Electronic Journal of Differential Equations, Vol. **2002**(2002), No. 71, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Positive periodic solutions of functional differential equations and population models *

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Abstract

In this paper, we employ Krasnosel'skii's fixed point theorem for cones to study the existence of positive periodic solutions to a system of infinite delay equations,

$$x'(t) = A(t)x(t) + f(t, x_t).$$

We prove two general theorems and establish new periodicity conditions for several population growth models.

1 Introduction

In this paper we study the existence of positive periodic solutions of the system of functional differential equations

$$\dot{x}(t) = A(t)x(t) + f(t, x_t) \tag{1.1}$$

in which $A(t) = \mathrm{diag}[a_1(t), a_2(t), \dots, a_n(t)]$, $a_j \in C(\mathbb{R}, \mathbb{R})$ is ω-periodic, $f(t, x_t)$ is a function defined on $\mathbb{R} \times BC$, and $f(t, x_t)$ is ω-periodic whenever x is ω-periodic, where BC denotes the Banach space of bounded continuous functions $\phi : \mathbb{R} \to \mathbb{R}^n$ with the norm $\|\phi\| = \sup_{\theta\in\mathbb{R}} \sum_{j=1}^{n} |\phi_j(\theta)|$ where $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$, $\omega > 0$ is a constant. If $x \in BC$, then $x_t \in BC$ for any $t \in \mathbb{R}$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in \mathbb{R}$.

One of the most used models, a prototype of (1.1), is the system of Volterra integrodifferential equations

$$\dot{x}_i(t) = x_i(t)\Big[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t) - \sum_{j=1}^{n} \int_{-\infty}^{t} C_{ij}(t, s)g_{ij}(x_j(s))ds\Big] \tag{1.2}$$

which governs the population growth of interacting species $x_j(t)$, $j = 1, 2, \dots, n$. The integral term here specifies how much weight to attach to the population at varies past times, in order to arrive at their present effect on the resources

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* *Mathematics Subject Classifications:* 34K13, 92B05.
Key words: Functional differential equations, positive periodic solution, population models.

Submitted July 5, 2002. Published July 30, 2002.
Jiang & Wei were partially supported by the National Natural Science Fundation of China.

availability. For an autonomous system (1.2), there may be a stable equilibrium point of the population. When the potential equilibrium point becomes unstable, there may exist a nontrivial periodic solution. Then the oscillation of solutions occurs. The existence of such stable periodic solution is of quite fundamental importance biologically since it concerns the long time survival of species. The study of such phenomena has become an essential part of qualitative theory of differential equations. For historical background, basic theory of periodicity, and discussions of applications of (1.1) to a variety of dynamical models, we refer to the reader to, for example, the work of Burton [1], Burton and Zhang [2], Cavani, Lizana, and Smith [3], Cushing [5], Gopalsamy [7], Hadeler [8], Hatvani and Krisztin [9], Hino, Murakami, Taito [10], Kuang [12], Makay [13], May [14], May and Leonard [15], Stech [17], and the references therein.

It is our view that one of the most important problems in the study of dynamical models and their applications is that of describing the nature of the solutions for a large range of parameters involved. From a numerical point of view, the existence of a periodic solution of the approximation scheme must also be studied. The usual approach to fulfill such requirements is to have a set of test equations which are as general as possible and for which explicit analytic conditions can be given. The next step is to characterize the numerical methods which show the same existence result under the same conditions when applied to the test equations.

In this part of our investigation, we provide a unified approach to the study of existence of positive periodic solutions of system (1.1) under general conditions and apply the results to some well-known models in population dynamics. In Section 2, we prove the main results concerning equation (1.1). Applications of these results to population models will be given in Section 3.

Let $\mathbb{R} = (-\infty, +\infty), \mathbb{R}_+ = [0, +\infty)$, and $\mathbb{R}_- = (-\infty, 0]$ respectively. For each $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, the norm of x is defined as $|x| = \sum_{j=1}^n |x_j|$. $\mathbb{R}^n_+ = \{(x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n : x_j \geq 0,\ j = 1, 2, \dots, n\}$. We say that x is "positive" whenever $x \in \mathbb{R}^n_+$. $BC(X, Y)$ denotes the set of bounded continuous functions $\phi : X \to Y$.

