "If $\delta > 0$ then any solution x of (1.6) with $x_0 = \phi$, which is defined on $[0, L]$, satisfies the inequality $|x(t) - y(t)| < \delta$ on $t \in [0, L]$". Using (5.2) (see Appendix A), formula (3.1) becomes

$$\forall \varepsilon \ (\forall^{st}\eta \ \varepsilon < \eta \implies \forall^{st}\delta \ B). \qquad (3.2)$$

In this formula L is standard and ε, η and δ range over the strictly positive real numbers. By (5.1) (see Appendix A), formula (3.2) is equivalent to

$$\forall \delta\ \exists^{fin} \eta'\ \forall \varepsilon\ (\forall \eta \in \eta'\ \varepsilon < \eta \implies B). \tag{3.3}$$

For η' a finite set, $\forall \eta \in \eta'\ \varepsilon < \eta$ is the same as $\varepsilon < \varepsilon_0$ for $\varepsilon_0 = \min \eta'$, and so formula (3.3) is equivalent to

$$\forall \delta\ \exists \varepsilon_0\ \forall \varepsilon\ (\varepsilon < \varepsilon_0 \implies B).$$

That is the statement of Theorem 2.2 holds for $L > 0$, L standard. By transfer, it holds for any $L > 0$. $\diamondsuit$

4 Proof of Theorem 3.1

4.1 Preliminary Lemmas

In this subsection we give some results we need for the proof of Theorem 3.1. Let $\varepsilon > 0$ be infinitesimal. Let $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be standard. The external formulations of conditions (H1)-(H3) are, respectively:

(H1') $\forall^{st}\tau \in \mathbb{R}_+ \quad \forall^{st}x_1, x_2 \in \mathbb{R}^d \quad \forall \tau' \in \mathbb{R}_+ \quad \forall x_1', x_2' \in \mathbb{R}^d$:

$$\tau' \simeq \tau,\ x_1' \simeq x_1,\ x_2' \simeq x_2 \Longrightarrow f(\tau', x_1', x_2') \simeq f(\tau, x_1, x_2).$$

(H2') $\forall^{st}x_1, x_2 \in \mathbb{R}^d \quad \forall x_1', x_2' \in \mathbb{R}^d \quad \forall \tau \in \mathbb{R}_+$:

$$x_1' \simeq x_1,\ x_2' \simeq x_2 \Longrightarrow f(\tau, x_1', x_2') \simeq f(\tau, x_1, x_2).$$

(H3') There is a standard function $f^o : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\forall^{st}x_1, x_2 \in \mathbb{R}^d\ \forall s \in \mathbb{R}_+\ \forall R \simeq +\infty :\ f^o(x_1, x_2) \simeq \frac{1}{R} \int_s^{s+R} f(\tau, x_1, x_2)d\tau.$$

We prove the following lemmas:

Lemma 4.1 *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be standard. Assume that hypotheses (H1')-(H3') hold. Then the function f^o is continuous and satisfies*

$$f^o(x_1, x_2) \simeq \frac{1}{R} \int_s^{s+R} f(\tau, x_1, x_2)d\tau$$

for all $x_1, x_2 \in \mathbb{R}^d$, x_1, x_2 nearstandard, all $s \in \mathbb{R}_+$ and all $R \simeq +\infty$.

Proof. Let $x_1, x_2, {}^ox_1, {}^ox_2 \in \mathbb{R}^d$ such that ${}^ox_1, {}^ox_2$ are standard and $x_1 \simeq {}^ox_1$, $x_2 \simeq {}^ox_2$. Let $s \in \mathbb{R}_+$. Let $\nu > 0$ be infinitesimal. By condition (H3) there exists $R_0 > 0$ such that, for $R > R_0$

$$\left| f^o(x_1, x_2) - \frac{1}{R} \int_s^{s+R} f(\tau, x_1, x_2)d\tau \right| < \nu.$$

