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# An embedding theorem for Campanato spaces *

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**Abstract**

The purpose of this paper is to give a Sobolev type embedding theorem for the spaces $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$. The homogeneous versions of these spaces contain well known spaces such as the Bounded Mean Oscillation spaces (BMO) and the Campanato spaces $\mathcal{L}^{2,\lambda}$. Our result extends some injections obtained by Campanato [3, 4], Strichartz [11], and Stein and Zygmund [10].

## 1 Introduction and statement of results

The main goal of this work is to give a Sobolev type embedding theorem for the appropriate scaled functions in $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ whose homogeneous version $\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ contains some well known spaces as special cases: John and Nirenberg space BMO (Bounded Mean Oscillation) and more generally Campanato spaces $\mathcal{L}^{2,\lambda}$ modulo polynomials [5].

It is well known that the homogeneous Triebel-Lizorkin $\dot{F}^s_{p,q}(\mathbb{R}^n)$ spaces coincide with BMO modulo polynomials for some values of $p, q$ and $s$. Namely, $BMO = \dot{F}^0_{\infty,2}$ [12, chap. 5] and thus $I^s(BMO) = \dot{F}^s_{\infty,2}$, where $I^s = \mathcal{F}^{-1}(|\cdot|^{-s}\mathcal{F})$ is the Riesz potential operator. Strichartz [11] discussed the connexion between $I^s(BMO)$ and the homogeneous Besov space $\dot{\Lambda}^s_{\infty,q}$ and proved the following injections (Theorem 3.4.)

$$\dot{\Lambda}^s_{\infty,2} \subseteq I^s(BMO) \subseteq \dot{\Lambda}^{s}_{\infty,\infty}$$

Let us mention that others classical embeddings have been obtained respectively by Stein and Zygmund [10] and Campanato [3, 4]; they proved that

$$H^{n/p}_p \hookrightarrow BMO \quad \text{and} \quad \mathcal{L}^{p,\lambda} \cong C^{\frac{\lambda-n}{p}} \quad \text{if } n < \lambda < n+p$$

here $H^{n/p}_p$ is the Bessel potential space and $\mathcal{L}^{p,\lambda}$ is the Campanato space (cf. Definition 1.5). In this paper we extend these injections to a more general frame (Theorem 1.8), by giving some embedding results between the spaces $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ (which include $I^s(BMO)$ and $I^s(\mathcal{L}^{2,\lambda})$ as special cases), Triebel-Lizorkin spaces

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$F^s_{p,q}(\mathbb{R}^n)$ (containing $H^{n/p}_p$ as particular case) and Besov-Peetre spaces $B^s_{p,q}(\mathbb{R}^n)$, and on the other side between the same spaces $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ and Hölder-Zygmund ones $C^s(\mathbb{R}^n)$. Such embeddings shed some light on duals of the closure of Schwartz space $\mathcal{S}(\mathbb{R}^n)$ in BMO and in Campanato spaces $\mathcal{L}^{2,\lambda}$ (Corollary 3.1).

The present article is organised as follows: first, we give the definition of the $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ spaces and their homogeneous counterparts $\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ which invoke a Littlewood-Paley partition of unity and some expressions with balls (or cubes) of $\mathbb{R}^n$, and we state the main result (Theorem 1.8). Next, we proceed to a discussion of some properties of these spaces: we give the connexion between the nonhomogeneous spaces $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ and the homogeneous ones, and we specify the differential dimension of the dotted spaces which depends on $p, q, s$, and $\lambda$. Finally, we prove Theorem 1.8 and Corollary 3.1.

To define these spaces, we will need a Littlewood-Paley partition of unity. Denote $x \in \mathbb{R}^n$ and $\xi$ its dual variable. Consider the cutoff function $\varphi \in C^\infty_0(\mathbb{R}^n)$, $\varphi \geq 0$ and $\varphi$ equal to 1 for $|\xi| \leq 1$, 0 for $|\xi| \geq 2$. Let $\theta(\xi) = \varphi(\xi) - \varphi(2\xi)$ which satisfies $\operatorname{supp}\theta \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$. For $j \in \mathbb{Z}$ we set

$$\dot{\Delta}_j u = \theta(2^{-j} D_x) u$$

and

$$\Delta_0 u = \varphi(D_x) u,\ \Delta_j u = \dot{\Delta}_j u \text{ if } j \geq 1$$

**Remark 1.1** We have $\varphi(\xi) + \sum_{k \geq 1} \theta(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^n$, thus the nonhomogeneous partition of $u \in \mathcal{S}'(\mathbb{R}^n)$ is given by the formula

$$u = \sum_{k \geq 0} \Delta_k u$$

Set $f(\xi) = \sum_{k \in \mathbb{Z}} \theta(2^{-k}\xi)$; for each fixed $\xi \neq 0$, $f(\xi)$ contains at most two nonvanishing terms, we deduce $f(\xi) = 1$ for any $|\xi| \geq 2$. Note that $f(2^{-j}\xi) = f(\xi)$ for all $j \in \mathbf{Z}$, and choose $j$ more and more large, to obtain $f(\xi) = 1$ for any $\xi \neq 0$. Then, for $u \in \mathcal{S}'(\mathbb{R}^n)$, with $0 \notin \operatorname{supp}\mathcal{F}u$, the homogeneous partition of $u$ is given by the formula

$$u = \sum_{k \in \mathbb{Z}} \dot{\Delta}_k u$$

Now, let us define the nonhomogeneous space $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$.

**Definition 1.2** Let $s \in \mathbb{R}, \lambda \geq 0$, $1 \leq p < +\infty$ and $1 \leq q < +\infty$. The space $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ denotes the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)} = \Big\{ \sup_B \frac{1}{|B|^{\lambda/n}} \sum_{j \geq J^+} 2^{jqs} \|\Delta_j u\|^q_{L^p(B)} \Big\}^{1/q} < +\infty \tag{1.1}$$

where $J^+ = \max(J, 0)$, $|B|$ is the measure of $B$ and the supremum is taken over all $J \in \mathbb{Z}$ and all balls $B$ of $\mathbb{R}^n$ with radius $2^{-J}$. When $p = q$, the space $\mathcal{L}^{\lambda,s}_{p,p}(\mathbb{R}^n)$ will be denoted $\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)$.

Note that the space $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ equipped with the norm (1.1) is a Banach space.

To give the homogeneous counterpart of the space $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$, we recall the notations of [12, chap. 5]. Let

$$Z(\mathbb{R}^n) = \big\{\varphi \in \mathcal{S}(\mathbb{R}^n); (D^\alpha \mathcal{F}\varphi)(0) = 0 \text{ for every multi-index } \alpha\big\}.$$

The space $Z(\mathbb{R}^n)$ is considered as a subspace of $\mathcal{S}(\mathbb{R}^n)$ with the induced topology, and $Z'(\mathbb{R}^n)$ its topological dual. We may identify $Z'(\mathbb{R}^n)$ with $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, where $\mathcal{P}(\mathbb{R}^n)$ is the set of all polynomials of $\mathbb{R}^n$ with complex coefficients i.e. $Z'(\mathbb{R}^n)$ is interpreted as $\mathcal{S}'(\mathbb{R}^n)$ modulo polynomials.