2 Existence of Positive Periodic Solutions

We establish the existence of positive periodic solutions of equation (1.1) by applying Krasnosel'skii's fixed point theorem (see [6],[11]) on cones. A compact operator will be constructed. It will be shown that the operator has a fixed point, which corresponds to a periodic solution of (1.1). We denote $f = (f_1, f_2, \dots, f_n)^T$ and assume

(H1) $\int_0^\omega a_j(s)ds \neq 0$ for $j = 1, 2, \dots, n$.

(H2) $f_j(t, \phi_t) \int_0^\omega a_j(s)ds \leq 0$ for all $(t, \phi) \in \mathbb{R} \times BC(\mathbb{R}, \mathbb{R}^n_+)$, $j = 1, 2, \dots, n$.

(H3) $f(t, x_t)$ is a continuous function of t for each $x \in BC(\mathbb{R}, \mathbb{R}^n_+)$.

(H4) For any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $[\phi, \psi \in BC,\ \|\phi\| \le L,\ \|\psi\| \le L,\ \|\phi - \psi\| < \delta,\ 0 \le s \le \omega]$ imply

$$|f(s, \phi_s) - f(s, \psi_s)| < \varepsilon. \tag{2.1}$$

Definition. Let X be a Banach space and K be a closed, nonempty subset of X. K is a cone if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \ge 0$

(ii) $u, -u \in K$ imply $u = 0$.

Theorem 2.1 (Krasnosel'skii [11]) *Let X be a Banach space, and let $K \subset X$ be a cone in X. Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*

$$\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|\Phi y\| \le \|y\|\ \forall y \in K \cap \partial\Omega_1$ and $\|\Phi y\| \ge \|y\|\ \forall y \in K \cap \partial\Omega_2$; or

(ii) $\|\Phi y\| \ge \|y\|\ \forall y \in K \cap \partial\Omega_1$ and $\|\Phi y\| \le \|y\|\ \forall y \in K \cap \partial\Omega_2$.

Then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

We now for $(t, s) \in \mathbb{R}^2$, $j = 1, 2, \dots, n$, we define

$$\sigma := \min\Big\{ \exp(-2\int_0^\omega |a_j(s)|ds), j = 1, 2, \dots, n\Big\} \tag{2.2}$$

$$G_j(t, s) = \frac{\exp(\int_s^t a_j(\nu)d\nu)}{\exp(-\int_0^\omega a_j(\nu)d\nu) - 1}. \tag{2.3}$$

We also define

$$G(t, s) = \operatorname{diag}[G_1(t, s), G_2(t, s), \dots, G_n(t, s)].$$

It is clear that $G(t, s) = G(t + \omega, s + \omega)$ for all $(t, s) \in \mathbb{R}^2$ and by (H2),

$$G_j(t, s) f_j(u, \phi_u) \ge 0$$

for $(t, s) \in \mathbb{R}^2$ and $(u, \phi) \in \mathbb{R} \times BC(\mathbb{R}, \mathbb{R}^n_+)$.

Next, we introduce two sets with ω and σ as given above.

$$\begin{gathered} C_\omega = \{x \in C(\mathbb{R}, \mathbb{R}^n) :\ x(t + \omega) = x(t),\ ; t \in \mathbb{R}\}, \\ K = \{x \in C_\omega : x_j(t) \ge \sigma\|x_j\|, t \in [0, \omega], x = (x_1, x_2, \dots, x_n)^T\}. \end{gathered} \tag{2.4}$$

One may readily verify that K is a cone.

Finally, we define an operator $\Phi : K \to K$ as

$$(\Phi x)(t) = \int_t^{t+\omega} G(t, s) f(s, x_s) ds$$

for $x \in K$, $t \in \mathbb{R}$, where $G(t, s)$ is defined following (2.3). We denote

$$(\Phi x) = \Big(\Phi_1 x, \Phi_2 x, \dots, \Phi_n x\Big)^T.$$

Lemma 2.2 *$\Phi : K \to K$ is well-defined.*

Proof. For each $x \in K$, since $f(t, x_t)$ is a continuous function of t, we have $(\Phi x)(t)$ is continuous in t and

$$\begin{aligned}(\Phi x)(t+\omega) &= \int_{t+\omega}^{t+2\omega} G(t+\omega, s) f(s, x_s) ds \\ &= \int_{t}^{t+\omega} G(t+\omega, v+\omega) f(v+\omega, x_{v+\omega}) dv \\ &= \int_{t}^{t+\omega} G(t, v) f(v, x_v) dv = (\Phi x)(t).\end{aligned}$$

