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A spectral mapping theorem for evolution semigroups on asymptotically almost periodic functions defined on the half line *

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Abstract

We prove that the evolution semigroup on $AAP_0(\mathbb{R}_+, X)$ is strongly continuous. Then we prove some properties of the generator of this evolution semigroup and show some applications in the theory of inequalities.

1 Introduction

Let X be a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X. The norms in X and in $\mathcal{L}(X)$ will be denoted by $\|\cdot\|$. Let A be a linear and bounded operator acting on X. We consider the system

$$\dot{u}(t) = Au(t) \quad t \geq 0 \tag{1.1}$$

and the Cauchy problem

$$\begin{gathered} \dot{u}(t) = Au(t) + e^{i\mu t}x \quad t \geq 0 \\ u(0) = 0 \end{gathered} \tag{1.2}$$

where $\mu \in \mathbb{R}$ and $x \in X$. It is well-known [12, 2] that the system (1.1) is exponentially stable; that is, there exist the constants $N > 0$ and $\nu > 0$ such that

$$\|e^{tA}\| \leq Ne^{-\nu t} \quad \text{for all } t \geq 0,$$

if and only if the solution of the Cauchy problem (1.2) is bounded for every $\mu \in \mathbb{R}$ and any $x \in X$, i.e., if and only if

$$\sup_{t>0} \| \int_0^t e^{-i\mu\xi} e^{\xi A} x d\xi \| < \infty, \quad \forall \mu \in \mathbb{R} \text{ and } \forall x \in X.$$

For unbounded linear operators, the above result is false, see e.g. [28, Example 3.1]. However some weaker results, described as follows, hold.

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Let $\mathbf{T} = \{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ be a strongly continuous semigroup on X and $A : D(A) \subset X \to X$ its infinitesimal generator. It is well-known that the Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) \\ u(0) &= x \in X \end{aligned} \tag{1.3}$$

is well-posed and the mild solution of (1.3) is defined by

$$u(t) = T(t)x, \quad t \geq 0. \tag{1.4}$$

For well-posedness of equations we refer the reader to [29, 30] and the references therein. The mild solution of the non-homogeneous Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) + f(t) \quad t \geq 0 \\ u(0) &= x \end{aligned} \tag{1.5}$$

is

$$u_f(t) = T(t)x + \int_0^t T(t-\xi)f(\xi)d\xi, \quad t \geq 0. \tag{1.6}$$

Particularly for $x = 0$, $y \in X$, $\mu \in \mathbb{R}$ and $f(t) := e^{i\mu t}y$, the solution $u_f(\cdot)$ can be written as

$$u_{\mu y}(t) = \int_0^t T(t-\xi)e^{i\mu\xi}y d\xi = e^{i\mu t}\int_0^t e^{-i\mu\xi}T(\xi)y d\xi.$$

In [28], it is shown that if $u_{\mu y}(\cdot)$ is bounded on $\mathbb{R}_+$ for every $\mu \in \mathbb{R}$ and all $y \in X$ then

$$\sigma(A) \subset \{\lambda \in \mathbf{C} : Re(\lambda) < 0\}. \tag{1.7}$$

Conversely if (1.7) holds and $\mathbf{T}$ is uniformly bounded (i.e. $\sup_{t\geq 0} \|T(t)\| < \infty$) then $u_{\mu y}(\cdot)$ is bounded on $\mathbb{R}_+$ for every $\mu \in \mathbb{R}$ and all $y \in X$. This last result is proven in [4, Proposition 2]. Another result of this type is due to Arendt and Batty in [1].

For $x \in X$, Let $\omega(x)$ the infimum of all $\omega \in \mathbb{R}$ for which there exists $M_\omega > 0$ such that $\|T(t)x\| \leq M_\omega e^{\omega t}$ for all $t \geq 0$. Let $\omega_1(\mathbf{T})$ the supremum of all $\omega(x)$ with $x \in D(A)$. Frank Neubrander [25] proved that $\omega_1(\mathbf{T})$ is the infimum of all $\omega \in \mathbb{R}$ with the property that

$$\{\mathrm{Re}(\lambda) > \omega\} \subset \rho(A) \quad \text{and there is} \quad R(\lambda, A)x = \lim_{t\to\infty} \int_0^t e^{-\lambda s}T(s)x ds$$

for every $\lambda \in \mathbf{C}$ with $Re(\lambda) > \omega$ and any $x \in X$. Neerven [23, 24] has shown that if

$$\sup_{\mu\in\mathbb{R}} \sup_{t>0} \| \int_0^t e^{-i\mu\xi}T(\xi)y d\xi\| = M(x) < \infty, \quad \forall \mu \in \mathbb{R} \text{ and } \forall x \in X \tag{1.8}$$

then $\omega_1(\mathbf{T}) < 0$; that is, if (1.8) holds then every solution of the system (1.1), starting in $D(A)$, is exponentially stable.