**Definition 1.3** Let $s \in \mathbb{R}, \lambda \geq 0$, $1 \leq p < +\infty$ and $1 \leq q < +\infty$. The dotted space $\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ denotes the set of all $u \in Z'(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)} = \Big\{\sup_B \frac{1}{|B|^{\lambda/n}} \sum_{j \geq J} 2^{jqs} \|\dot{\Delta}_j u\|^q_{L^p(B)}\Big\}^{1/q} < +\infty \tag{1.2}$$

where the supremum is taken over all $J \in \mathbb{Z}$ and all balls $B$ of $\mathbb{R}^n$ with radius $2^{-J}$.

If $p = q$, the space $\dot{\mathcal{L}}^{\lambda,s}_{p,p}(\mathbb{R}^n)$ will be denoted $\dot{\mathcal{L}}^{p,\lambda,s}(\mathbb{R}^n)$. If $P$ is a polynomial of $\mathcal{P}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, it follows immediatly that

$$\|u + P\|_{\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)} = \|u\|_{\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)}$$

This shows that the norm (1.2) is well defined. Further, the space $\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ equipped with this norm is a Banach space.

**Remark 1.4** The supremum given in expressions (1.1) and (1.2) may be taken over all $J \in \mathbb{Z}$ and all cubes $B$ of $\mathbb{R}^n$ of length side $2^{-J}$.

Now we define Campanato spaces modulo polynomials $\dot{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)$.

**Definition 1.5** Let $\lambda \geq 0$ and $1 \leq p < +\infty$.

(i) The space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ denotes the set of (classes of) functions $u \in L^p_{loc}(\mathbb{R}^n)$ such that

$$\|u\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = \Big\{\sup_B \frac{1}{|B|^{\lambda/n}} \int_B |u - m_B u|^p dx\Big\}^{1/p} < +\infty$$

where $m_B u = \frac{1}{|B|}\int_B u(y)dy$ is the mean value of $u$ and the supremum is taken over all balls $B$ of $\mathbb{R}^n$.

Note that $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ is a Banach space modulo constants equal to $\{0\}$ if $\lambda > n+p$.

(ii) The space $\dot{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)$ denotes the set of functions $u \in Z'(\mathbb{R}^n)$ such that $\|u\|_{\dot{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)} < +\infty$, where

$$\|u\|_{\dot{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)} = \begin{cases} \inf_{P\in\mathcal{P}} \|u+P\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} & \text{if } u \in L^1_{loc}(\mathbb{R}^n) \\ +\infty & \text{if } u \in Z'(\mathbb{R}^n) \text{ but} \\ & \text{not locally integrable} \end{cases}$$

the infimum is taken over the set $\mathcal{P}$ of all polynomials of $\mathbb{R}^n$ with complex coefficients.

**Remark 1.6** (i) Modulo polynomials, the space $\dot{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)$ is the Campanato space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$. For $0 \le \lambda < n+p$, $\dot{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)$ can be defined as the set of all equivalence classes modulo $\mathcal{P}$ of elements of $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$, equipped with the following norm $\|U\|_{\dot{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)} = \|u\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)}$ where $u$ is the unique (modulo a constant) element of $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ belonging to the class $U$.

(ii) Using a result of John and Nirenberg [9] it is classical that if $\lambda = n$, then for every $1 \le p < +\infty$, $\mathcal{L}^{p,n}(\mathbb{R}^n)$ coincides with the space $BMO\,(\mathbb{R}^n)$ of all functions $u \in L^1_{loc}\,(\mathbb{R}^n)$ such that there exists a constant $C > 0$ satisfying the following inequality

$$\frac{1}{|B|}\int_B |u - m_B u| dx \le C$$

for all balls $B$ of $\mathbb{R}^n$. We show in [5] that in general $\dot{\mathcal{L}}^{2,\lambda,0}(\mathbb{R}^n) \equiv \dot{\mathcal{L}}^{2,\lambda}(\mathbb{R}^n)$ provided $0 \le \lambda < n+2$; therefore the characterization of Littlewood-Paley type of Campanato spaces $\dot{\mathcal{L}}^{2,\lambda}(\mathbb{R}^n)$ is given. This result is not true in general for $p \ne 2$, nevertheless if $p \ge 2$ and $0 \le \lambda < n+p$ we show that $\dot{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)$ is continuously embedded in $\dot{\mathcal{L}}^{p,\lambda,0}(\mathbb{R}^n)$.

Now, recall the definition of some function spaces.

**Definition 1.7** Let $s \in \mathbb{R}$, $1 \le p < +\infty$ and $1 \le q < +\infty$.

(i) We denote $F^s_{p,q}(\mathbb{R}^n)$ the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{F^s_{p,q}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} \Big(\sum_{j\ge 0} 2^{jqs}|\Delta_j u|^q\Big)^{p/q} dx\Big)^{1/p} < +\infty$$

(ii) The space $B^s_{\infty,q}(\mathbb{R}^n)$ denotes the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{B^s_{\infty,q}(\mathbb{R}^n)} = \Big\{\sum_{k\ge 0} 2^{kqs}\|\Delta_k u\|^q_{L^\infty(\mathbb{R}^n)}\Big\}^{1/q} < +\infty$$

(iii) We denote $C^s(\mathbb{R}^n)$ the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{C^s(\mathbb{R}^n)} = \sup_{j\ge 0} 2^{js}\|\Delta_j u\|_{L^\infty(\mathbb{R}^n)} < +\infty$$

(iv) If we replace in (i), (ii) and (iii) $\mathcal{S}'(\mathbb{R}^n)$ by $\mathcal{Z}'(\mathbb{R}^n)$, $\Delta_j$ by $\dot{\Delta}_j$ and $j \geq 0$ by $j \in \mathbb{Z}$, we obtain respectively the definition of the homogeneous spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$, $\dot{B}^s_{\infty,q}(\mathbb{R}^n)$ and $\dot{C}^s(\mathbb{R}^n)$.

Here is the main result of this paper.

**Theorem 1.8** *Let $s \in \mathbb{R}$, $1 \leq p < +\infty$, $1 \leq q < +\infty$ and $\lambda \geq 0$. We have the following continuous embeddings*

$$\mathcal{L}^{\lambda, s+\frac{n}{p}-\frac{\lambda}{q}}_{p,q}(\mathbb{R}^n) \hookrightarrow C^s(\mathbb{R}^n)$$

$$F^{s+\frac{n}{p}}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{\lambda, s+\frac{n}{p}-\frac{\lambda}{q}}_{p,q}(\mathbb{R}^n) \text{ provided } q \geq p$$

$$F^{s+\frac{n}{p}}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{p,\lambda, s-\frac{\lambda-n}{p}}(\mathbb{R}^n) \text{ provided } p \geq q.$$

$$B^{s-\frac{n}{p}+\frac{\lambda}{q}}_{\infty,q}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n) \text{ provided } \lambda \geq n\frac{q}{p}$$

*and finally if $\lambda \geq n$ then*

$$\sum_{j\geq 0} 2^{jq(s+\frac{\lambda-n}{q})}|\Delta_j u|^q \in L^\infty(\mathbb{R}^n) \text{ implies } u \in \mathcal{L}^{q,\lambda,s}(\mathbb{R}^n)$$

*We have also the same continuous embeddings if we replace $B, C, F$ and $\mathcal{L}$ respectively by the dotted spaces $\dot{B}, \dot{C}, \dot{F}$ and $\dot{\mathcal{L}}$.*

**Remark 1.9** (i) In the case $s = 0$, S. Campanato [3, 4] showed that if $n < \lambda < n + p$ we have $\mathcal{L}^{p,\lambda} \cong C^{\frac{\lambda-n}{p}}$ and $\mathcal{L}^{p,n+p} = Lip$ (cf. [8] too).