Hence for some $R \simeq +\infty$ we have

$$f^o(x_1, x_2) \simeq \frac{1}{R} \int_s^{s+R} f(\tau, x_1, x_2) d\tau.$$

By condition (H2') we have $f(\tau, x_1, x_2) \simeq f(\tau, {}^o x_1, {}^o x_2)$ for $\tau \in \mathbb{R}_+$. Therefore

$$f^o(x_1, x_2) \simeq \frac{1}{R} \int_s^{s+R} f(\tau, {}^o x_1, {}^o x_2) d\tau.$$

From (H3') we deduce that $f^o(x_1, x_2) \simeq f^o({}^o x_1, {}^o x_2)$. Thus f^o is continuous. Moreover for $s \in \mathbb{R}_+$ and $R \simeq +\infty$ we have

$$\begin{aligned}
f^o(x_1, x_2) &\simeq f^o({}^o x_1, {}^o x_2) \\
&\simeq \frac{1}{R} \int_s^{s+R} f(\tau, {}^o x_1, {}^o x_2) d\tau \\
&\simeq \frac{1}{R} \int_s^{s+R} f(\tau, x_1, x_2) d\tau.
\end{aligned}$$

$\diamondsuit$

Lemma 4.2 *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be standard. Assume that hypotheses (H1')-(H3') hold. Let $\varepsilon > 0$ be infinitesimal. Let $\alpha > 0$ be infinitesimal such that $\alpha/\varepsilon \simeq \infty$. Then*

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} f(\tau, x_1, x_2) d\tau \simeq T.f^o(x_1, x_2)$$

for all $x_1, x_2 \in \mathbb{R}^d$, x_1, x_2 nearstandard, all $t \in \mathbb{R}_+$ and all $T \in [0, 1]$.

Proof. Let x_1, x_2 nearstandard in $\mathbb{R}^d$, $t \in \mathbb{R}_+$ and $T \in [0, 1]$.
Case 1. T is such that $T\alpha/\varepsilon$ is unlimited. By Lemma 4.1 we have

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} f(\tau, x_1, x_2) d\tau \simeq T.f^o(x_1, x_2).$$

Case 2. T is such that $T\alpha/\varepsilon$ is limited. By means of Lemma 4.1 we obtain

$$\begin{aligned}
&\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} f(\tau, x_1, x_2) d\tau \\
&= \frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, x_1, x_2) d\tau - \frac{\varepsilon}{\alpha} \int_{t/\varepsilon + T\alpha/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon + (1-T)\alpha/\varepsilon} f(\tau, x_1, x_2) d\tau \\
&\simeq f^o(x_1, x_2) - (1 - T)f^o(x_1, x_2) = T.f^o(x_1, x_2).
\end{aligned}$$

$\diamondsuit$

Lemma 4.3 *Let $X : [-r, 0] \times [0, 1] \to \mathbb{R}^d$, $F : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $G : \mathbb{R}_+ \to \mathbb{R}^d$ be continuous fonctions. Let $\alpha > 0$ be infinitesimal. Suppose that*

i) $\alpha X(-r, T) \simeq 0 \ \forall T \in [0, 1]$.

ii) $X(0, T) = \int_0^T F(s, X(0, s), X(-r, s)) ds \ \forall T \in [0, 1]$.

iii) $F(T, x_1, x_2) \simeq G(T) \ \forall T \in [0, 1] \ \forall x_1, x_2 \in \mathbb{R}^d$: $\alpha x_1 \simeq 0 \simeq \alpha x_2$.

iv) $\int_0^T G(s) ds$ *is limited for all* $T \in [0, 1]$.

Then $X(0, \cdot)$ *is limited on* $[0, 1]$ *and satisfies*

$$X(0, T) \simeq \int_0^T G(s) ds \quad \forall T \in [0, 1].$$

Proof.   Suppose there exists $\overline{T} \in [0, 1]$ such that $X(0, \overline{T}) \simeq \infty$. Let T be such that $0 \leq T \leq \overline{T}$, $X(0, T) \simeq \infty$ and $\alpha X(0, s) \simeq 0$ for all $s \in [0, T]$. Then we have