However, there can be solutions of the system (1.1) starting in $X \setminus D(A)$ which are not exponentially stable, even if (1.8) holds, see e.g. [6, Example 2]. Moreover, in [23, Corollary 5 and the proof of Theorem 4] it is shown that if (1.8) holds then the operator resolvent $R(\lambda, A)$ exists and the function $\lambda \mapsto R(\lambda, A)$ is uniformly bounded on $\{Re(\lambda) > 0\}$. Combining this fact with the Gearhart's famous stability theorem [14] (see also Herbst [15], Howland [16], Huang [17], Prüss [27] Weiss [32]) follows that if X is a complex Hilbert space and (1.8) holds then

$$\omega_0(\mathbf{T}) := \lim_{t\to\infty} \frac{\ln \|T(t)\|}{t}$$

is negative, i.e. in these conditions every solution of the system (1.1) is exponentially stable. This and related results are explicitly presented in a very recent paper of Phong [31]. It seems that the last stability result, having (1.8) as hypothesis, cannot be extended for periodic evolution families, but a weaker result holds, see Theorem 2.4 below.

For a well-posed, non-autonomous Cauchy problem

$$\begin{gathered} \dot{u}(t) = A(t)u(t) \quad t \ge 0 \\ u(0) = x \in X \end{gathered} \tag{1.9}$$

with (possibly unbounded) linear operators $A(t)$, the mild solutions lead to an evolution family on $\mathbb{R}_+$, $\mathcal{U} = \{U(t,s) : t \ge s \ge 0\} \subset \mathcal{L}(X)$; that is:

(e1) $U(t,r) = U(t,s)U(s,r)$ for all $t \ge s \ge r \ge 0$ and $U(t,t) = I$ for any $t \ge 0$, (I is the identity operator in $\mathcal{L}(X)$)

(e2) The maps $(t,s) \mapsto U(t,s)x : \{(t,s) : t \ge s \ge 0\} \to X$ are continuous for each $x \in X$.

An evolution family is *exponentially bounded* if there exist $\omega \in \mathbb{R}$ and $M_\omega > 0$ such that

$$\|U(t,s)\| \le M_\omega e^{\omega(t-s)}, \quad \forall t \ge s \ge 0. \tag{1.10}$$

An evolution family is *exponentially stable* if (1.10) holds with some negative ω.

If the evolution family $\mathcal{U}$ verifies the condition

(e3) $U(t,s) = U(t-s,0)$ for all $t \ge s \ge 0$,

then the family $\mathbf{T} = \{U(t,0) : t \ge 0\} \subset \mathcal{L}(X)$ is a strongly continuous semigroup on X. In this case the estimate (1.10) holds automatically.

If the Cauchy problem (1.9) is q-periodic, i.e. $A(t+q) = A(t)$ for $t \ge 0$, then the corresponding evolution family $\mathcal{U}$ is q-periodic, that is,

(e4) $U(t+q, s+q) = U(t,s)$ for all $t \ge s \ge 0$.

Every q-periodic evolution family is exponentially bounded [9, Lemma 4.1]. For a locally Bochner integrable function $f : \mathbb{R}_+ \to X$, the mild solution of the well-posed, inhomogeneous Cauchy problem

$$\begin{gathered} \dot{u}(t) = A(t)u(t) + f(t), \quad t \ge 0 \\ u(0) = x \end{gathered} \tag{1.11}$$

is

$$u_f(t,x) := U(t,0)x + \int_0^t U(t,\tau)f(\tau)d\tau, (t \geq 0). \tag{1.12}$$

We also consider evolution families on the line. We shall use the same notation as in the case of evolution families on $\mathbb{R}_+$ with the mention that variables s and t can take any value in $\mathbb{R}$. For more details about the strongly continuous semigroups and evolution families we refer to [13].