(ii) If $\lambda = n$ and $p = q = 2$, we obtain the theorem 3.4. of R. S. Strichartz [11], namely $\dot{B}^{s}_{\infty,2}(\mathbb{R}^n) \hookrightarrow I^s(BMO) \hookrightarrow \dot{C}^{s}(\mathbb{R}^n)$, where $I^s$ is the Riesz potential operator and $I^s(BMO) = \dot{\mathcal{L}}^{2,n,s}(\mathbb{R}^n)$, cf. [5], BMO is defined modulo polynomials. We note that Theorem 1.8 generalise Theorem 2.1 of [6].

(iii) Choosing $s = 0$, $p = q = 2$ and $\lambda = n$ in the third embedding, we find a result due to E. M. Stein and A. Zygmund [10]

$$\dot{H}^{\frac{n}{2}} = \dot{F}^{\frac{n}{2}}_{2,2} \hookrightarrow \dot{\mathcal{L}}^{2,n,0}(\mathbb{R}^n) = BMO \text{ modulo polynomials}$$

# 2 Properties of the spaces

We start with some helpful lemmas needed in the further considerations.

**Lemma 2.1** *Let $1 \leq p < +\infty$, and $A$ a real $< 0$. If $(a_{j\nu})_{j,\nu}$ is a sequence of positive real numbers satisfying $(a_{j\nu})_j \in l^p$ for all $\nu \geq 1$, then there exists a constant $C > 0$ such that*

$$\sum_{j\geq 1}\Big(\sum_{\nu\geq 1} 2^{\nu A} a_{j\nu}\Big)^p \leq C \sup_{\nu\geq 1}\sum_{j\geq 1} a^p_{j\nu}.$$

**Proof** Let $\frac{1}{p}+\frac{1}{q}=1$. Using Hölder's inequality we get

$$\begin{aligned}\sum_{j\geq 1}\Big(\sum_{\nu\geq 1}2^{\nu A}a_{j\nu}\Big)^p &\leq \sum_{j\geq 1}\Big(\sum_{\nu\geq 1}2^{\frac{\nu}{2}A}(2^{\frac{\nu}{2}A}a_{j\nu})\Big)^p\\ &\leq \sum_{j\geq 1}\Big(\Big(\sum_{\nu\geq 1}2^{\frac{\nu}{2}Aq}\Big)^{p/q}\sum_{\nu\geq 1}2^{\frac{\nu}{2}Ap}a_{j\nu}^p\Big)\\ &\leq C\sum_{j\geq 1}\sum_{\nu\geq 1}2^{\frac{\nu}{2}Ap}a_{j\nu}^p\\ &\leq C\sum_{\nu\geq 1}2^{\frac{\nu}{2}Ap}\sum_{j\geq 1}a_{j\nu}^p \leq C\sup_{\nu\geq 1}\sum_{j\geq 1}a_{j\nu}^p\end{aligned}$$

It is known that $\dot{\Delta}_l$ is uniformly bounded on $L^p(\mathbb{R}^n)$. On the other hand, on $L^p(B), B\subset\mathbb{R}^n$, we have the following result. We denote $\alpha B$, $\alpha>0$, the ball with the same center $x_0$ as $B$ and of radius $\alpha r$, $r$ is the radius of $B$.

**Lemma 2.2** *For each integer $M>0$, there exists a constant $C_M>0$, such that for any $J\in\mathbb{Z}$, $x_0\in\mathbb{R}^n$, for any ball $B$ centered at $x_0$ and of radius $2^{-J}$, for any $l\in\mathbb{Z}$ and $u\in L^p(\mathbb{R}^n), 1\leq p\leq+\infty$,*

$$\|A_l u\|_{L^p(B)}\leq C_M\Big\{\|u\|_{L^p(2B)}+\sum_{\nu\geq -J+1}2^{-(l+\nu)M}\|u\|_{L^p(F_\nu)}\Big\}$$

*holds with $F_\nu=\big\{x\in\mathbb{R}^n; 2^\nu\leq|x-x_0|\leq 2^{\nu+1}\big\}$ and $A_l$ denotes any product of $\Delta_l$, $S_l$ and the dotted operators.*

**Proof** We give the proof for $A_l=\dot{\Delta}_l$. Remark that

$$u=\chi_{2B}u+\sum_{\nu\geq -J+1}\chi_{F_\nu}u$$

where $\chi_\Omega$ is the characteristic function of the set $\Omega$. Thus

$$\begin{aligned}\|\dot{\Delta}_l u\|_{L^p(B)} &\leq \|\dot{\Delta}_l\chi_{2B}u\|_{L^p(B)}+\sum_{\nu\geq -J+1}\|\dot{\Delta}_l\chi_{F_\nu}u\|_{L^p(B)}\\ &\leq C\|u\|_{L^p(2B)}+\sum_{\nu\geq -J+1}\|\dot{\Delta}_l\chi_{F_\nu}u\|_{L^p(B)},\end{aligned}$$

where $C$ is a constant independent of $l$ and $B$. Now

$$\dot{\Delta}_l\chi_{F_\nu}u(x)=2^{nl}\int_{F_\nu}u(y)\mathcal{F}^{-1}\theta(2^l(x-y))dy$$

For $x\in B$ and $y\in F_\nu$, we have $|x-y|\sim|x_0-y|\sim 2^\nu$. Since $\theta\in\mathcal{S}(\mathbb{R}^n)$, for every integer $N$, there exists $C_N>0$ such that

$$|\mathcal{F}^{-1}\theta\left(2^l(x-y)\right)|\leq C_N 2^{-(l+\nu)N}$$

We deduce

$$\begin{aligned} |\dot{\Delta}_l \chi_{F_\nu} u(x)| &\leq C_N 2^{nl} 2^{-(l+\nu)N} \int_{F_\nu} |u(y)| dy \\ &\leq C_N 2^{nl} 2^{-(l+\nu)N} \|u\|_{L^p(F_\nu)} |F_\nu|^{1-\frac{1}{p}} \\ &\leq C_N 2^{-(l+\nu)(N-n)} 2^{-\nu \frac{n}{p}} \|u\|_{L^p(F_\nu)} \end{aligned}$$

It follows

$$\|\dot{\Delta}_l \chi_{F_\nu} u(x)\|_{L^p(B)} \leq C_N 2^{-(l+\nu)(N-n)} 2^{-\frac{n}{p}(\nu+J)} \|u\|_{L^p(F_\nu)}$$

Therefore

$$\|\dot{\Delta}_l u\|_{L^p(B)} \leq C\|u\|_{L^p(2B)} + C_N \sum_{\nu \geq -J+1} 2^{-(l+\nu)(N-n)} \|u\|_{L^p(F_\nu)}$$

Put $M = N - n > 0$, and the proof is complete. ♢

**Remark 2.3** Note that

$$|\mathcal{F}^{-1}\theta\left(2^l (x-y)\right)| \leq \|\mathcal{F}^{-1}\theta\|_{L^\infty(\mathbb{R}^n)}$$

Thus we may improve the statement of the last lemma by replacing the term $2^{-(l+\nu)M}$ by $inf(1, 2^{-(l+\nu)M})$.

Here is a classical lemma [1].