Thus, $(\Phi x) \in C_\omega$. Observe that

$$p_j := \frac{\exp(-\int_0^\omega |a_j(\nu)| d\nu)}{\left|\exp(-\int_0^\omega a_j(\nu) d\nu) - 1\right|} \le |G_j(t, s)| \le \frac{\exp(\int_0^\omega |a_j(\nu)| d\nu)}{|\exp(-\int_0^\omega a_j(\nu) d\nu) - 1|} =: q_j \tag{2.5}$$

for all $s \in [t, t+\omega]$. Hence, for $x \in K$, we have

$$\|\Phi_j x\| \le q_j \int_0^\omega |f_j(s, x_s)| ds \tag{2.6}$$

and

$$(\Phi_j x)(t) \ge p_j \int_0^\omega |f_j(s, x_s)| ds \ge \frac{p_j}{q_j} \|\Phi_j x\| \ge \sigma \|\Phi_j x\|.$$

Therefore, $(\Phi x) \in K$. This completes the proof. □

Lemma 2.3 *$\Phi : K \to K$ is completely continuous, and $x = x(t)$ is an ω-periodic solution of (1.1) whenever x is a fixed point of Φ.*

Proof. We first show that Φ is continuous. By (H4), for any $L > 0$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $[\phi,\ \psi \in C_\omega,\ \|\phi\| \le L,\ \|\psi\| \le L,\ \|\phi - \psi\| < \delta]$ imply

$$\sup_{0 \le s \le \omega} |f(s, \phi_s) - f(s, \psi_s)| < \frac{\varepsilon}{q\omega}$$

where $q = \max_{1 \le j \le n} q_j$. If x, $y \in K$ with $\|x\| \le L$, $\|y\| \le L$, and $\|x - y\| < \delta$, then

$$\begin{aligned}|(\Phi x)(t) - (\Phi y)(t)| &\le \int_t^{t+\omega} |G(t, s)| |f(s, x_s) - f(s, y_s)| ds \\ &\le q \int_0^\omega |f(s, x_s) - f(s, y_s)| ds < \varepsilon\end{aligned}$$

for all $t \in [0, \omega]$, where $|G(t, s)| = \max_{1 \le j \le n} |G_j(t, s)|$. This yields $\|(\Phi x) - (\Phi y)\| < \varepsilon$. Thus, Φ is continuous.

Next, we show that f maps bounded sets into bounded sets. Indeed, let $\varepsilon = 1$. By (H4), for any $\mu > 0$ there exists $\delta > 0$ such that $[x,\, y \in BC,\ \|x\| \le \mu,\ \|y\| \le \mu,\ \|x-y\| < \delta]$ imply

$$|f(s,x_s) - f(s,y_s)| < 1.$$

Choose a positive integer N such that $\frac{\mu}{N} < \delta$. Let $x \in BC$ and define $x^k(t) = x(t)\frac{k}{N}$ for $k = 0,1,2,\dots,N$. If $\|x\| \le \mu$, then

$$\|x^k - x^{k-1}\| = \sup_{t\in\mathbb{R}} |x(t)\frac{k}{N} - x(t)\frac{k-1}{N}| \le \|x\|\frac{1}{N} \le \frac{\mu}{N} < \delta.$$

Thus,

$$|f(s,x_s^k) - f(s,x_s^{k-1})| < 1$$

for all $s \in [0,\omega]$. This yields

$$\begin{aligned} |f(s,x_s)| &\le \sum_{k=1}^{N} |f(s,x_s^k) - f(s,x_s^{k-1})| + |f(s,0)| \\ &< N + \sup_{s\in[0,\omega]} |f(s,0)| =: M_\mu. \end{aligned} \tag{2.7}$$

It follows from (2.6) that for $t \in [0,\omega]$,

$$\|\Phi x\| = \sup_{\theta\in\mathbb{R}} \sum_{j=1}^{n} |(\Phi_j x)(\theta)| \le \sum_{j=1}^{n} q_j \int_0^\omega |f_j(s,x_s)|ds \le q\omega M_\mu.$$