$$X(0, T) = \int_0^T F(s, X(0, s), X(-r, s)) ds \simeq \int_0^T G(s) ds.$$

Hence $X(0, T)$ is limited. This is a contradiction. $\diamond$

Lemma 4.4 *Let* $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *be standard. Assume that hypotheses (H1')-(H3') hold. Let* $\phi \in \mathcal{C}_o$ *be standard. Let* $\varepsilon > 0$ *be infinitesimal. Let* $\alpha > 0$ *be infinitesimal such that* $\alpha/\varepsilon \simeq \infty$. *Let* x *be a solution of (1.6) with* $x_0 = \phi$. *Suppose that* x *is defined on* $[0, r]$ *and* $x(t)$ *is nearstandard for all* $t \in [0, r]$. *Then for all* $t_0 \in [0, r - \alpha]$ *we have*

i) $\dfrac{x(t_0 + \alpha) - x(t_0)}{\alpha} \simeq f^o(x(t_0), x(t_0 - r))$.

ii) $x(t_0 + \alpha T) \simeq x(t_0) \ \forall T \in [0, 1]$.

Proof.   Let $t_0 \in [0, r - \alpha]$. Define a change of variables as follows:

$$\begin{gathered} T = \frac{t - t_0}{\alpha}, \quad X(0, T) = \frac{x(t_0 + \alpha T) - x(t_0)}{\alpha}, \\ X(-r, T) = \frac{x(t_0 + \alpha T - r) - x(t_0 - r)}{\alpha}, \ T \in [0, 1], \end{gathered} \tag{4.1}$$

where

$$X(\theta, T) = \frac{x(t_0 + \alpha T + \theta) - x(t_0 + \theta)}{\alpha}, \quad \text{for } \theta \in [-r, 0] \text{ and } T \in [0, 1].$$

We have $\alpha X(-r, T) \simeq 0 \ \forall T \in [0, 1]$. Indeed, let $T \in [0, 1]$. Taking into account that $t_0 + \alpha T - r \in [-r, 0]$ and $x_{|_{[-r, 0]}} \equiv \phi$, using the S-continuity of ϕ, we have

$$x(t_0 + \alpha T - r) = \phi(t_0 + \alpha T - r) \simeq \phi(t_0 - r) = x(t_0 - r)$$

which is equivalent to $\alpha X(-r, T) \simeq 0$.

Now, under the change of variables (4.1) equation (1.6) becomes

$$\frac{dX}{dT}(0, T) = f\big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}T, x(t_0) + \alpha X(0, T), x(t_0 - r) + \alpha X(-r, T)\big).$$

As $x(t_0)$ and $x(t_0 - r)$ are nearstandard, so are $x(t_0) + \alpha X_1$ and $x(t_0 - r) + \alpha X_2$, for all X_1, X_2 in $\mathbb{R}^d$ such that αX_1 and αX_2 are infinitesimal. Then by (H2') we have

$$f\big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}T, x(t_0) + \alpha X_1, x(t_0 - r) + \alpha X_2\big) \simeq f\big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}T, x(t_0), x(t_0 - r)\big)$$

for all $T \in [0, 1]$ and all $X_1, X_2 \in \mathbb{R}^d$ such that αX_1 and αX_2 are infinitesimal. On the other hand, for $T \in [0, 1]$, by Lemma 4.2 we have

$$
\begin{aligned}
\int_0^T f\big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}\tau, x(t_0), x(t_0 - r)\big)d\tau &= \frac{\varepsilon}{\alpha}\int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(\tau, x(t_0), x(t_0 - r))d\tau \\
&\simeq T.f^o(x(t_0), x(t_0 - r))
\end{aligned}
$$

which is limited for all $T \in [0, 1]$. Therefore by Lemma 4.3 we have

$$X(0, T) \simeq T.f^o(x(t_0), x(t_0 - r)) \quad \forall T \in [0, 1].$$

Hence we deduce that

$$\frac{x(t_0 + \alpha) - x(t_0)}{\alpha} = X(0, 1) \simeq f^o(x(t_0), x(t_0 - r))$$

and

$$x(t_0 + \alpha T) - x(t_0) = \alpha X(0, T) \simeq 0 \quad \forall T \in [0, 1],$$