**Lemma 2.4** *Let $\lambda > 0$, $1 \leq p \leq q \leq +\infty$. For any $u \in L^p(\mathbb{R}^n)$ with* $\operatorname{supp} \mathcal{F}u \subset \{|\xi| \leq \lambda\}$ *and for any multi-index $\alpha$, there exists $C_\alpha > 0$ such that*

$$\|D_x^\alpha u\|_{L^q(\mathbb{R}^n)} \leq C_\alpha \lambda^{|\alpha| + n(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p(\mathbb{R}^n)}$$

**Lemma 2.5** *Let $R$ be a real $> 1$; there exists $C > 0$ such that for any $\lambda > 0$ and $1 \leq p \leq +\infty$, for any $u \in L^p(\mathbb{R}^n)$ with* $\operatorname{supp} \mathcal{F}u \subset \{\lambda \leq |\xi| \leq R\lambda\}$ *and for any $k \in \mathbb{N}$ we have*

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C^k \lambda^{-k} \sup_{|\alpha|=k} \|D_x^\alpha u\|_{L^p(\mathbb{R}^n)}$$

**Proof** The above lemma gives that $D_x^\alpha u \in L^p(\mathbb{R}^n)$ for all multi-index $\alpha$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi(\xi) = 1$ on a neighbourhood of the annulus $\{1 \leq |\xi| \leq R\}$ and $\psi(\xi) = 0$ on a neighbourhood of 0. Setting $\psi_\lambda(\xi) = \psi(\frac{\xi}{\lambda})$ and $\psi_{\lambda,j} = \frac{\xi_j}{|\xi|^2}\psi_\lambda$ we obtain $\sum_{j=1}^n \xi_j \psi_{\lambda,j}(\xi) = 1$ on a neighbourhood of $\operatorname{supp} \mathcal{F}u$. We deduce

$$u = \sum_{j=1}^n F_{\lambda,j} * D_{x_j} u \quad \text{where } F_{\lambda,j} = \mathcal{F}^{-1}\psi_{\lambda,j} \in \mathcal{S}(\mathbb{R}^n) \tag{2.2}$$

We have $F_{\lambda,j}(x) = \lambda^{n-1}F_{1,j}(\lambda x)$, $\|F_{\lambda,j}\|_{L^1} = \lambda^{-1}\|F_{1,j}\|_{L^1}$ and using (2.2) we deduce lemma 2.5 for $k = 1$. For the general case we do an induction on $k > 1$. $\diamondsuit$

The following lemma yields the homogeneous decomposition modulo polynomials of a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ without taking in account the condition $0 \notin \operatorname{supp} \mathcal{F}u$ that we have met in remark 1.1.

**Lemma 2.6** *For any $u \in \mathcal{S}'(\mathbb{R}^n)$ we have the decomposition*

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \quad \text{in } Z'(\mathbb{R}^n).$$

**Proof** The series $\sum_{j=k}^{+\infty} \dot{\Delta}_j u$ converges in $\mathcal{S}'(\mathbb{R}^n)$ for all $k \in \mathbb{Z}$. It suffices to show that for any $\psi \in Z(\mathbb{R}^n)$,

$$\lim_{k\to-\infty} \Big\langle \sum_{j=k}^{+\infty} \dot{\Delta}_j u, \check{\psi} \Big\rangle_{\mathcal{S}'\times\mathcal{S}} = \Big\langle u, \check{\psi} \Big\rangle_{\mathcal{S}'\times\mathcal{S}}$$

where $\check{\psi}(x) = \psi(-x)$ for $x \in \mathbb{R}^n$. Now

$$\sum_{j=k}^{+\infty} \dot{\Delta}_j u - u = \sum_{j=k}^{0} \dot{\Delta}_j u - \Delta_0 u = \varphi(2^{-k} D_x) u$$

which is a smooth function. Thus

$$(2\pi)^n \Big\langle \sum_{j=k}^{+\infty} \dot{\Delta}_j u - u, \check{\psi} \Big\rangle_{\mathcal{S}'\times\mathcal{S}} = \big\langle \mathcal{F}u, \varphi(2^{-k}.)\mathcal{F}\psi \big\rangle_{\mathcal{S}'\times\mathcal{S}}$$

Since $\mathcal{F}u \in \mathcal{S}'(\mathbb{R}^n)$, there exists a constant $C > 0$ and an integer $N$ such that

$$\begin{aligned}
&(2\pi)^n \Big| \Big\langle \sum_{j=k}^{+\infty} \dot{\Delta}_j u - u, \check{\psi} \Big\rangle_{\mathcal{S}'\times\mathcal{S}} \Big| \\
&\le C \sup_{|\alpha|+|\beta|\le N} \sup_{\xi\in\mathbb{R}^n} |\xi^\alpha D_\xi^\beta (\varphi(2^{-k}\xi)\mathcal{F}\psi(\xi))| \\
&\le C' \sup_{|\alpha|+|\beta|\le N} \sup_{|\xi|\le 2^{k+1}} \sum_{\gamma\le\beta} 2^{k(|\alpha|-|\beta-\gamma|)} |(D_\xi^{\beta-\gamma}\varphi)(2^{-k}\xi)| |D_\xi^\gamma \mathcal{F}\psi(\xi)|
\end{aligned}$$

By the assumption $\psi \in \mathcal{Z}(\mathbb{R}^n)$ and Taylor's formula, for all nonnegative large integer $M$

$$(2\pi)^n \Big| \Big\langle \sum_{j=k}^{+\infty} \dot{\Delta}_j u - u, \check{\psi} \Big\rangle_{\mathcal{S}'\times\mathcal{S}} \Big| \le C 2^{k(M-N)} \underset{k\to-\infty}{\longrightarrow} 0$$

The proof is complete. $\diamondsuit$

Now we state the connexion between $\mathcal{L}_{p,q}^{\lambda,s}(\mathbb{R}^n)$ and $\dot{\mathcal{L}}_{p,q}^{\lambda,s}(\mathbb{R}^n)$.

**Lemma 2.7** *Let $1 \le p, q < +\infty$, $\lambda \ge 0$ and $s \in \mathbb{R}$.*

*(i) If the class of $u$ modulo $\mathcal{P}$ belongs to $\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ and if $\Delta_0 u \in L^p(\mathbb{R}^n)$, then $u \in \mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$.*

*(ii) $L^p(\mathbb{R}^n) \cap \dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n) \subset \mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ with the same meaning as (i).*

*(iii) $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n) \subset \dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ provided $s > 0$.*

**Remark 2.8** It follows that if $s > 0$ then

$$L^p(\mathbb{R}^n) \cap \dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap \mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n).$$

**Proof of Lemma 2.7** (i) If $u \in \dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$, then $u \in \mathcal{S}'(\mathbb{R}^n)$ and there exists a constant $M > 0$ such that for any ball $B$ of $\mathbb{R}^n$ with radius $2^{-J}$ with $J \in \mathbb{Z}$,

$$\frac{1}{|B|^{\lambda/n}} \sum_{l \ge J} 2^{lsq} \|\dot{\Delta}_l u\|^q_{L^p(B)} \le M < +\infty\,. \tag{2.3}$$

Let $J \in \mathbb{Z}$ and $B$ be a ball of $\mathbb{R}^n$ with radius $2^{-J}$. If $J \ge 1$, then inequality (2.3) gives

$$\frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{l \ge J^+} 2^{lsq} \|\Delta_l u\|^q_{L^p(B)} \le M$$

