Electronic Journal of Differential Equations, Vol. **2002**(2002), No. 67, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Boundedness and almost periodicity for some state-dependent delay differential equations *

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Abstract

This work is devoted to the study the existence and uniqueness of bounded solutions for state-dependent delay differential equations. We also study the existence of periodic and almost periodic solutions.

1 Introduction

Differential delay equations, or functional differential equations, have been used in modelling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case is called distributed delay. However, complicated situations in which the delay depends on the unknown functions have been proposed in modelling in recent years. These equations are frequently called equations with state-dependent delay. Many works related to this topics have been published; see the references in this article.

In this work we study the existence of bounded, periodic, and almost periodic solutions of the state-dependent delay differential of the form

$$\begin{aligned} &\frac{d}{dt}x(t) = F(t, x(t), x(t-\rho(x_t))), \quad \text{for } t \geq 0 \\ &x_0 = \varphi \end{aligned} \tag{1.1}$$

where φ is a given function in the space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$. This space is denoted by $C = C([-\tau, 0]; \mathbb{R}^n)$ and endowed with the uniform norm topology. For every $t \geq 0$, the history function $x_t \in C$ is defined by

$$x_t(\theta) = x(t+\theta), \quad \text{for } \theta \in [-\tau, 0].$$

The function F is a continuous from $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$ and ρ is a positive bounded continuous function on C, τ is the maximal delay defined by

$$\tau = \sup_{\varphi \in C} \rho(\varphi).$$

* *Mathematics Subject Classifications:* 34K05, 34K12, 34K13, 34K14.
Key words: State-dependent delay, bounded solutions, almost periodic solutions.

Submitted March 25, 2002. Published July 15, 2002.
This research was supported by grant 00-412 RG/MATHS/AF/AC from TWAS.

According to the book by Hale [9], if F is continuous, then (1.1) has at least one maximal solution $x(., \varphi)$ which is defined on some interval $[0, t_\varphi)$ and if t_φ is finite then

$$\overline{\lim_{t \to t_\varphi}} |x(t, \varphi)| = \infty.$$

The uniqueness may not hold here, because the right-hand side of (1.1) is not locally Lipschitz. Even if F is lipschitzian with respect to the second or the third arguments, uniqueness may not hold. Consider the equation

$$\frac{d}{dt} x(t) = x(t - \sigma(x(t))), \tag{1.2}$$

where $\sigma : \mathbb{R} \to [0, 1]$ is smooth, $\sigma'(0) \neq 0$, and $\sigma(0) = 1$.

Note that the right-hand side of (1.2) can be written as $G(\varphi) = \varphi(-\sigma(\varphi(0)))$, for $\varphi \in C([-1, 0], \mathbb{R})$, and G is not locally Lipschitz in a neighborhood of zero. In fact assume that there exist positive constants k and ρ such that

$$|G(\varphi_1) - G(\varphi_2)| \leq k|\varphi_1 - \varphi_2|, \quad \text{for} |\varphi_1|, |\varphi_2| < \rho.$$

Let $\varphi(\theta) = \varepsilon(-1 + \sqrt{1 + \theta})$, for $\theta \in [-1, 0]$, where ε is a positive constant such that $|\varphi| < \rho$. Let $\varkappa \in [-1, 0]$ such that $|\varphi| + |\varkappa| < \rho$, then

$$|G(\varphi + \varkappa) - G(\varphi)| \leq k|\varkappa|,$$

which implies

$$|\varepsilon\sqrt{1 - \sigma(\varkappa)} + \varkappa| \leq k|\varkappa| \quad \text{and} \quad |\frac{\varepsilon(\sigma(\varkappa) - 1)}{\varkappa\sqrt{1 - \sigma(\varkappa)}}| \leq (1 + k) \; .$$