We recall the notion of evolution semigroup. For more details we refer the reader to [10, 11] and references therein. Let us consider the following spaces:

- $BUC(\mathbb{R}, X)$ is the space of all X-valued, bounded and uniformly continuous functions on the real line endowed with the sup-norm.
- $C_0(\mathbb{R}, X)$ is the subspace of $BUC(\mathbb{R}, X)$ consisting of all functions f such that $\lim_{|t|\to\infty} f(t) = 0$.
- $AP(\mathbb{R}, X)$ is the space of all almost periodic functions, that is, the smallest closed subspace of $BUC(\mathbb{R}, X)$ containing the functions of the form, [20],

$$t \mapsto e^{i\mu t}x, \quad \mu \in \mathbb{R} \text{ and } x \in X\,.$$

Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators on X. For every $t \geq 0$ and each $F \in C_0(\mathbb{R}, X)$ the function

$$s \mapsto (\mathcal{T}(t)F)(s) := U(s, s-t)F(s-t) : \mathbb{R} \to X \tag{1.13}$$

belongs to $C_0(\mathbb{R}, X)$ and the family $\mathcal{T} = \{\mathcal{T}(t) : t \geq 0\}$ is a strongly continuous semigroup on $C_0(\mathbb{R}, X)$, [19]. If $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ is a q-periodic evolution family, $t \geq 0$, and $G \in AP(\mathbb{R}, X)$ then the function given by

$$s \mapsto (\mathcal{S}(t)G)(s) := U(s, s-t)G(s-t) : \mathbb{R} \to X, \tag{1.14}$$

belongs to $AP(\mathbb{R}, X)$ and the one-parameter family $\mathcal{S} = \{\mathcal{S}(t) : t \geq 0\}$ is a strongly continuous semigroup on $AP(\mathbb{R}, X)$, [21]. $\mathcal{T}$ and $\mathcal{S}$ are called evolution semigroups on $C_0(\mathbb{R}, X)$ and $AP(\mathbb{R}, X)$, respectively. In the following we will consider spaces consisting of functions defined on $\mathbb{R}_+$. $AP(\mathbb{R}_+, X)$ and $C_0(\mathbb{R}_+, X)$ are the spaces consisting of all functions $g : \mathbb{R}_+ \to X$ for which there exists $G \in AP(\mathbb{R}, X)$, respectively $G \in C_0(\mathbb{R}, X)$, such that $G(s) = g(s)$ for all $s \geq 0$. $C_{00}(\mathbb{R}_+, X)$ is the subspace of $C_0(\mathbb{R}_+, X)$ consisting of all functions f for which $f(0) = 0$, and $AAP_0(\mathbb{R}_+, X)$ is the space of all X-valued functions h such that $h(0) = 0$ and there exist $f \in C_0(\mathbb{R}_+, X)$ and $g \in AP(\mathbb{R}_+, X)$ such that $h = f + g$. For each $h \in AAP_0(\mathbb{R}_+, X)$ and every $t \geq 0$ consider the function $T(t)h$ given by

$$[T(t)h](s) = \begin{cases} U(s, s-t)h(s-t), & s \geq t \\ 0, & 0 \leq s < t. \end{cases} \tag{1.15}$$

2 Results

Lemma 2.1 *Let $\mathbf{T} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup and $A : D(A) \subset X \to X$ its infinitesimal generator. If $\mathbf{T}$ is uniformly stable, i.e. there exists a positive constant M such that $\sup_{t\geq 0} \|T(t)\| = M < \infty$, then*

$$\|Ax\|^2 \leq 4M^2\|A^2x\|\|x\|, \quad \text{for all } x \in D(A^2).$$

The proof of this lemma can be found in [18].

Lemma 2.2 *The semigroup $\mathbf{T} = \{T(t) : t \geq 0\}$ described in (1.15) is defined on $AAP_0(\mathbb{R}_+, X)$ and is strongly continuous. This semigroup is called evolution semigroup associated to $\mathcal{U}$ on the space $AAP_0(\mathbb{R}_+, X)$.*

Proof. Let $h = f + g$ with $f \in C_0(\mathbb{R}_+, X)$ and $g \in AP(\mathbb{R}_+, X)$ such that $h(0) = 0$ and let $F \in C_0(\mathbb{R}, X)$ and $G \in AP(\mathbb{R}, X)$ such that $F(s) = f(s)$ and $G(s) = g(s)$ for all $s \geq 0$. It is easy to see that for each $t \geq 0$, we have

$$T(t)h = (1_{[0,\infty)})\mathcal{S}(t)G) + (1_{[t,\infty)}T(t)f - 1_{[0,t)}\mathcal{S}(t)G).$$