If $J \le 0$, we have

$$\begin{aligned}\frac{1}{|B|^{\lambda/n}} \sum_{l \ge J^+} 2^{lsq} \|\Delta_l u\|^q_{L^p(B)} &= \frac{1}{|B|^{\lambda/n}} \|\Delta_0 u\|^q_{L^p(B)} + \\ \frac{1}{|B|^{\lambda/n}} \sum_{l \ge 1} 2^{lsq} \|\Delta_l u\|^q_{L^p(B)} &\le C 2^{J\lambda} \|\Delta_0 u\|^q_{L^p(B)} + M \\ &\le C \|\Delta_0 u\|^q_{L^p(\mathbb{R}^n)} + M\end{aligned}$$

Since $\Delta_0 u \in L^p(\mathbb{R}^n)$, we obtain

$$\frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{l \ge J^+} 2^{lsq} \|\Delta_l u\|^q_{L^p(B)} \le C' M$$

Hence $u \in \mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$.
(ii) follows from (i).
(iii) Let $u \in \mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$. There exists a constant $M > 0$ such that for any ball $B$ of $\mathbb{R}^n$ with radius $2^{-J}$ and $J \in \mathbb{Z}$,

$$\frac{1}{|B|^{\lambda/n}} \sum_{l \ge J^+} 2^{lsq} \|\Delta_l u\|^q_{L^p(B)} \le M < +\infty\,.$$

We have to show (2.3). Let $J \in \mathbb{Z}$ and $B$ a ball of $\mathbb{R}^n$ of radius $2^{-J}$. If $J \geq 1$, then $J^+ = J$ and (2.3) is valid. If $J \leq 0$ (i.e. $J^+ = 0$). We have

$$\begin{aligned} \frac{1}{|B|^{\lambda/n}} \sum_{l \geq J} 2^{lsq} \|\dot{\Delta}_l u\|^q_{L^p(B)} &= \frac{1}{|B|^{\lambda/n}} \sum_{l=J}^{0} 2^{lsq} \|\dot{\Delta}_l u\|^q_{L^p(B)} + \\ \frac{1}{|B|^{\lambda/n}} \sum_{l \geq 1} 2^{lsq} \|\dot{\Delta}_l u\|^q_{L^p(B)} &\leq \frac{1}{|B|^{\lambda/n}} \sum_{l=J}^{0} 2^{lsq} \|\dot{\Delta}_l u\|^q_{L^p(B)} + M \end{aligned} \tag{2.4}$$

Using the nonhomogeneous decomposition of $u$ we have for $l \leq 0$

$$\dot{\Delta}_l u = \sum_{k \geq 0} \Delta_k \dot{\Delta}_l u = \Delta_0 \dot{\Delta}_l u + \Delta_1 \dot{\Delta}_l u$$

and then

$$\|\dot{\Delta}_l u\|_{L^p(B)} \leq \|\Delta_0 \dot{\Delta}_l u\|_{L^p(B)} + \|\Delta_1 \dot{\Delta}_l u\|_{L^p(B)}. \tag{2.5}$$

Lemma 2.2 gives for $M$ large,

$$\|\dot{\Delta}_l \Delta_0 u\|^q_{L^p(B)} \leq C_M \{ \|\Delta_0 u\|^q_{L^p(2B)} + 2^{-(l-J)Mq} [\sum_{\nu \geq 1} 2^{-\nu M} \|\Delta_0 u\|_{L^p(B_{\nu - J})}]^q \}$$

where $B_{\nu - J} = 2^{\nu+1} B$. Thus

$$\begin{aligned} &\frac{1}{|B|^{\lambda/n}} \sum_{l=J}^{0} 2^{lsq} \|\dot{\Delta}_l \Delta_0 u\|^q_{L^p(B)} \\ &\leq \frac{C}{|B|^{\lambda/n}} \|\Delta_0 u\|^q_{L^p(2B)} \sum_{l=J}^{0} 2^{lsq} + \frac{1}{|B|^{\lambda/n}} \sum_{l=J}^{0} 2^{lsq} \Big( \sum_{\nu \geq 1} 2^{-\nu M} \|\Delta_0 u\|_{L^p(B_{\nu - J})} \Big)^q \end{aligned}$$

Since $s > 0$, $\sum_{l=J}^{0} 2^{lsq} \leq C$

$$\begin{aligned} \frac{1}{|B|^{\lambda/n}} \sum_{l=J}^{0} 2^{lsq} \|\dot{\Delta}_l \Delta_0 u\|^q_{L^p(B)} &\leq \frac{C}{|B|^{\lambda/n}} \|\Delta_0 u\|^q_{L^p(2B)} \\ &+ C \Big( \sum_{\nu \geq 1} 2^{\nu(-M + \frac{\lambda}{q})} \frac{1}{|B_{\nu - J}|^{\frac{\lambda}{nq}}} \|\Delta_0 u\|_{L^p(B_{\nu - J})} \Big)^q \\ &\leq CM + C \sup_{\nu} \frac{1}{|B_{\nu - J}|^{\lambda/n}} \|\Delta_0 u\|^q_{L^p(B_{\nu - J})} \leq CM \end{aligned} \tag{2.6}$$

In the same way we have

$$\frac{1}{|B|^{\lambda/n}} \sum_{l=J}^{0} 2^{lsq} \|\dot{\Delta}_l \Delta_1 u\|^q_{L^p(B)} \leq CM \tag{2.7}$$

From (2.4), (2.5), (2.6) and (2.7) it follows

$$\frac{1}{|B|^{\lambda/n}} \sum_{l \geq J} 2^{lsq} \|\dot{\Delta}_l u\|^q_{L^p(B)} \leq CM$$

and then $u \in \dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$. $\diamondsuit$

Now we give the differential dimension of $\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$.

**Proposition 2.9** *For any $u \in \dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ and any $t > 0$, we have*

$$\|u(t^{-1}\cdot)\|_{\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)} \approx t^d \|u\|_{\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)}$$

*where $d = \frac{n}{p} - \frac{\lambda}{q} - s$.*

**Proof** Let $t = 2^N$, $N \in \mathbb{Z}$ and set $v(x) = u(\frac{x}{2^N})$. From

$$\dot{\Delta}_l v(x) = (\dot{\Delta}_{l+N} u)(\frac{x}{2^N})$$

we get

$$\|\dot{\Delta}_l v\|_{L^p(B)} = 2^{N\frac{n}{p}} \|\dot{\Delta}_{l+N} u\|_{L^p(2^{-N}B)}$$

where $B$ is a ball of $\mathbb{R}^n$ with radius $2^{-J}$, $J \in \mathbb{Z}$. Then we have

$$\begin{aligned}
\|v\|^q_{\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)} &\approx \sup_B 2^{J\lambda} \sum_{l \geq J} 2^{lsq} 2^{N\frac{n}{p}q} \|\dot{\Delta}_{l+N} u\|^q_{L^p(2^{-N}B)} \\
&\approx 2^{N\frac{n}{p}q} 2^{-N\lambda} 2^{-Nsq} \sup_B 2^{(J+N)\lambda} \sum_{l+N \geq J+N} 2^{(l+N)sq} \|\dot{\Delta}_{l+N} u\|^q_{L^p(2^{-N}B)} \\
&\approx 2^{(\frac{n}{p} - \frac{\lambda}{q} - s)qN} \sup_B 2^{J\lambda} \sum_{l \geq J} 2^{lsq} \|\dot{\Delta}_l u\|^q_{L^p(B)}
\end{aligned}$$

**Lemma 2.10** *Let $1 \leq p \leq p' < +\infty$, $1 \leq q' \leq q < +\infty$ and $s \in \mathbb{R}$. Further, let $\lambda$ and $\mu \geq 0$ such that $\frac{n}{p'} - \frac{\mu}{q'} \geq \frac{n}{p} - \frac{\lambda}{q}$. Then we have the continuous embedding*

$$\mathcal{L}^{\mu, s + \frac{n}{p'} - \frac{\mu}{q'}}_{p',q'}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{\lambda, s + \frac{n}{p} - \frac{\lambda}{q}}_{p,q}(\mathbb{R}^n).$$

*We also have the same result for the dotted spaces $\dot{\mathcal{L}}$.*

In particular, if $p = p'$ and $q = q'$ then $\mathcal{L}^{\mu, s - \frac{\mu}{q}}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{\lambda, s - \frac{\lambda}{q}}_{p,q}(\mathbb{R}^n)$ holds for all $\mu \leq \lambda$. Furthermore if $p = p' = q = q'$ we get $\mathcal{L}^{p, \lambda, s + \frac{\alpha}{p}}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{p, \lambda + \alpha, s}(\mathbb{R}^n)$ for all $\alpha \geq 0$ and $\lambda \geq 0$.