Letting $\varkappa$ approach zero, we obtain a contradiction. Therefore, the right-hand side of equation (1.2) is not locally Lipschitz near zero. The uniqueness has been proved for lipschitzian initial data in [18]. However, the standard argument for uniqueness can not be applied in this example. The following counter example explains more the situation

$$\begin{aligned} &\frac{d}{dt} x(t) = x(t - x(t)), \quad t \in [0, 1] \\ &x(\theta) = \sqrt{|\theta|} + 1, \quad \theta \in [-1, 0] \, . \end{aligned} \tag{1.3}$$

Then equation (1.3) has two solutions namely

$$x_1(t) = t + \frac{t^2}{4} \text{ and } x_2(t) = t, \; t \in [0, 1] \, .$$

In fact one has $t - x_1(t) = -\frac{t^2}{4}$ and $t - x_2(t) = 0$, it follows that

$$x_1'(t) = 1 + \frac{t}{2} = \varphi(t - x_1(t)) \text{ and } x_2'(t) = 1 = \varphi(t - x_2(t)), t \in [0, 1] \, .$$

Differential equations with state-dependent delay have been the subject for several works. In [1] the author proved the existence and periodicity for some state-dependent delay differential equation. In [3] it has been proved also the existence of oscillatory and periodic solutions for some state dependent delay differential equations arising from population dynamics. In [4], it has been proved the stability of some state-dependent model arising form epidemic problems.

The organization of this work is as follows: in section 2 we recall some preliminaries results about ordinary differential equations that will be used in the work. In section 3, we study the problem of the existence of bounded and almost periodic solutions of equation (1.1). The remaining section is devoted to some example.

2 Preliminaries

We define

$$[x, y] = \lim_{h\to 0^+} \frac{|x + hy| - |x|}{h}, \quad \text{for } x, y \in \mathbb{R}^n.$$

Lemma 2.1 *[13] Let x, y and z be in $\mathbb{R}^n$. Then the following properties hold*

(i) $[x, y] = \inf_{h>0} \frac{|x+hy|-|x|}{h}$

(ii) $|[x, y]| \leq |y|$

(iii) $[x, y + z] \leq [x, y] + [x, z]$,

(iv) Let u be a function from a real interval J to $\mathbb{R}^n$ such that $u'(t_0)$ for an interior point t_0 of J. Then $D_+|u(t_0)|$ exists and

$$D_+|u(t_0)| = [u(t_0), u'(t_0)],$$

where $D_+|u(t_0)|$ denotes the right derivative of $|u(t)|$ at t_0.

Let $B(0, \rho) = \{x \in \mathbb{R}^n : |x| \leq \rho\}$. The following result will be used in the sequel.

Theorem 2.2 *[12] Let $\mathcal{H}$ be an $\mathbb{R}^n$-valued function defined on $\mathbb{R}\times\mathbb{R}^n$. Suppose that there exist positive constants p, r, M such that $\frac{M}{p} < r$, $\mathcal{H}$ is continuous on $\mathbb{R} \times B(0, r)$, $|\mathcal{H}(t, 0)| \leq M$, for $t \in \mathbb{R}$, and*

$$[x - y, \mathcal{H}(t, x) - \mathcal{H}(t, y)] \leq -p|x - y|, \text{ for } t \in \mathbb{R} \quad \text{and} \quad x, y \in B(0, r). \tag{2.1}$$

Then the equation

$$\frac{d}{dt}x(t) = \mathcal{H}(t, x(t)), \tag{2.2}$$

has a unique solution u defined on $\mathbb{R}$ such that $|u(t)| \leq \frac{M}{p}$, for all $t \in \mathbb{R}$. Moreover, if v is another solution of (2.2) on $\mathbb{R}$ such that $|v(t_0)| \leq \frac{M}{p}$, for some t_0, then

$$|v(t)| \leq \frac{M}{p} \quad \text{and} \quad |u(t) - v(t)| \leq e^{-p(t-t_0)}|u(t_0) - v(t_0)|, \text{ for } t \geq t_0.$$