Here $\{\mathcal{S}(t)\}_{t\geq 0}$ is the evolution semigroup on $AP(\mathbb{R}, X)$ given in (1.14) and 1_J is the characteristic function of the interval J. If we put $g_1 := 1_{[0,\infty)}\mathcal{S}(t)G$ and $f_1 := 1_{[t,\infty)}T(t)f - 1_{[0,t]}\mathcal{S}(t)G$ then $f_1 \in C_0(\mathbb{R}_+, X)$, $g_1 \in AP(\mathbb{R}_+, X)$ and $(f_1 + g_1)(0) = 0$, so $T(t)$ is defined on $AAP_0(\mathbb{R}_+, X)$ for every $t \geq 0$. Moreover, for all $h \in AAP_0(\mathbb{R}_+, X)$, we have:

$$\begin{aligned}
\sup_{s\geq 0} \|T(t)h - h\| &\leq \sup_{s\geq t} \|(T(t)h - h)(s)\| + \sup_{s\in[0,t]} \|(T(t)h - h)(s)\| \\
&\leq \sup_{s\geq t} \|(\mathcal{S}(t)G - G)(s)\| + \sup_{s\geq t} \|(\mathcal{T}(t)F - F)(s)\| + \sup_{s\in[0,t]} \|h(s)\| \\
&\leq \|\mathcal{S}(t)G - G\|_{AP(\mathbb{R},X)} + \|\mathcal{T}(t)F - F\|_{C_0(\mathbb{R},X)} + \sup_{s\in[0,t]} \|h(s)\|.
\end{aligned}$$

Thus $\|T(t)h-h\|_{AAP_0(\mathbb{R}_+,X)}$ tends to 0 as $t \to 0$; i.e., the semigroup $\mathbf{T}$ is strongly continuous. □

Lemma 2.3 *Let $\mathcal{U} = \{U(t, s) : (t, s) \in \Delta\}$ be a q-periodic evolution family of bounded linear operators on X, $\mathbf{T} = \{T(t) : t \geq 0\}$ the evolution semigroup associated to $\mathcal{U}$ on the space $AAP_0(\mathbb{R}_+, X)$, given in (1.15), and $(A, D(A))$ its infinitesimal generator. Let u, f in $AAP_0(\mathbb{R}_+, X)$. The following two statements are equivalent.*

1. *$u \in D(A)$ and $Au = -f$*
2. *$u(t) = \int_0^t U(t, s)f(s)ds$ for all $t \geq 0$.*

The proof of this lemma follows from lemma 2.2 using an argument given in [22, lemma 1.1].

By $P_q^0(\mathbb{R}_+, X)$ we will denote the space of all q-periodic, X-valued functions f on $\mathbb{R}_+$, such that $f(0) = 0$.

Theorem 2.4 *Let $\mathcal{U}, \mathbf{T}$ and $(A, D(A))$ as in Lemma 2.3. The following five statements are equivalent.*

(i) *$\mathcal{U}$ is uniformly exponentially stable*

(ii) *A is an invertible operator*

(iii) *For every $f \in AAP_0(\mathbb{R}_+, X)$ the function $t \mapsto u_f(t,0) = \int_0^t U(t,s)f(s)ds$ belongs to $AAP_0(\mathbb{R}_+, X)$*

(iv) *For every $f \in AAP_0(\mathbb{R}_+, X)$ the function $u_f(\cdot, 0)$ is bounded on $\mathbb{R}_+$*

(v) *For every $f \in P_q^0(\mathbb{R}_+, X)$ and $\mu \in \mathbb{R}$ the function $t \mapsto \int_0^t U(t,s)e^{-i\mu s}f(s)ds$ is bounded on $\mathbb{R}_+$.*