**Proof** Let $u \in \mathcal{L}^{\mu, s + \frac{n}{p'} - \frac{\mu}{q'}}_{p',q'}(\mathbb{R}^n)$. Let $B$ be a ball of $\mathbb{R}^n$ with radius $2^{-J}$, $J \in \mathbb{Z}$. Since $p \leq p'$ and $\frac{q}{q'} \geq 1$ we obtain

$$\begin{aligned}
&\frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{l \geq J^+} 2^{l(s + \frac{n}{p} - \frac{\lambda}{q})q} \|\Delta_l u\|^q_{L^p(B)} \\
&\leq \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{l \geq J^+} 2^{l(s + \frac{n}{p} - \frac{\lambda}{q})q} \left( \|\Delta_l u\|_{L^{p'}(B)} |B|^{\frac{1}{p} - \frac{1}{p'}} \right)^q
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{|B|^{\frac{\lambda}{n}}}\sum_{l\ge J^+}\Big(2^{l(s+\frac{n}{p}-\frac{\lambda}{q})q'}\|\Delta_l u\|^{q'}_{L^{p'}(B)}\Big)^{q/q'}|B|^{q\left(\frac{1}{p}-\frac{1}{p'}\right)}\\
&\le \frac{1}{|B|^{\frac{\lambda}{n}}}\Big(\sum_{l\ge J^+}2^{l(s+\frac{n}{p}-\frac{\lambda}{q})q'}\|\Delta_l u\|^{q'}_{L^{p'}(B)}\Big)^{q/q'}|B|^{q\left(\frac{1}{p}-\frac{1}{p'}\right)}
\end{aligned}$$

Now $|B|\sim 2^{-nJ}$, $l\ge J^+\ge J$ and the assumption $\frac{n}{p'}-\frac{\mu}{q'}\ge\frac{n}{p}-\frac{\lambda}{q}$ yield

$$\begin{aligned}
&\Big\{\frac{1}{|B|^{\frac{\lambda}{n}}}\sum_{l\ge J^+}2^{l(s+\frac{n}{p}-\frac{\lambda}{q})q}\|\Delta_l u\|^{q}_{L^{p}(B)}\Big\}^{1/q}\\
&\le \Big(\frac{1}{|B|^{\frac{\mu}{n}}}\sum_{l\ge J^+}2^{l(s+\frac{n}{p}-\frac{\lambda}{q})q'}\|\Delta_l u\|^{q'}_{L^{p'}(B)}\Big)^{1/q'}|B|^{\frac{1}{p}-\frac{1}{p'}}|B|^{\frac{\mu}{nq'}-\frac{\lambda}{nq}}\\
&\le C\Big(\frac{1}{|B|^{\frac{\mu}{n}}}\sum_{l\ge J^+}2^{J\left((\frac{n}{p'}-\frac{\mu}{q'})-(\frac{n}{p}-\frac{\lambda}{q})\right)q'}2^{l(s+\frac{n}{p}-\frac{\lambda}{q})q'}\|\Delta_l u\|^{q'}_{L^{p'}(B)}\Big)^{1/q'}\\
&\le C\Big(\frac{1}{|B|^{\frac{\mu}{n}}}\sum_{l\ge J^+}2^{l(s+\frac{n}{p'}-\frac{\mu}{q'})q'}\|\Delta_l u\|^{q'}_{L^{p'}(B)}\Big)^{1/q'}
\end{aligned}$$

The proof of lemma 2.10 is complete. ◇

**Lemma 2.11** *The derivation $D^\alpha_x$ is a bounded operator from $\mathcal{L}^{\lambda,s+|\alpha|}_{p,q}(\mathbb{R}^n)$ to $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ and from $\dot{\mathcal{L}}^{\lambda,s+|\alpha|}_{p,q}(\mathbb{R}^n)$ to $\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$.*

**Proof** Let $|\alpha|=1$. We have

$$\begin{aligned}
\|D_x u\|^q_{\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)} &= \|D_x\sum_{l\ge 0}\Delta_l u\|^q_{\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)}\\
&=\sup_B\frac{1}{|B|^{\lambda/n}}\sum_{j\ge J^+}2^{jsq}\|\sum_{l=j-1}^{j+1}D_x\Delta_j\Delta_l u\|^q_{L^p(B)}\\
&\le C\sup_B\frac{1}{|B|^{\lambda/n}}\sum_{j\ge J^+}2^{jsq}\sum_{l\sim j}\|D_x\Delta_j\Delta_l u\|^q_{L^p(B)}
\end{aligned}$$

(in the case of the dotted spaces we use lemma 2.6). Let $B$ a ball of $\mathbb{R}^n$ of radius $2^{-J}$, $J\in\mathbf{Z}$. Set $f_l=\Delta_l u$. Note that

$$f_l=\chi_{2B}f_l+\sum_{\nu\ge -J+1}\chi_{F_\nu}f_l$$

and use lemma 2.4 to get

$$\begin{aligned}
\|D_x\Delta_j f_l\|_{L^p(B)} &\le\|D_x\Delta_j\chi_{2B}f_l\|_{L^p(B)}+\sum_{\nu\ge -J+1}\|D_x\Delta_j\chi_{F_\nu}f_l\|_{L^p(B)}\\
&\le C2^j\|\Delta_j\chi_{2B}f_l\|_{L^p(\mathbb{R}^n)}+\sum_{\nu\ge -J+1}\|D_x\Delta_j\chi_{F_\nu}f_l\|_{L^p(B)} \qquad (2.8)\\
&\le C2^j\|f_l\|_{L^p(2B)}+\sum_{\nu\ge -J+1}\|D_x\Delta_j\chi_{F_\nu}f_l\|_{L^p(B)}
\end{aligned}$$

On the other hand,

$$D_x\Delta_j\chi_{F_\nu}f_l\,(x) = 2^{(n+1)j}\int_{F_\nu} f_l\,(y)\,\psi\left(2^j(x-y)\right)dy$$

where $\psi = D_x\mathcal{F}^{-1}\theta$. Therefore, if $x \in B$ and $y \in F_\nu, \nu \geq -J+1$, then $|x-y| \sim |x_0-y| \sim 2^\nu$. Since $\psi \in \mathcal{S}(\mathbb{R}^n)$, for each integer $N$ there exists a constant $C_N$ such that

$$\begin{aligned}|D_x\Delta_j\chi_{F_\nu}f_l\,(x)\,| &\leq C_N 2^{j(n+1-N)}2^{-\nu N}\int_{F_\nu}|f_l(y)|\,dy\\ &\leq C_N 2^{j(n+1-N)}2^{-\nu N}\|f_l\|_{L^p(F_\nu)}|F_\nu|^{1-\frac{1}{p}}\end{aligned}$$