Finally, for $t \in \mathbb{R}$ we have

$$\begin{aligned} \frac{d}{dt}(\Phi x)(t) &= G(t,t+\omega)f(t+\omega,x_{t+\omega}) - G(t,t)f(t,x_t) + A(t)(\Phi x)(t) \\ &= A(t)(\Phi x)(t) + [G(t,t+\omega) - G(t,t)]f(t,x_t) \\ &= A(t)(\Phi x)(t) + f(t,x_t). \end{aligned} \tag{2.8}$$

Combine (2.6), (2.7), and (2.8) to obtain

$$\begin{aligned} |\frac{d}{dt}(\Phi x)(t)| &\le \|A\| q \int_0^\omega |f(s,x_s)|ds + |f(t,x_t)| \\ &\le \|A\| q\omega M_\mu + M_\mu \end{aligned}$$

where $\|A\| = \max_{1\le j\le n} \|a_j\|$. Hence, $\{(\Phi x) : x \in K, \|x\| \le \mu\}$ is a family of uniformly bounded and equicontinuous functions on $[0,\omega]$. By a theorem of Ascoli-Arzela (Royden [16, p.169]), the function Φ is completely continuous. It is clear from (2.8) that $x = x(t)$ is an ω-periodic solution of (1.1) whenever x is a fixed point of Φ. This proves the lemma.

Theorem 2.4 *Assume (H1)-(H4) and that there are positive constants R_1 and R_2 with $R_1 < R_2$ such that*

$$\sup_{\|\phi\|=R_1,\, \phi\in K} \int_0^\omega |f(s,\phi_s)|ds \le R_1/q \tag{2.9}$$

$$\inf_{\|\phi\|=R_2,\, \phi\in K} \int_0^\omega |f(s,\phi_s)|ds \ge R_2/p \tag{2.10}$$

for all $t \in [0,\omega]$ with $p = \min_{1\le j\le n} p_j$ and $q = \max_{1\le j\le n} q_j$, where p_j, q_j are defined in (2.5). Then equation (1.1) has an ω-periodic solution x with $R_1 \le \|x\| \le R_2$.

Proof. Let $x \in K$ and $\|x\| = R_1$. By (2.9), we have

$$|(\Phi x)(t)| \le q\int_t^{t+\omega} |f(s,x_s)|ds \le qR_1/q = R_1.$$

This implies that $\|(\Phi x)\| \le \|x\|$ for $x \in K \cap \partial\Omega_1$, $\Omega_1 = \{\psi \in C_\omega : \|\psi\| < R_1\}$. If $x \in K$ and $\|x\| = R_2$, then

$$(\Phi_j x)(t) \ge p_j \int_t^{t+\omega} |f_j(s,x_s)|ds.$$

Use (2.10) to obtain

$$|(\Phi x)(t)| \ge p\int_t^{t+\omega} |f(s,x_s)|ds \ge p(R_2/p) = R_2$$

Thus, $\|(\Phi x)\| \ge \|x\|$ for $x \in K\cap\partial\Omega_2$, $\Omega_2 = \{\psi \in C_\omega : \|\psi\| < R_2\}$. By Theorem 2.1, Φ has a fixed point in $K \cap (\bar\Omega_2 \setminus \Omega_1)$. It follows from Lemma 2.3 that (1.1) has an ω-periodic solution x with $R_1 \le \|x\| \le R_2$. This completes the proof. $\square$

Corollary 2.5 *Assume (H1)-(H4) and*

$$\lim_{\phi\in K,\, \|\phi\|\to 0} \frac{\int_0^\omega |f(s,\phi_s)|ds}{\|\phi\|} = 0, \tag{2.11}$$

$$\lim_{\phi\in K,\, \|\phi\|\to \infty} \frac{\int_0^\omega |f(s,\phi_s)|ds}{\|\phi\|} = \infty. \tag{2.12}$$

Then (1.1) has an ω-periodic solution.

Theorem 2.6 *Assume (H1)-(H4) and that there are positive constants R_1 and R_2 with $R_1 < R_2$ such that*

$$\inf_{\|\phi\|=R_1,\, \phi\in K} \int_0^\omega |f(s,\phi_s)|ds \ge R_1/p, \tag{2.13}$$

$$\sup_{\|\phi\|=R_2,\, \phi\in K} \int_0^\omega |f(s,\phi_s)|ds \le R_2/q. \tag{2.14}$$

Then equation (1.1) has an ω-periodic solution x with $R_1 \le \|x\| \le R_2$.