Proof. $(i) \Rightarrow (ii)$ Let $\mathcal{X} := AAP_0(\mathbb{R}_+, X)$. Then

$$\begin{aligned}
\|T(t)\|_{\mathcal{L}(\mathcal{X})} &= \sup\{\sup_{s\geq t} \|U(s, s-t)h(s-t)\| : \|h\|_{\mathcal{X}} = 1\} \\
&\leq \sup\{M_\omega e^{\omega t} \sup_{s\geq t} \|h(s-t)\| : \|h\|_{\mathcal{X}} = 1\} \leq M_\omega e^{\omega t},
\end{aligned}$$

for all $t \geq 0$. Thus $\omega_0(\mathbf{T}) := \lim_{t\to\infty} \frac{\ln(\|T(t)\|)}{t} \leq \omega < 0$ and by the general theory of linear semigroups [26, p. 4-5] it follows that $0 \in \rho(A)$; that is, A is an invertible operator.
$(ii) \Rightarrow (iii)$ follows from Lemma 2.3.
$(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (v)$ are obvious.
$(v) \Rightarrow (i)$ follows as in [5, Theorem 4]; see also [8, Theorem 2.1], or [21]. $\square$

Remarks: **1.** Let $A(\cdot)$ be a q-periodic operator-valued function on $\mathbb{R}_+$ and $\{U(t,s) : t \geq s \geq 0\}$ the q-periodic evolution family associated to it. Since the function $t \mapsto \int_0^t U(t,s)f(s)ds$ is the mild solution of the abstract Cauchy problem

$$\begin{gathered}
\dot{u}(t) = A(t)u(t) + f(t) \quad (t \geq 0) \\
u(0) = 0,
\end{gathered} \tag{2.1}$$

the equivalence between **(i)** and **(iii)** from Theorem 2.4 can be interpreted in the following way:

> The system $\dot{u}(t) = A(t)u(t)$ is exponentially stable if and only if for every input $f \in AAP_0(\mathbb{R}_+, X)$ the mild solution of the Cauchy problem (2.1), belongs to $AAP_0(\mathbb{R}_+, X)$.

2. A related result on the individual stability, where the space $AAP(\mathbb{R}_+, X)$ is also involved, can be found in [3, Proposition 2.9].
3. If every solution of the q-periodic homogeneous system $\dot{v}(t) = A(t)v(t)$, $t \in \mathbb{R}_+$ is exponentially stable then for each $f \in P_q^0(\mathbb{R}_+, X)$ and every $\mu \in \mathbb{R}$

there is a mild solution $u(\cdot)$ of the Cauchy problem

$$\dot{v}(t) = A(t)v(t) + e^{i\mu t}f(t), \quad (t \geq 0)$$
$$v(0) = x$$

such that the function $t \mapsto e^{-i\mu t}u(t)$ is q-periodic on $\mathbb{R}_+$. Indeed the family $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ is exponentially stable, so the resolvent set $\rho(V)$ contains all complex number z with $|z| \geq 1$. Here $V := U(q,0)$ denotes the monodromy operator associated to $\mathcal{U}$. Then for each $\mu \in \mathbb{R}$ we have that $e^{i\mu q} \in \rho(V)$. Let $y := \int_0^q U(q,\tau)e^{i\mu\tau}$ and $x = (e^{i\mu q} - V)^{-1}(y)$. Using (1.12) follows that

$$u(t) = U(t,0)x + \int_0^t U(t,\tau)e^{i\mu\tau}f(\tau)d\tau.$$

Thus

$$u(t+q) = U(t,0)Vx + U(t,0)y + e^{i\mu q}\int_q^t U(t,\tau)e^{i\mu\tau}f(\tau)d\tau.$$

In the end we obtain that $e^{-i\mu q}u(t+q) = u(t)$ for every $t \geq 0$, and now it is easy to see that the function $e^{-i\mu\cdot}u(\cdot)$ is q-periodic.

4. The proof of Theorem 2.4 depends of Lemma 2.3 which also depends of the strongly continuity of the evolution semigroup $\mathbf{T}$, defined in (1.15). Thus the condition, $h(0) = 0$, which appears in the definition of the space $AP_0(\mathbb{R}_+, X)$, is essentially in the proof of Theorem 2.4, because it is involved in the proof of Lemma 2.2. □

An immediate consequence of Theorem 2.4 is the spectral mapping theorem for the evolution semigroup $\mathbf{T}$ on $AAP_0(\mathbb{R}_+, X)$. Similar results, but for the evolution semigroup on $C_{00}(\mathbb{R}_+, X)$, can be found in [22, Theorem 2.2, Corollary 2.4]. Recall that $\sigma(L)$ denotes the *spectrum* of the linear operator L acting on X, and $\rho(L) := \mathbf{C} \setminus \sigma(L)$ is the resolvent set of L. The *spectral radius* of L is $r(L) := \sup\{|\lambda| : \lambda \in \sigma(L)\}$ and the *spectral bound* is $s(L) := \sup\{Re(\lambda) : \lambda \in \sigma(L)\}$.