We deduce

$$\|D_x\Delta_j\chi_{F_\nu}f_l\|_{L^p(B)} \leq C_N 2^{j(n+1-N)}|B|^{1/p}2^{\nu\left(-N+\frac{\lambda}{q}-\frac{n}{p}+n\right)}\frac{1}{|F_\nu|^{\frac{\lambda}{nq}}}\|f_l\|_{L^p(F_\nu)}$$

Thus

$$\begin{aligned}&\sum_{\nu\geq -J+1}\|D_x\Delta_j\chi_{F_\nu}f_l\|_{L^p(B)}\\ &\leq C_N 2^{j(n+1-N)}2^{-J\frac{n}{p}}\sum_{\nu\geq -J+1}2^{\nu\left(-N+\frac{\lambda}{q}-\frac{n}{p}+n\right)}\frac{\|f_l\|_{L^p(F_\nu)}}{|F_\nu|^{\frac{\lambda}{nq}}} \qquad (2.9)\\ &\leq C2^j2^{(j-J)(n-N)}2^{-J\frac{\lambda}{q}}\sum_{\nu\geq 1}2^{\nu\left(-N+\frac{\lambda}{q}-\frac{n}{p}+n\right)}\frac{\|f_l\|_{L^p(F_{\nu-J})}}{|F_{\nu-J}|^{\frac{\lambda}{nq}}}\end{aligned}$$

Choose $N$ large and use inequalities (2.8), (2.9) and lemma 2.1 to deduce

$$\begin{aligned}&\frac{1}{|B|^{\lambda/n}}\sum_{j\geq J^+}2^{jsq}\sum_{l\sim j}\|D_x\Delta_j\Delta_l u\|^q_{L^p(B)}\\ &\leq C\frac{1}{|B|^{\lambda/n}}\sum_{j\geq J^+}2^{j(s+1)q}\|f_j\|^q_{L^p(2B)}\\ &\quad +C_N\sum_{j\geq J^+}\Big\{\sum_{\nu\geq 1}2^{\nu\left(-N+\frac{\lambda}{q}-\frac{n}{p}+n\right)}\frac{2^{j(s+1)}}{|F_{\nu-J}|^{\frac{\lambda}{nq}}}\|f_j\|_{L^p(F_{\nu-J})}\Big\}^q\\ &\leq C\frac{1}{|3B|^{\lambda/n}}\sum_{j\geq (J-2)^+}2^{j(s+1)q}\|\Delta_j u\|^q_{L^p(2B)}\\ &\quad +C'_N\sup_{\nu\geq 1}\frac{1}{|F_{\nu-J}|^{\lambda/n}}\sum_{j\geq(J-\nu-1)^+}2^{j(s+1)q}\|\Delta_j u\|^q_{L^p(F_{\nu-J})}\end{aligned}$$

Finally,

$$\|D_x u\|^q_{\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)} \leq C\|u\|^q_{\mathcal{L}^{\lambda,s+1}_{p,q}(\mathbb{R}^n)}$$

which completes the proof of lemma 2.11.

# 3 Proof of theorem 1.8

For the first embedding, let $u \in \mathcal{L}_{p,q}^{\lambda,s+\frac{n}{p}-\frac{\lambda}{q}}(\mathbb{R}^n)$. Let $j \geq 1$ and $x \in \mathbb{R}^n$ be fixed. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\mathcal{F}\phi = 1$ on $\frac{1}{2} \leq |\xi| \leq 2$.

From $\mathcal{F}\Delta_j u(\xi) = \mathcal{F}(\Delta_j u)(\xi)\mathcal{F}\phi(2^{-j}\xi)$ we deduce

$$\Delta_j u(x) = 2^{nj} \int_{\mathbb{R}^n} \Delta_j u(y)\phi(2^j(x-y))dy \tag{3.1}$$

(in the case $j = 0$, we assume $\mathcal{F}\phi = 1$ on $|\xi| \leq 2$). We decompose $\mathbb{R}^n$ into disjunctive cubes with the side-length $2^{-j}$,

$$Q_\nu = \left\{ y \in \mathbb{R}^n : \|y - x - 2^{-j}\nu\|_\infty \leq 2^{-j-1} \right\},$$

$\nu \in \mathbb{Z}^n$, and we set $a_\nu = \sup_{\|y-\nu\|_\infty \leq \frac{1}{2}} |\phi(y)|$ which is a rapidly decreasing sequence. Then, we have

$$\begin{aligned}
|\Delta_j u(x)| &\leq 2^{nj} \sum_\nu \int_{Q_\nu} |\Delta_j u(y)\phi(2^j(x-y))|dy \\
&\leq 2^{nj} \sum_\nu a_\nu \int_{Q_\nu} |\Delta_j u(y)|dy \\
&\leq 2^{nj} \sum_\nu a_\nu \|\Delta_j u\|_{L^p(Q_\nu)} |Q_\nu|^{1-\frac{1}{p}} \\
&\leq C2^{nj} \sum_\nu a_\nu \Big( 2^{-j(s+\frac{n}{p}-\frac{\lambda}{q})} \|u\|_{\mathcal{L}_{p,q}^{\lambda,s+\frac{n}{p}-\frac{\lambda}{q}}(\mathbb{R}^n)} |Q_\nu|^{\frac{\lambda}{nq}} \Big) |Q_\nu|^{1-\frac{1}{p}} \\
&\leq C2^{nj} \sum_\nu a_\nu 2^{-j(s+\frac{n}{p}-\frac{\lambda}{q})} \|u\|_{\mathcal{L}_{p,q}^{\lambda,s+\frac{n}{p}-\frac{\lambda}{q}}(\mathbb{R}^n)} 2^{-jn(\frac{\lambda}{nq}+1-\frac{1}{p})} \\
&\leq C2^{-js} \|u\|_{\mathcal{L}_{p,q}^{\lambda,s+\frac{n}{p}-\frac{\lambda}{q}}(\mathbb{R}^n)}.
\end{aligned}$$

The first embedding is proved.

Now let $u \in F_{p,q}^{s+\frac{n}{p}}(\mathbb{R}^n)$ and $B$ a ball of $\mathbb{R}^n$ centered at $x_0$ with radius $2^{-J}$, $J \in \mathbf{Z}$. The assumption $\frac{q}{p} \geq 1$ gives

$$\begin{aligned}
&\frac{1}{|B|^{\frac{\lambda}{nq}}} \Big\{ \sum_{j \geq J^+} 2^{jq(s+\frac{n}{p}-\frac{\lambda}{q})} \Big( \int_B |\Delta_j u|^p dx \Big)^{q/p} \Big\}^{1/q} \\
&= \frac{1}{|B|^{\frac{\lambda}{nq}}} \Big\| \Big( \int_B 2^{jp(s+\frac{n}{p}-\frac{\lambda}{q})} |\Delta_j u|^p dx \Big)_{j \geq J^+} \Big\|_{l^{\frac{q}{p}}}^{1/p} \\
&\leq \frac{1}{|B|^{\frac{\lambda}{nq}}} C \Big\{ \int_{\mathbb{R}^n} \Big\| \Big( 2^{jp(s+\frac{n}{p}-\frac{\lambda}{q})} |\Delta_j u|^p \Big)_{j \geq J^+} \Big\|_{l^{\frac{q}{p}}} dx \Big\}^{1/p} \\
&\leq C 2^{J\frac{\lambda}{q}} \Big\{ \int_{\mathbb{R}^n} \Big( \sum_{j \geq J^+} 2^{jq(s+\frac{n}{p}-\frac{\lambda}{q})} |\Delta_j u|^q \Big)^{p/q} \Big\}^{1/p} \\
&\leq C \Big\{ \int_{\mathbb{R}^n} \Big( \sum_{j \geq J^+} 2^{J\lambda} 2^{jq(s+\frac{n}{p}-\frac{\lambda}{q})} |\Delta_j u|^q \Big)^{p/q} \Big\}^{1/p} \leq C\|u\|_{F_{p,q}^{s+\frac{n}{p}}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore the second embedding yields true.