The proof will be omitted here since it is similar to that of Theorem 2.4. We now state a corollary to the above theorem.

Corollary 2.7 *Assume (H1)-(H4) and that*

$$\lim_{\phi\in K,\, \|\phi\|\to 0} \frac{\int_0^\omega |f(s,\phi_s)|ds}{\|\phi\|} = \infty, \tag{2.15}$$

$$\lim_{\phi\in K,\, \|\phi\|\to \infty} \frac{\int_0^\omega |f(s,\phi_s)|ds}{\|\phi\|} = 0. \tag{2.16}$$

Then (1.1) has an ω-periodic solution.

3 Applications

We now apply the main result obtained in the previous section to some well-known models in population dynamics. First, we consider the scalar integrodifferential equation

$$\dot N(t) = \alpha(t)N(t)\Big[1 - \frac{1}{N_0(t)}\int_{-\infty}^0 B(s)N(t+s)ds\Big] \tag{3.1}$$

which governs the growth of the population $N(t)$ of a single species whose members compete among themselves for a limited amount of food and living spaces. Equation (3.1) is a modification of the simple logistic equation

$$\dot N(t) = \alpha N(t)\Big[1 - \frac{N(t)}{N_0}\Big] \tag{3.2}$$

where α is the intrinsic per capita growth rate and N_0 is the total carrying capacity. We refer the reader to May [14] for a detailed model construction from (3.2) to (3.1). Suppose that

(P1) $\alpha(t)$, $N_0(t)$ are real-valued and ω-periodic functions on $\mathbb{R}$ with $N_0(t)$ positive and $\int_0^\omega \alpha(t)dt > 0$.

(P2) $B(t)$ is nonnegative and piecewise continuous on $\mathbb{R}^-$ with $\int_{-\infty}^0 B(t)dt = 1$.

Theorem 3.1 *Under assumptions (P1) and (P2), equation (3.1) has at least one positive ω-periodic solution.*

Let $a(t) = \alpha(t)$ and

$$f(t,x_t) = -\frac{x(t)\alpha(t)}{N_0(t)}\int_{-\infty}^0 B(s)x(t+s)ds.$$

It is clear that $f(t,x_t)$ is ω-periodic whenever x is ω-periodic. We need to show that (H1)-(H4) hold. In fact, $\int_0^\omega a(t)dt > 0$ and $f(t,\phi_t) \le 0$ for all

$(t,\phi) \in \mathbb{R} \times BC(\mathbb{R},\mathbb{R}_+)$. Thus, (H1) and (H2) are satisfied. To verify (H3), let $x \in BC(\mathbb{R},\mathbb{R}_+)$ with $\|x\| \leq B_1$. Notice that $x(t)\alpha(t)/N_0(t)$ is continuous. By Lebesgue Dominant Convergence Theorem, one can easily see that $\int_{-\infty}^0 B(s)x(t+s)ds$ is continuous for all $t \in \mathbb{R}$. Thus, $f(t,x_t)$ is continuous in t. Now let $x,y \in BC(\mathbb{R},\mathbb{R}_+)$ with $\|x\| \leq L$, $\|y\| \leq L$ for some $L > 0$. Then

$$\begin{aligned}
&|f(t,x_t) - f(t,y_t)| \\
&= \Big|\frac{x(t)\alpha(t)}{N_0(t)}\int_{-\infty}^0 B(s)x(t+s)ds - \frac{y(t)\alpha(t)}{N_0(t)}\int_{-\infty}^0 B(s)y(t+s)ds\Big| \\
&\leq \Big|\frac{x(t)\alpha(t)}{N_0(t)}\Big|\int_{-\infty}^0 B(s)|x(t+s)-y(t+s)|ds \\
&\quad +\Big|\frac{(x(t)-y(t))\alpha(t)}{N_0(t)}\Big|\int_{-\infty}^0 B(s)|y(t+s)|ds \\
&\leq \frac{L\|\alpha\|}{N_{0*}}\sup_{s\in\mathbb{R}_-}|x(t+s)-y(t+s)| + \frac{|x(t)-y(t)|\|\alpha\|L}{N_{0*}},
\end{aligned}$$

where $N_{0*} = \inf\{N(s) : 0 \leq s \leq \omega\}$. For any $\varepsilon > 0$, choose $\delta = \varepsilon N_{0*}/(2L\|a\|)$. If $\|x-y\| < \delta$, then

$$|f(t,x_t) - f(t,y_t)| < L\|\alpha\|\delta/N_{0*} + \delta\|\alpha\|L/N_{0*} = 2L\|\alpha\|\delta/N_{0*} = \varepsilon.$$

This implies that (H4) holds.