Definition A continuous function $\mathcal{H}$ from $\mathbb{R} \times B(0,r)$ to $\mathbb{R}^n$ is said to be almost periodic in t uniformly with respect to x in $B(0,r)$ if for each $\varepsilon > 0$, there exists a positive number l such that any interval of length l contains a τ for which

$$|\mathcal{H}(t+\tau,x) - \mathcal{H}(t,x)| < \varepsilon \quad \text{for } t \in \mathbb{R},\ x \in B(0,r).$$

For a sequence α in $\mathbb{R}$, we write $\alpha' \subset \alpha$ to indicate that α' is a subsequence of α. For a function $\mathcal{H} : \mathbb{R} \times B(0,r) \to \mathbb{R}^n$, we write

$$T_\alpha \mathcal{H} = \mathcal{G},$$

to indicate that $\lim_n \mathcal{H}(t+\alpha_n, x) = \mathcal{G}(t,x)$, the mode of convergence will be made clear at each use of the symbol.

Theorem 2.3 ([7]) *A continuous function $\mathcal{H}$ from $\mathbb{R} \times B(0,r)$ to $\mathbb{R}^n$ is said to be almost periodic in t uniformly with respect to x in $B(0,r)$ if and only if for every real sequence α there exists a subsequence α' such that $T_{\alpha'}\mathcal{H} = \mathcal{G}$ uniformly in any $\mathbb{R} \times B(0,r)$. Furthermore, $\mathcal{G}$ is also almost periodic in t uniformly with respect to x in $B(0,r)$.*

Definition [7] The hull of $\mathcal{H}$, denoted by $H(\mathcal{H})$ is the set of continuous functions $\mathcal{G}$ in $\mathbb{R} \times B(0,r)$ with values in $\mathbb{R}^n$ such that there exists a sequence of real numbers α such that $T_\alpha \mathcal{H} = \mathcal{G}$ uniformly in any $\mathbb{R} \times B(0,r)$.

Theorem 2.4 ([7]) *A continuous function $\mathcal{H}$ from $\mathbb{R} \times B(0,r)$ to $\mathbb{R}^n$ is said to be almost periodic function in t uniformly with respect to x in $B(0,r)$, if and only if for any real sequences α, β, there exist two subsequences α', β' such that*

$$T_{\alpha'+\beta'}\mathcal{H} = T_{\alpha'}T_{\beta'}\mathcal{H},\ \text{pointwise on } \mathbb{R} \times B(0,r).$$

The following Proposition is a consequence of the uniqueness of the bounded solution.

Proposition 2.5 *Assume that assumptions of Theorem 2.2 hold. If $\mathcal{H}$ is almost periodic in t uniformly with respect to x in $B(0,r)$. Then the only bounded solution of equation (2.2) in $B(0, \frac{M}{p})$ is almost periodic.*

Proof It is sufficient to show that for all $\mathcal{G} \in H(\mathcal{H})$, the limit equation

$$\frac{d}{dt}x(t)) = \mathcal{G}(t, x(t)), \tag{2.3}$$

satisfies condition (2.1). Let $\mathcal{G} \in H(\mathcal{H})$ be such that for some sequence α we have $T_\alpha \mathcal{H} = \mathcal{G}$ uniformly in any $\mathbb{R} \times B(0,r)$. Then we have

$$\begin{aligned}
&[x-y, \mathcal{G}(t,x) - \mathcal{G}(t,y)] \\
&= \Big[x-y, \mathcal{G}(t,x) - \mathcal{H}(t+\alpha_n,x) + \mathcal{H}(t+\alpha_n,x) - \mathcal{G}(t,y) \\
&\quad +\mathcal{H}(t+\alpha_n,y) - \mathcal{H}(t+\alpha_n,y)\Big] \\
&\le -p|x-y| + |\mathcal{G}(t,x) - \mathcal{H}(t+\alpha_n,x)| + |\mathcal{G}(t,y) - \mathcal{H}(t+\alpha_n,y)|.
\end{aligned}$$