Theorem 2.5 *Let $\mathcal{U}$ be a q-periodic evolution family of bounded linear operators on the Banach space X. Then the evolution semigroup $\mathbf{T}$ on $AAP_0(\mathbb{R}_+, X)$ satisfies the spectral mapping theorem, as follows:*

$$e^{t\sigma(A)} = \sigma(T(t)) \setminus \{0\}, \quad t \geq 0.$$

Moreover, $\sigma(A) = \{\lambda \in \mathbf{C} : Re(\lambda) \leq s(A)\}$ *and*

$$\sigma(T(t)) = \{\lambda \in \mathbf{C} : |\lambda| \leq r(T(t))\}, \text{ for all } t > 0.$$

The proof of this theorem follows from Theorem 2.4 using an argument given in [22, Corollary 2.4].

Another application of Theorem 2.4 is the following inequality of Landau's type. For more details about theorems of this form, see [7].

Theorem 2.6 *Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be a q-periodic evolution family of bounded linear operators acting on X and let $f \in \mathcal{X} := AAP_0(\mathbb{R}_+, X)$. Suppose that the following two conditions are satisfied:*

(i) $u_f(\cdot,0) = \int_0^{\cdot} U(\cdot,s)f(s)ds$ belongs to $\mathcal{X}$

(ii) $v_f(\cdot) := \int_0^{\cdot}(\cdot - s)U(\cdot,s)f(s)ds$ belongs to $\mathcal{X}$.

If $\sup\{\|U(t,s)\| : t \geq s \geq 0\} = M < \infty$ *then*

$$\|u_f(\cdot,0)\|_{\mathcal{X}}^2 \leq 4M^2\|f\|_{\mathcal{X}} \cdot \|v_f(\cdot)\|_{\mathcal{X}}. \tag{2.2}$$

Proof. Let $\mathbf{T}$ the evolution semigroup associated to $\mathcal{U}$ on the space $\mathcal{X}$ and $(A, D(A))$ its infinitesimal generator. From Lemma 2.3 results that $u_f(\cdot,0)$ belongs to $D(A)$ and $Au_f(\cdot,0) = -f$. Using Fubini's theorem it is easy to see that $v_f(t) = \int_0^t U(t,r)u_f(r,0)dr$ for every $t \geq 0$. Then from Lemma 2.3 follows that $v_f(\cdot) \in D(A^2)$ and $A^2v_f(\cdot) = f$. Now the inequality (2.2) can be easily obtained from Lemma 2.1. □

For $U(t,s) = I$, Theorem 2.6 can be generalized in the following sense.

Proposition 2.7 *Let f be a X-valued, locally Bochner integrable function on $\mathbb{R}_+$ and g, h the mappings on $\mathbb{R}_+$ given by*

$$g(t) := \int_0^t f(s)ds \quad and \quad h(t) = \int_0^t (t-s)f(s)ds.$$

If $\sup\{|f(t)| : t \geq 0\} = M_1 < \infty$ *and* $\sup\{|h(t)| : t \geq 0\} = M_3 < \infty$ *then*

$$|g(r)|^2 \leq 4M_1M_3, \quad \forall r \geq 0. \tag{2.3}$$

Proof For every $t \geq 0$ and any X-valued function F on $\mathbb{R}_+$ let us consider the function F_t given by

$$F_t(s) = \begin{cases} F(s-t), & s \geq t \\ 0, & 0 \leq s < t. \end{cases}$$

With this notation, we have

$$h_t(r) - h(r) + tg(r) = \int_0^t (t-s)f_s(r)ds, \quad \forall t \geq 0, \text{ and } \forall r \geq 0. \tag{2.4}$$

Passing to the norm in this equation, we obtain

$$\|g(r)\| \leq \frac{2M_3}{t} + \frac{tM_1}{2}, \quad \forall t > 0. \tag{2.5}$$

If $M_1 = 0$ or $M_3 = 0$ then $g = 0$ and (2.3) holds with equality. If $M_1 > 0$ and $M_3 > 0$ then (2.3) can be obtained from (2.5) with $t = \sqrt{4M_3/M_1}$. □

Remark. If f is a continuous function then Proposition 2.7 follows directly and easily by [18], because $g'(t) = f(t)$ and $h'(t) = g(t)$ for all $t \geq 0$. The author thanks to the referee who brought to the author's attention about this fact.

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