which completes the proof. $\diamondsuit$

Lemma 4.5 *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be standard. Assume that hypotheses (H1')-(H3') hold. Let $\phi \in \mathcal{C}_o$ be standard. Let $\varepsilon > 0$ be infinitesimal. Let $\alpha > 0$ be infinitesimal such that $\alpha/\varepsilon \simeq \infty$. Let x be a solution of (1.6) with $x_0 = \phi$. Suppose that x is defined on $[0, 2r]$ and $x(t)$ is nearstandard for all $t \in [0, 2r]$. Then for all $t_0 \in [0, 2r - \alpha]$ we have*

i) $\dfrac{x(t_0 + \alpha) - x(t_0)}{\alpha} \simeq f^o(x(t_0), x(t_0 - r)).$

ii) $x(t_0 + \alpha T) \simeq x(t_0) \ \forall T \in [0, 1].$

Proof. Let $t_0 \in [0, 2r - \alpha] = [0, r - \alpha] \cup [r - \alpha, 2r - \alpha]$.
Case 1. $t_0 \in [0, r - \alpha]$. By Lemma 4.4, the conclusion of Lemma 4.5 holds.
Case 2. $t_0 \in [r - \alpha, 2r - \alpha]$. Consider the change of variables (4.1). We will show that $\alpha X(-r, T) \simeq 0 \ \forall T \in [0, 1]$. Let $T \in [0, 1]$. As $t_0 - r \in [-\alpha, r - \alpha] = [-\alpha, 0] \cup [0, r - \alpha]$, consider each of the following cases:

1. $t_0 - r \in [-\alpha, 0]$. Then $t_0 + \alpha T - r \in [-\alpha, \alpha] = [-\alpha, 0] \cup [0, \alpha]$.

a. $t_0 + \alpha T - r \in [-\alpha, 0]$. Since $x_{|_{[-r,0]}} \equiv \phi$ and ϕ is S-continuous it follows that

$$x(t_0 + \alpha T - r) = \phi(t_0 + \alpha T - r) \simeq \phi(t_0 - r) = x(t_0 - r),$$

that is, $\alpha X(-r, T) \simeq 0$.

b. $t_0 + \alpha T - r \in [0, \alpha]$. By Lemma 4.4, ii) we have

$$x(t_0 + \alpha T - r) \simeq x(0).$$

On the other hand, taking into account that $x_{|_{[-r,0]}} \equiv \phi$, by use of the S-continuity of ϕ we obtain

$$x(t_0 - r) = \phi(t_0 - r) \simeq \phi(0).$$

Hence $x(t_0 + \alpha T - r) \simeq x(t_0 - r)$, that is, $\alpha X(-r, T) \simeq 0$.

2. $t_0 - r \in [0, r - \alpha,]$. By Lemma 4.4, ii) we have

$$x(t_0 + \alpha T - r) \simeq x(t_0 - r),$$

that is, $\alpha X(-r, T) \simeq 0$.

At this stage, the rest of the proof is the same one as in Lemma 4.4.   ◇

By induction we have the following lemma.

Lemma 4.6 *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be standard. Assume that hypotheses (H1')-(H3') hold. Let $\phi \in \mathcal{C}_o$ be standard. Let $\varepsilon > 0$ be infinitesimal. Let $\alpha > 0$ be infinitesimal such that $\alpha/\varepsilon \simeq \infty$. Let x be a solution of (1.6) with $x_0 = \phi$. Suppose that x is defined on $[0, mr]$ and $x(t)$ is nearstandard for all $t \in [0, mr]$, where m is a standard positive integer. Then for all $t_0 \in [0, mr - \alpha]$ we have*

i) $\dfrac{x(t_0 + \alpha) - x(t_0)}{\alpha} \simeq f^o(x(t_0), x(t_0 - r)).$

ii) $x(t_0 + \alpha T) \simeq x(t_0) \ \forall T \in [0, 1].$

Using this lemma, it is easy to prove the following result.