Letting n tend to infinity, we obtain

$$[x-y, \mathcal{G}(t,x)-\mathcal{G}(t,y)] \le -p|x-y|,$$

for all $t \in \mathbb{R}$ and $x, y \in B(0,r)$. It follows that for all $\mathcal{G}$ in $H(\mathcal{H})$, condition (2.1) is satisfied and for all $\mathcal{G} \in H(\mathcal{H})$ the limit equation

$$\frac{d}{dt}x(t) = \mathcal{G}(t, x(t)), \tag{2.4}$$

has only one solution $x_{\mathcal{G}}$ on $B(0, M/p)$. We will show that the only bounded solution $x_{\mathcal{H}}$ of (2.2) is almost periodic. Let α be a sequence of real numbers such that $T_\alpha \mathcal{H} = \mathcal{G}$ uniformly in any $\mathbb{R} \times B(0,r)$. If we put $x_n(t) = x_{\mathcal{H}}(t+\alpha_n)$, for $t \in \mathbb{R}$, then for $n \ge 0$, x_n satisfies the equation

$$\frac{dx_n(t)}{dt} = \mathcal{H}(t+\alpha_n, x_n(t)),$$

and $(x_n)_n$ is equicontinuous and bounded, by Ascoli-Arzela's theorem, there exists a subsequence $(x'_n)_n$ of $(x_n)_n$ such that $(x'_n)_n$ converges uniformly in any bounded set of $\mathbb{R}$. Let y be the limit function of $(x'_n)_n$, then

$$\frac{dx_n(t)}{dt} \to \mathcal{G}(t, y(t)), \text{ uniformly in bounded sets of } \mathbb{R} \text{ as } n \to \infty.$$

It follows that

$$\frac{dy(t)}{dt} = \mathcal{G}(t, y(t)), \quad t \in \mathbb{R}.$$

By the uniqueness of the bounded solution in $B(0, M/p)$ of the limit equation (1.1), we deduce that $y = x_{\mathcal{G}}$. We conclude that for any sequence of real numbers α there exists a subsequence $\alpha' \subset \alpha$ such that

$$T_{\alpha'} x_{\mathcal{G}} = x_{T_{\alpha'}\mathcal{G}}, \text{ pointwise.}$$

By Theorem 2.4, for two sequences α, β, there exists two subsequence $\alpha' \subset \alpha$ and $\beta' \subset \beta$ such that

$$T_{\alpha'+\beta'}\mathcal{H} = T_{\alpha'}T_{\beta'}\ \mathcal{H}, \text{ pointwise in } \mathbb{R} \times B(0,r).$$

From this, we deduce that

$$T_{\alpha'+\beta'} x_{\mathcal{H}} = x_{T_{\alpha'+\beta'}\mathcal{H}} = x_{T_{\alpha'}T_{\beta'}\mathcal{H}} = T_{\alpha'}T_{\beta'} x_{\mathcal{H}}, \text{ pointwise.}$$

In view of the Theorem 2.4, we deduce that $x_{\mathcal{H}}$ is almost periodic. $\diamondsuit$

Corollary 2.6 *Assume that assumptions of Theorem 2.2 hold. If $\mathcal{H}$ is p-periodic in t, then the only bounded solution of equation (2.2) in $B(0, M/p)$ is p-periodic.*

Proof Let u be the only bounded solution of equation (2.2) in $B(0, M/p)$, then by periodicity $u(.+p)$ is also bounded solution of (2.2) in $B(0, M/p)$ and from the uniqueness of the bounded solution in $B(0, M/p)$ we get that $u = u(.+p)$.

3 Boundedness and almost periodicity

We suppose that

(H1) $F : \mathbb{R} \times B(0, r) \times B(0, r) \to \mathbb{R}^n$ is continuous and $\rho : C_r \to \mathbb{R}^+$ is lipschitzian, where $C_r = \{\varphi \in C : |\varphi| \leq r\}$.

(H2) There exist positive constants M and N such that

(i) $|F(t, 0, u)| \leq M$, $|F(t, x, 0)| \leq N$, for $t \in \mathbb{R}$ and $x, y \in B(0, r)$,

(ii) There exist positive constants p, L with $p > M/r$, such that

$$[x - y, F(t, x, u) - F(t, y, v)] \leq -p|x - y| + L|u - v|,$$

for $t \in \mathbb{R}$ and $x, y, u, v \in B(0, r)$.