For the third embedding, use $\frac{p}{q} \geq 1$ to obtain

$$\begin{aligned}
\frac{1}{|B|^{\frac{\lambda}{n}}}\int_B \sum_{j\geq J^+} 2^{jp(s-\frac{\lambda-n}{p})}|\Delta_j u|^p dx &= \frac{1}{|B|^{\frac{\lambda}{n}}}\int_B \sum_{j\geq J^+} \Big(2^{jq(s-\frac{\lambda-n}{p})}|\Delta_j u|^q\Big)^{p/q} dx \\
&\leq C2^{J\lambda}\int_B \Big(\sum_{j\geq J^+} 2^{jq(s-\frac{\lambda-n}{p})}|\Delta_j u|^q\Big)^{p/q} dx \\
&\leq \int_B \Big(\sum_{j\geq J^+} 2^{J\lambda\frac{q}{p}}2^{jq(s-\frac{\lambda-n}{p})}|\Delta_j u|^q\Big)^{p/q} dx \\
&\leq C\int_{\mathbb{R}^n} \Big(\sum_{j\geq J^+} 2^{jq(s+\frac{n}{p})}|\Delta_j u|^q\Big)^{p/q} dx \\
&\leq C\|u\|^p_{F^{s+\frac{n}{p}}_{p,q}(\mathbb{R}^n)}
\end{aligned}$$

Let $B$ be a ball of $\mathbb{R}^n$ centered at $x_0$ with radius $2^{-J}$ for $J \in \mathbb{Z}$. Let $u \in B^{s-\frac{n}{p}+\frac{\lambda}{q}}_{\infty,q}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\frac{1}{|B|^{\frac{\lambda}{n}}}\sum_{j\geq J^+} 2^{jqs}\|\Delta_j u\|^q_{L^p(B)} &\leq C2^{J\lambda}\sum_{j\geq J^+} 2^{jqs}|B|^{\frac{q}{p}}\|\Delta_j u\|^q_{L^\infty(B)} \\
&\leq C\sum_{j\geq J^+} 2^{jqs}2^{J(\lambda-n\frac{q}{p})}\|\Delta_j u\|^q_{L^\infty(B)} \\
&\leq C\sum_{j\geq 0} 2^{jq(s-\frac{n}{p}+\frac{\lambda}{q})}\|\Delta_j u\|^q_{L^\infty(B)}
\end{aligned}$$

Therefore the fourth injection is proved. For the last assertion,

$$\begin{aligned}
\frac{1}{|B|^{\lambda/n}}\sum_{j\geq J^+} 2^{jqs}\|\Delta_j u\|^q_{L^q(B)} &\leq C\frac{1}{|B|^{\lambda/n}}\int_B \sum_{j\geq J^+} 2^{jqs}|\Delta_j u|^q dx \\
&\leq C\sup_{x\in\mathbb{R}^n}\sum_{j\geq J^+} 2^{jqs}|\Delta_j u(x)|^q|B|^{-\frac{\lambda}{n}+1} \\
&\leq C\sup_{x\in\mathbb{R}^n}\sum_{j\geq 0} 2^{jq(s+\frac{\lambda-n}{q})}|\Delta_j u|^q
\end{aligned}$$

and the proof of theorem 1.8 is complete. ♢

Now we can state a partial result on the topological dual of $\overset{\circ}{\mathcal{L}}{}^{\lambda,s}_{p,q}(\mathbb{R}^n)$, the closure of Schwartz space $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$.

**Corollary 3.1** *Let $s \in \mathbb{R}$, $\lambda \geq 0$, $1 \leq p < +\infty$, $1 \leq q < +\infty$, $1 < p' \leq +\infty$ and $1 < q' \leq +\infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. We have*

$$F^{-s-\frac{\lambda}{q}+\frac{n}{p}}_{1,1}(\mathbb{R}^n) \hookrightarrow (\overset{\circ}{\mathcal{L}}{}^{\lambda,s}_{p,q}(\mathbb{R}^n))' \hookrightarrow F^{-s-\frac{\lambda}{q}}_{p',q'}(\mathbb{R}^n) \text{ provided } p \leq q. \tag{3.2}$$

$$F_{1,1}^{-s-\frac{\lambda}{p}+\frac{n}{p}}(\mathbb{R}^n) \hookrightarrow (\overset{\circ}{\mathcal{L}}^{p,\lambda,s}(\mathbb{R}^n))' \hookrightarrow F_{p',q'}^{-s-\frac{\lambda}{p}}(\mathbb{R}^n) \text{ provided } q \leq p \tag{3.3}$$

*in particular*

$$F_{1,1}^{\frac{n}{2}-\frac{\lambda}{2}}(\mathbb{R}^n) \hookrightarrow \left(\overset{\circ}{\mathcal{L}}^{2,\lambda}(\mathbb{R}^n)\right)' \hookrightarrow F_{2,q'}^{-\frac{\lambda}{2}}(\mathbb{R}^n) \text{ for all } q' \geq 2.$$

*We have also the same injections for the dotted spaces.*

**Proof** We prove only (3.3). Applying the first and the third embedding of theorem 1.8 with $s+\frac{\lambda-n}{p}$ instead of $s$ we obtain

$$F_{p,q}^{s+\frac{\lambda}{p}}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{p,\lambda,s}(\mathbb{R}^n) \hookrightarrow C^{s+\frac{\lambda-n}{p}}(\mathbb{R}^n) \equiv B_{\infty,\infty}^{s+\frac{\lambda-n}{p}}(\mathbb{R}^n)$$

hence

$$F_{p,q}^{s+\frac{\lambda}{p}}(\mathbb{R}^n) \hookrightarrow \overset{\circ}{\mathcal{L}}^{p,\lambda,s}(\mathbb{R}^n) \hookrightarrow \overset{\circ}{B}_{\infty,\infty}^{s+\frac{\lambda-n}{p}}(\mathbb{R}^n)$$

where $\overset{\circ}{B}_{\infty,\infty}^{s+\frac{\lambda-n}{p}}(\mathbb{R}^n)$ denotes the closure of $\mathcal{S}(\mathbb{R}^n)$ in $B_{\infty,\infty}^{s+\frac{\lambda-n}{p}}(\mathbb{R}^n)$. Now

$$\left(\overset{\circ}{B}_{\infty,\infty}^{s+\frac{\lambda-n}{p}}(\mathbb{R}^n)\right)' = F_{1,1}^{-s-\frac{\lambda}{p}+\frac{n}{p}}(\mathbb{R}^n)$$

yields the result. ◇

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