Lemma 4.7 *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be standard. Assume that hypotheses (H1')-(H3') hold. Let $\phi \in \mathcal{C}_o$ be standard. Let $\varepsilon > 0$ be infinitesimal. Let $\alpha > 0$ be infinitesimal such that $\alpha/\varepsilon \simeq \infty$. Let x be a solution of (1.6) with $x_0 = \phi$. Let $L_1 > 0$ be standard such that x is defined on $[0, L_1]$ and $x(t)$ is nearstandard for all $t \in [0, L_1]$. Then for all $t_0 \in [0, L_1 - \alpha]$ we have*

i) $\dfrac{x(t_0 + \alpha) - x(t_0)}{\alpha} \simeq f^o(x(t_0), x(t_0 - r)).$

ii) $x(t_0 + \alpha T) \simeq x(t_0) \ \forall T \in [0, 1].$

As a consequence of Lemma 4.7 we have:

Lemma 4.8 *Assume that all hypotheses in Lemma 4.7 hold. Let $\{t_n : n = 0, \ldots, N_o + 1\}$ be the infinitesimal partition of $[0, L_1 - \alpha]$ given by $t_0 = 0$, $t_{N_o} \leq L_1 - \alpha < t_{N_o+1}$, and for $n \in \{0, \ldots, N_o + 1\}$, $t_n = n\alpha$. Then for all $n \in \{0, \ldots, N_o\}$ we have*

i) $\dfrac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \simeq f^o(x(t_n), x(t_n - r))$.

ii) $x(t) \simeq x(t_n) \quad \forall t \in [t_n, t_{n+1}]$.

Lemma 4.9 *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be standard. Assume that hypotheses (H1')-(H3') hold. Let $\phi \in \mathcal{C}_o$ be standard. Let $\varepsilon > 0$ be infinitesimal. Let x be a solution of (1.6) with $x_0 = \phi$. Let $L_1 > 0$ be standard such that x is defined on $[0, L_1]$ and $x(t)$ is nearstandard for all $t \in [0, L_1]$. Then x is S-continuous on $[0, L_1]$.*

Proof. Let $\alpha > 0$, $\alpha \simeq 0$ and $\alpha/\varepsilon \simeq \infty$. Let $t, t' \in [0, L_1] = [0, L_1 - \alpha] \cup [L_1 - \alpha, L_1]$ such that $t \leq t'$ and $t \simeq t'$.

Case 1. $t, t' \in [0, L_1 - \alpha]$. Then there exist positive integers p and q, $p, q \in \{0, \ldots, N_o\}$, such that $t \in [t_p, t_{p+1}]$ and $t' \in [t_q, t_{q+1}]$, where the t_n are determined by Lemma 4.8. By Lemma 4.8, i) we have

$$\begin{aligned}
x(t_q) - x(t_p) &= \sum_{k=p}^{q-1} (x(t_{k+1}) - x(t_k)) \\
&= \sum_{k=p}^{q-1} (t_{n+1} - t_n)[f^o(x(t_k), x(t_k - r)) + \eta_k], \quad \eta_k \simeq 0.
\end{aligned} \tag{4.2}$$

Let $\eta = \max\{|\eta_p|, \ldots, |\eta_{q-1}|\}$ and $m = \max\{|f^o(x(t_k), x(t_k-r))| : k = \overline{p, q-1}\}$. We have $\eta \simeq 0$ and $m = |f^o(x(t_s), x(t_s-r))|$ for some $s \in \{p, \ldots, q-1\}$. As f^o is standard and continuous and $x(t_s)$, $x(t_s - r)$ are nearstandard, $f^o(x(t_s), x(t_s - r))$ is nearstandard and so is m. By Lemma 4.8, ii), (4.2) implies that

$$|x(t') - x(t)| \simeq |x(t_q) - x(t_p)| \leq (m + \eta)(t_q - t_p) \simeq 0.$$

Case 2. $t, t' \in [L_1 - \alpha, L_1]$. By lemma 4.7, ii) we have

$$x(t) \simeq x(L_1 - \alpha) \simeq x(t').$$

Case 3. $t \in [0, L_1 - \alpha]$ and $t' \in [L_1 - \alpha, L_1]$. Taking into account that in this case $t \simeq L_1 - \alpha \simeq t'$, we have first

$$x(t) \simeq x(L_1 - \alpha) \quad \text{(see Case 1 above)}$$

and next

$$x(L_1 - \alpha) \simeq x(t') \quad \text{(see Case 2 above)}.$$

Hence, $x(t) \simeq x(t')$. Thus, in all cases we have $x(t) \simeq x(t')$, that is, x is S-continuous on $[0, L_1]$. $\diamond$