For a lipschitzian function h from (a, b) to $\mathbb{R}^n$, we define

$$\mathrm{Lip}(h) = \sup \Big\{ \Big| \frac{h(s) - h(t)}{s - t} \Big| : s, t \in (a, b) \text{ and } s \neq t \Big\}.$$

Theorem 3.1 *Assume that (H1) and (H2) hold. Then for a lipschitzian function $\varphi \in C$ such that $|\varphi| \leq M/p$ and $\mathrm{Lip}(\varphi) \leq N + Lr$, equation (1.1) has at least one solution defined on $\mathbb{R}^+$ which is bounded by M/p.*

Proof By condition (H2-i) we have

$$|F(t, x, u)| \leq N + Lr \quad \text{for } t \in \mathbb{R} \text{ and } x, y \in B(0, r). \tag{3.1}$$

Let $\varphi \in C$ be such that $|\varphi| \leq M/p$, for $T > 0$ and $C([-\tau, T]; \mathbb{R}^n)$ be the space of continuous function from $[-\tau, T]$ to $\mathbb{R}^n$ provided with the uniform norm topology. Let

$$S_\varphi = \Big\{ y \in C([-\tau, T]; \mathbb{R}^n) : y_0 = \varphi, |y| \leq \frac{M}{p} \text{ and } \mathrm{Lip}(y) \leq N + Lr \Big\}.$$

Then S_φ is a convex compact set in $C([-\tau, T]; \mathbb{R}^n)$. For $f \in S_\varphi$, we consider the equation

$$\begin{aligned} &\frac{d}{dt} x(t) = F(t, x(t), f(t - \rho(f_t))), \quad \text{for } t \geq 0 \\ &x(0) = \varphi(0) \end{aligned} \tag{3.2}$$

By Theorem 2.2, equation (3.2) has only one solution x defined on $\mathbb{R}^+$ which is bounded by M/p, moreover $\mathrm{Lip}(x) \leq N + Lr$ and $x \in S_\varphi$. Define the operator $\mathcal{K}$ on S_φ by

$$(\mathcal{K}f)(t) = \begin{cases} \varphi(t) \text{ if } t \in [-\tau, 0] \\ x(t) \text{ if } t \in [0, T] \end{cases}$$

where x is the only solution of equation (4.1) in S_φ. Then $\mathcal{K}$ takes S_φ to itself. We still need to prove the continuity of $\mathcal{K}$. Let $f, g \in S_\varphi$ and $x = \mathcal{K}f$ et $y = \mathcal{K}g$, then by Lemma 2.1 we have

$$\begin{aligned} D_+|x(t) - y(t)| &= [x(t) - y(t), x'(t) - y'(t)] \\ &= [x(t) - y(t), F(t, x, f(t - \rho(f_t))) - F(t, y, g(t - \rho(g_t)))] \end{aligned}$$

By (H2) we obtain that

$$D_+|x(t) - y(t)| \leq -p|x(t) - y(t)| + L|f(t - \rho(f_t)) - g(t - \rho(g_t))|,$$

it follows that

$$D_+|x(t) - y(t)| \leq -p|x(t) - y(t)| + L\left((N + Lr)\,\mathrm{Lip}(\rho) + 1\right)|f - g|. \tag{3.3}$$

To solve this differential inequality, we need the following Lemma.

Lemma 3.2 *[14] Let D be an open set of $\mathbb{R}^2$ and θ is a continuous function from D to $\mathbb{R}$. Consider the scalar differential equation*

$$\begin{aligned} &\frac{d}{dt} w(t) = \theta(t, w(t) \\ &w(t_0) = w_0 \end{aligned} \tag{3.4}$$

and ϱ is a solution of equation (3.4) which is defined on $[t_0, t_1[$. Let z be a continuous function from $[t_0, t_1[$ to $\mathbb{R}$ such that $(t, z(t)) \in D$, for $t \in [t_0, t_1[$, $z(t_0) \leq w_0$ and

$$D_+ z(t) \leq \theta(t, z(t)), \quad \text{for } t \in [t_0, t_1[.$$

Then $z(t) \leq \varrho(t)$, for $t \in [t_0, t_1[$.

Let v be the solution of the following differential equation

$$\begin{aligned} &v'(t) = \alpha(t)v(t) + \beta(t), \quad t \geq a \\ &w(a) = v_0 \geq 0 \end{aligned} \tag{3.5}$$

Using the variation of constants formula, we can see that the solution of (3.5) is

$$v(t) = v_0 \exp\Big(\int_a^t \alpha(s)ds\Big) + \int_a^t \exp\Big(\int_u^t p(s)ds\Big)\beta(u)du, \quad \text{for } t \in [a, b]$$

Applying Lemma 3.2 to inequality (3.3) we obtain that

$$|x(t) - y(t)| \leq e^{-pt}|x(0) - y(0)| + \frac{L\left((N + Lr)\,\mathrm{Lip}(\rho) + 1\right)}{p}|f - g|, \text{ for } t \geq 0.$$

On the other hand $x(0) = y(0)$, which gives

$$|\mathcal{K}f - \mathcal{K}g| \leq \frac{L\left((N + Lr)\operatorname{Lip}(\rho) + 1\right)}{p}|f - g|, \tag{3.6}$$

this implies that $\mathcal{K}$ is continuous in S_φ. By Schauder's fixed point theorem, we deduce that $\mathcal{K}$ has at least one fixed point which is solution of equation (1.1) in S_φ. This implies that equation (1.1) has at least a solution which is defined on $\mathbb{R}^+$ and the solution is bounded by M/p. ♢

For the uniqueness we have the following proposition.

Proposition 3.3 *Assume that (H1) and (H2) hold with*

$$\frac{L\left((N + Lr)\operatorname{Lip}(\rho) + 1\right)}{p} < 1. \tag{3.7}$$

Then for any lipschitzian function $\varphi \in C$ *such that* $|\varphi| \leq M/p$ *and* $\operatorname{Lip}(\varphi) \leq N + Lr$*, equation (1.1) has a unique solution bounded by* M/p *on* $\mathbb{R}^+$ *.*

Proof The proof is just a consequence from inequality (3.6), it follows that $\mathcal{K}$ is a strict contraction in S_φ and $\mathcal{K}$ has only one fixed point in S_φ which is the unique solution of equation (1.1). ♢

For the existence of almost periodic solution, we assume that

(H3) F is almost periodic in t uniformly with respect to $x, y \in B(0, r)$.

Proposition 3.4 *Assuming that (H1), (H2) and (H3) hold. If*

$$\frac{L\left((N + Lr)\operatorname{Lip}(\rho) + 1\right)}{p} < 1,$$

then equation (1.1) has an almost periodic solution that is bounded by M/p*.*

Proof Let $AP(\mathbb{R}^n)$ be the space of almost periodic functions endowed with the uniform norm topology. Let

$$\Lambda = \left\{x \in AP(\mathbb{R}, \mathbb{R}^n) : |x| \leq \frac{M}{p} \text{ and } \operatorname{Lip}(x) \leq N + Lr\right\}.$$

For $f \in \Lambda$, consider the equation

$$\frac{d}{dt}x(t) = F(t, x(t), f(t - \rho(f_t))). \tag{3.8}$$

By Proposition 2.5, equation (3.8) has only one almost periodic solution x that is bounded by M/p and $\operatorname{Lip}(x) \leq N + Lr$, it follows that $x \in \Lambda$. Define $\mathcal{L}$ on Λ by

$$\mathcal{L}f = x.$$

Then $\mathcal{L}$ takes Λ into itself. It is sufficient to prove that $\mathcal{L}$ is a strict contraction on Λ. So we have

$$\begin{aligned} D_+|x(t)-y(t)| &= [x(t)-y(t), x'(t)-y'(t)] \\ &= [x(t)-y(t), F(t,x,f(t-\rho(f_t))) - F(t,y,g(t-\rho(g_t)))] \end{aligned}$$