4.2   Proof of Theorem 2

We are now able to prove our main result (there is not much work left). We assume that the hypotheses in Theorem 3.1 hold. The proof will be given in two steps:

Step 1. Let $L > 0$, L standard and $L \in J$, and let x be a solution of (1.6) with $x_0 = \phi$. We suppose that x is defined on $[0, L]$. Let K be a standard tubular neighborhood of radius ρ around the trajectory of y on $[0, L]$. Let $L_1 > 0$ be standard such that

$$L_1 \leq L \quad \text{and} \quad x([0, L_1]) \subset K. \tag{4.3}$$

Consider the standard function $z : [-r, L_1] \to \mathbb{R}^d$ defined by

$$z(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ {}^o x(t), & t \in [0, L_1]. \end{cases}$$

We will prove that z is a solution of (2.1). Then, by the uniqueness of the solutions of (2.1) (hypothesis (H4)), we deduce that $z \equiv y$ on $[-r, L_1]$ so that $x(t) \simeq {}^o x(t) = z(t) = y(t)$ for $t \in [0, L_1]$, which completes the first part of the proof.

Indeed, as $x(t) \in K$ for all $t \in [0, L_1]$ and K is standard and compact, $x(t)$ is nearstandard (in K) for all $t \in [0, L_1]$. By Lemma 4.9, x is S-continuous on $[0, L_1]$ and then z is continuous. Let us show that for all $t \in [0, L_1]$

$$z(t) = \phi(0) + \int_0^t f^o(z(\tau), z(\tau - r)) d\tau.$$

Let $\alpha > 0$, $\alpha \simeq 0$ and $\alpha/\varepsilon \simeq \infty$. Let $t \in [0, L_1] = [0, L_1 - \alpha] \cup [L_1 - \alpha, L_1]$ be standard.

Case 1. $t \in [0, L_1 - \alpha]$. Then there exists $p \in \{0, \ldots, N_o\}$ such that $t \in [t_p, t_{p+1}]$ where the t_n are determined by Lemma 4.8. By Lemma 4.8 we have

$$\begin{aligned} z(t) - \phi(0) &\simeq x(t_p) - x(0) \\ &= \sum_{k=0}^{p-1} (x(t_{k+1}) - x(t_k)) \\ &= \sum_{k=0}^{p-1} (t_{k+1} - t_k)[f^o(x(t_k), x(t_k - r)) + \eta_k], \quad \eta_k \simeq 0. \end{aligned} \tag{4.4}$$

As f^o is standard and continuous, and $x(t_k) \simeq z(t_k)$, $x(t_k - r) \simeq z(t_k - r)$ with $x(t_k)$, $x(t_k - r)$ nearstandard, we have $f^o(x(t_k), x(t_k - r)) \simeq f^o(z(t_k), z(t_k - r))$. Then (4.4) implies that

$$\begin{aligned} z(t) - \phi(0) &\simeq \sum_{k=0}^{p-1} (t_{k+1} - t_k)[f^o(z(t_k), z(t_k - r)) + \beta_k + \eta_k], \quad \beta_k \simeq 0 \\ &\simeq \sum_{k=0}^{p-1} (t_{k+1} - t_k) f^o(z(t_k), z(t_k - r)) \\ &\simeq \int_0^t f^o(z(\tau), z(\tau - r)) d\tau. \end{aligned}$$