By (H2) we have

$$D_+|x(t)-y(t)| \le -p|x(t)-y(t)| + L|f(t-\rho(f_t)) - g(t-\rho(g_t))|.$$

Therefore,

$$D_+|x(t)-y(t)| \le -p|x(t)-y(t)| + L\left((N+Lr)\operatorname{Lip}(\rho)+1\right)|f-g|$$

By Lemma 3.2 we obtain that for $t \ge a$,

$$|x(t)-y(t)| \le e^{-(t-a)}|x(a)-y(a)| + \frac{L\left((N+Lr)\operatorname{Lip}(\rho)+1\right)}{p}|f-g|.$$

Letting a tend to $-\infty$, one has

$$|\mathcal{L}f - \mathcal{L}g| \le \frac{L\left((N+Lr)\operatorname{Lip}(\rho)+1\right)}{p}|f-g|.$$

By the contraction mapping theorem, $\mathcal{L}$ has a unique fixed point in Λ which must be the unique almost periodic solution of equation (1.1) in Λ. ♢

For the periodicity by using the same argument as above, we obtain the following statement.

Corollary 3.5 *Assuming that (H1), (H2) hold and F is p-periodic in t. If*

$$\frac{L\left((N+Lr)\operatorname{Lip}(\rho)+1\right)}{p} < 1.$$

Then equation (1.1) has a p-periodic solution which is bounded by M/p .

When the uniqueness of solutions with initial data holds, the periodic solutions can be obtained by the use of Poincaré map. So we have the following statement.

Proposition 3.6 *Assume that (H1), (H2) hold with F being p-periodic in t and Lipschitz continuous with respect to x and u in $B(0,r)$. If $\tau_0 = \inf_{\varphi\in C}\rho(\varphi) > 0$, then (1.1) has a p-periodic solution bounded by M/p.*

Proof Let $\varphi \in C$ such that $|\varphi| \le M/p$ and $\operatorname{Lip}(\varphi) \le N + Lr$, then equation (1.1) has a unique solution on $\mathbb{R}^+$. In fact, we proceed by steps, if we take $t \in [0,\tau_0]$, then (1.1) becomes

$$\begin{aligned} &\frac{d}{dt}x(t) = F(t,x(t),\varphi(t-\rho(x_t))), \quad \text{for } t \ge 0 \\ &x_0 = \varphi \in C \end{aligned} \tag{3.9}$$

From the Lipschitz condition of F, φ and ρ, we deduce that the right hand of equation (3.9) is Lipschitz continuous with respect to the second argument. It is well known that (3.9) has a unique solution on $[0, \tau_0]$. We proceed in the same way in $[\tau_0, 2\tau_0], \dots, [n\tau_0, (n+1)\tau_0]$. Now we deduce the uniqueness of the solution $x(., \varphi)$. Consider the convex set

$$K = \Big\{\varphi \in C : |\varphi| \le \frac{M}{p} \text{ and } \mathrm{Lip}(\varphi) \le N + Lr\Big\}.$$

Then K is compact in C. Let $\mathcal{P}$ be the Poincaré map defined on K by

$$\mathcal{P}\varphi = x_p(., \varphi)$$

From Theorem 3.1 it follows that the solution is bounded by M/p and from inequality (3.1) we get $\mathrm{Lip}(x_t(., \varphi)) \le N + Lr$, for every $t \ge 0$. We conclude that $\mathcal{P}K \subset K$, from the local Lipschitz conditions, we get that $\mathcal{P}$ is continuous and by Schauder's fixed point Theorem, we deduce that $\mathcal{P}$ has at least one fixed point which gives a p-periodic solution of equation (1.1).