Case 2. $t \in [L_1 - \alpha, L_1]$. As t is standard, $t = L_1$. Consider the interval $[0, L_1 - \alpha]$. We have $L_1 - \alpha \in [t_{N_o}, t_{N_o+1}]$ where the t_n, $n \in \{0, \dots, N_o\}$, are determined by Lemma 4.8. As $t_{N_o+1} \simeq t_{N_o}$, $z(t_{N_o})$ and $z(t_{N_o} - r)$ are nearstandard, and f^o is standard and continuous, we have $(t_{N_o+1} - t_{N_o})f^o(z(t_{N_o}), z(t_{N_o} - r)) \simeq 0$ so that

$$
\begin{aligned}
z(L_1) - \phi(0) &\simeq z(L_1 - \alpha) - \phi(0) \simeq x(t_{N_o}) - x(0) \\
&\simeq \sum_{n=0}^{N_o-1} (t_{n+1} - t_n)f^o(z(t_n), z(t_n - r)) \quad \text{(see Case 1 above)} \\
&\simeq \sum_{n=0}^{N_o} (t_{n+1} - t_n)f^o(z(t_n), z(t_n - r)) \\
&\simeq \int_0^{L_1} f^o(z(\tau), z(\tau - r))d\tau.
\end{aligned}
$$

Thus, in all cases we have

$$
z(t) \simeq \phi(0) + \int_0^t f^o(z(\tau), z(\tau - r))d\tau. \tag{4.5}
$$

As both sides of (4.5) are standard we have

$$
z(t) = \phi(0) + \int_0^t f^o(z(\tau), z(\tau - r))d\tau \tag{4.6}
$$

and by transfer (4.6) holds for all $t \in [0, L_1]$, that is, z is a solution of (2.1).
Step 2. It remains to prove that L satisfies (4.3). Suppose that this is false. Then there exists $t_1 \in (0, L]$ such that

$$
x(t_1) \in \partial K, \tag{4.7}
$$

where ∂K is the boundary of K. We may suppose that t_1 is the first time such that (4.7) holds. Since x and y are continuous, $x(0) = y(0)$, and (by (4.7)) $|x(t_1) - y(t_1)| = \rho$, we deduce that there exists $t_2 \in (0, t_1)$ such that $|x(t_2) - y(t_2)| = \rho/2$. It is clear that $0 \not\simeq t_2 \not\simeq t_1$. Let $^o t_2$ be the standard part of t_2. We have $|x(^o t_2) - y(^o t_2)| \simeq \rho/2$ which implies that $x(^o t_2) \in K$. However, this contradicts the conclusion of Step 1 above, that is, $x(t) \simeq y(t)$ for $t \in [0, {}^o t_2]$ since $x([0, {}^o t_2]) \subset K$. The proof is complete. $\diamondsuit$

Remark 4.10 It appears clearly throughout the proof of Theorem 3.1 that it is not necessary to consider the whole space $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$. One can restrict the domain of definition of f to $\mathbb{R}_+ \times D \times D$ for any standard $D \subset \mathbb{R}^d$ with the assumption that y lies on $[0, L]$ in the interior of D.

5 Internal Set Theory

In IST we adjoin to ordinary mathematics (say ZFC) a new undefined unary predicate *standard* (st). The axioms of IST are the usual axioms of ZFC plus

three others which govern the use of the new predicate. Hence, *all theorems* of ZFC *remain valid* in IST. What is new in IST is an addition, not a change. We call a formula of IST *external* in the case where it involves the new predicate *st*; otherwise, we call it *internal*. Thus internal formulas are the formulas of ZFC. The theory IST is a *conservative extension* of ZFC, that is, every internal theorem of IST is a theorem of ZFC. Some of the theorems which are proved in IST are external and can be reformulated so that they become internal. Indeed, there is a *reduction algorithm* which reduces any external formula $F(x_1, \ldots, x_n)$ of IST without other free variables than $x_1, \ldots, x_n$, to an internal formula $F'(x_1, \ldots, x_n)$ with the same free variables, such that $F \equiv F'$, that is, $F \Longleftrightarrow F'$ for all standard values of the free variables. In other words, any result which may be formalized within IST by a formula $F(x_1, \ldots, x_n)$ is equivalent to the classical property $F'(x_1, \ldots, x_n)$, provided the parameters $x_1, \ldots, x_n$ are restricted to standard values. Here is the reduction of the frequently occurring formula $\forall x \, (\forall^{st} y \, A \Longrightarrow \forall^{st} z \, B)$ where A and B are internal formulas

$$\forall x \, (\forall^{st} y \, A \Longrightarrow \forall^{st} z \, B) \;\equiv\; \forall z \, \exists^{fin} y' \, \forall x \, (\forall y \in y' \, A \Longrightarrow B). \qquad (5.1)$$