4 Examples

As an application, we study the existence of bounded and almost periodic solutions of the scalar state-dependent delay differential equation

$$\begin{aligned} &\frac{d}{dt}x(t) = -x(t)g(x(t)) + \gamma \sin x(t - |\cos x(t)|) + \sin(t) + \sin(\sqrt{2}t), \quad \text{for } t \ge 0 \\ &x_0 = \varphi \end{aligned} \tag{4.1}$$

where $\gamma > 0$. In this example, we assume that

(H4) $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Also we assume that for $\chi(x) = xg(x)$,

$$p_0 = \inf_{\xi \in [-1,1]} \inf \chi'(\xi) > 0.$$

This assumption is satisfied for example for $g(x) = a + bx$, with $a, b > 0$ and $a - 2b > 0$. In this case $p_0 = a - 2b$.

Equation (4.1) can be written as (1.1) with

$$F(t, x, u) = -x(a + bx) + \gamma \sin(u) + \sin(t) + \sin(\sqrt{2}t), \quad \text{for } t, x, u \in \mathbb{R}.$$

and

$$\rho(\varphi) = |\cos \varphi(0)|, \text{ for } \varphi \in C\left([-1, 0]; \mathbb{R}\right).$$

Moreover we assume that

(H5) $\gamma < a - 2b - 2$.

Proposition 4.1 *Assume that (H4) and (H5) hold. Then for a lipschitzian function $\varphi \in C([-1,0];\mathbb{R})$ such that $|\varphi| \leq \frac{\gamma+2}{a-2b}$ and $\mathrm{Lip}(\varphi) \leq (a+b+\gamma+2)$, equation (4.1) has at least one solution defined on $\mathbb{R}^+$ and bounded by $\frac{\gamma+2}{a-2b}$. Moreover if*

$$\gamma < \frac{a-2b}{a+b+\gamma+3}. \tag{4.2}$$

Then (4.1) has only one almost periodic solution in $B(0,\frac{\gamma+2}{a-2b})$.

Proof It is sufficient to prove that (H1), (H2) and (H3) hold. By a simple computation we can see that

$$[x-y, F(t,x,u)-F(t,y,v)] \leq -\operatorname{sgn}(x-y)\,(xg(x)-yg(y)) + \gamma|u-v|,$$

for $t \in \mathbb{R}$ and $x, y, u, v \in B(0,1)$, where

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

From the monotonicity of χ we have

$$\operatorname{sgn}(x-y) = \operatorname{sgn}(xg(x)-yg(y)).$$

It follows that

$$[x-y, F(t,x,u)-F(t,y,v)] \leq -\operatorname{sgn}(xg(x)-yg(y))\,(xg(x)-yg(y)) + \gamma|u-v|,$$

for $t \in \mathbb{R}$ and $x, y, u, v \in B(0,1)$. Which implies that

$$[x-y, F(t,x,u)-F(t,y,v)] \leq -|xg(x)-yg(y)| + \gamma|u-v|,$$

for $t \in \mathbb{R}$ and $x, y, u, v \in B(0,1)$. Using the fact that $a-2b>0$, we deduce that

$$[x-y, F(t,x,u)-F(t,y,v)] \leq -(a-2b)|x-y| + \gamma|u-v|,$$

for $t \in \mathbb{R}$ and $x, y, u, v \in B(0,1)$. Consequently, assumptions (H1) and (H2) hold with

$$M = \gamma+2, \quad N = a+b+2, \quad L = \gamma, \quad r = 1, \quad \tau = 1, \quad p = a-2b.$$

Then by Theorem 3.1 we deduce that (4.1) has at least one solution defined on $\mathbb{R}^+$ which is bounded by $(\gamma+2)/(a-2b)$. Moreover assumption (H3) is also satisfied and (3.7) is equivalent to (4.2). It follows by Proposition 3.4 that (4.1) has only one almost periodic solution in $B(0,\frac{\gamma+2}{a-2b})$.

Acknowledgments The authors express their sincere gratitude to the anonymous referee who carefully read the manuscript and made remarks leading to improvements on the presentation of this paper.

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