A real number x is called *infinitesimal* when $|x| < a$ for all standard $a > 0$, *limited* when $|x| \le a$ for some standard a, *appreciable* when it is limited and not infinitesimal, and *unlimited*, when it is not limited. We use the following notations: $x \simeq 0$ for x infinitesimal, $x \simeq +\infty$ for x unlimited positive, $x \gg 0$ for x non infinitesimal positive. Thus we have

$$
\begin{array}{rcl}
x \simeq 0 & \Longleftrightarrow & \forall^{st} a > 0 \; |x| < a \\
x \gg 0 & \Longleftrightarrow & \exists^{st} a > 0 \; x \ge a \\
x \text{ limited} & \Longleftrightarrow & \exists^{st} a > 0 \; |x| \le a \\
x \simeq +\infty & \Longleftrightarrow & \forall^{st} a > 0 \; x > a.
\end{array}
\qquad (5.2)
$$

Let (E, d) be a standard metric space. Two points x and y in E are called *infinitely close*, denoted $x \simeq y$, when $d(x,y) \simeq 0$. If there exists in that space a standard x_0 such that $x \simeq x_0$, the element x is called *nearstandard* in E and the standard point x_0 is called the *standard part* of x (it is unique) and is also denoted ${}^o x$. A vector in $\mathbb{R}^d$ (d standard) is said to be *infinitesimal* (resp. *limited*, *unlimited*) if its norm $|x|$ is infinitesimal (resp. limited, unlimited), where $|.|$ is a norm in $\mathbb{R}^d$.

We may not use external formulas to define subsets. The notations $\{x \in \mathbb{R} : x \text{ is limited}\}$ or $\{x \in \mathbb{R} : x \simeq 0\}$ are not allowed. Moreover we can prove that

Lemma 5.1 *There do not exist subsets $\mathcal{L}$ and $\mathcal{I}$ of $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, x is in $\mathcal{L}$ if and only if x is limited, or x is in $\mathcal{I}$ if and only if x is infinitesimal.*

It happens sometimes in classical mathematics that a property is assumed, or proved, on a certain domain, and that afterwards it is noticed that the character of the property and the nature of the domain are incompatible. So actually the property must be valid on a large domain. In the same manner, in Nonstandard Analysis, the result of Lemma 5.1 is frequently used to prove that the validity of a property exceeds the domain where it was established in direct way. Suppose

that we have shown that A holds for every limited x, then we know that A holds for some unlimited x, for otherwise we could let $\mathcal{L} = \{x \in \mathbb{R} : A\}$. This statement is called the *Cauchy principle*. It has the following consequence.

Lemma 5.2 (Robinson's Lemma) *Let g be a real function such that $g(t) \simeq 0$ for all limited $t \in \mathbb{R}_+$, then there exists $\omega \simeq +\infty$ such that $g(t) \simeq 0$ for all $t \in [0, \omega]$.*

Proof. The set of all s such that for all $t \in [0, s]$ we have $|g(t)| < 1/s$ contains all limited $s \geq 1$. By the Cauchy principle it must contain some unlimited ω. $\diamond$

We conclude this section with the following remark.

Remark 5.3 The use of Nonstandard Analysis in perturbation theory of differential equations goes back to the seventies with the Reebian school (cf. [20, 21, 27] and the references therein). It gave birth to the nonstandard perturbation theory of differential equations which has became today a well-established tool in asymptotic theory (see the special five-digits classification 34E18 of the 2000 Mathematics Subject Classification). To have an idea of the rich literature on the subject, the reader is referred to [3, 4, 5, 23, 24].

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