We now verify (2.11) and (2.12) for equation (3.1). For $\phi \in K$, we have $\phi(t) \geq \sigma\|\phi\|$ for all $t \in [0,\omega]$. This yields

$$\frac{|f(t,\phi)|}{\|\phi\|} \leq \sup_{\tau\in[0,\omega]}\frac{\alpha(\tau)}{N_0(\tau)}\int_{-\infty}^0 B(s)ds\|\phi\| \to 0$$

as $\|\phi\| \to 0$ and

$$\frac{|f(t,\phi)|}{\|\phi\|} \geq \inf_{\tau\in[0,\omega]}\frac{\alpha(\tau)}{N_0(\tau)}\int_{-\infty}^0 B(s)ds(\sigma^2\|\phi\|) \to +\infty$$

as $\|\phi\| \to \infty$. Thus, (2.11) and (2.12) are satisfied. By Corollary 2.5, equation (3.1) has a positive ω-periodic solution.

Next, consider a hematopoiesis model (Weng and Liang [19])

$$\dot{N}(t) = -\gamma(t)N(t) + \alpha(t)\int_0^{+\infty} B(s)e^{-\beta(t)N(t-s)}ds \tag{3.3}$$

where $N(t)$ is the number of red blood cells at time t, $\alpha,\beta,\gamma \in C(\mathbb{R},\mathbb{R})$ are ω-periodic, and $B \in L^1(\mathbb{R}^+)$ is nonnegative and piecewise continuous. This is a generalized model of the red cell system introduced by Wazewska-Czyzewska and Lasota [18]

$$\dot{n}(t) = -\gamma n(t) + \alpha e^{-\beta n(t-r)} \tag{3.4}$$

where α, β, γ, r are constants with $r > 0$. Periodicity in equation (3.4) has been investigated extensively in the literature (see Chow [4]). Using Corollary 2.7 with $a(t) = -\gamma(t)$ and

$$f(t, \phi_t) = \alpha(t) \int_0^{+\infty} B(s) e^{-\beta(t)\phi(t-s)} ds,$$

we obtain the following theorem whose proof is essentially the same as that of Theorem 3.1, and therefore will be omitted.

Theorem 3.2 *Suppose that $\alpha(t) > 0$, $\beta(t) \geq 0$ for all $t \in \mathbb{R}^+$, $\int_0^\omega \gamma(t)dt > 0$. $\int_0^{+\infty} B(s)ds > 0$. Then equation (3.3) has a positive ω-periodic solution.*

We now consider the Volterra equation mentioned in Section 1.1.

$$\dot{x}_i(t) = x_i(t)\Big[a_i(t) - \sum_{j=1}^n b_{ij}(t)x_j(t) - \sum_{j=1}^n \int_{-\infty}^t C_{ij}(t,s) g_{ij}(x_j(s)) ds\Big] \tag{3.5}$$

where $x_i(t)$ is the population of the ith species, a_i, $b_{ij} \in C(\mathbb{R}, \mathbb{R})$ are ω-periodic and $C_{ij}(t,s)$ is piecewise continuous on $\mathbb{R}^2$.

Theorem 3.3 *Suppose that the following conditions hold for $i, j = 1, 2, \ldots, n$.*

(i) $\displaystyle\int_0^\omega a_i(s)ds \neq 0,$

(ii) $\displaystyle b_{ij}(t)\int_0^\omega a_i(s)ds \geq 0,\ C_{ij}(t,s)\int_0^\omega a_i(\nu)d\nu \geq 0$ *for all* $(t,s) \in \mathbb{R}^2$,

(iii) $g_{ij} \in C(\mathbb{R}^+, \mathbb{R}^+)$ *is increasing with* $g_{ij}(0) = 0$,

(iv) $\displaystyle\int_0^\omega b_{ii}(s)ds \neq 0,$

(v) $C_{ij}(t+\omega, s+\omega) = C_{ij}(t,s)$ *for all* $(t,s) \in \mathbb{R}^2$ *with* $\sup_{t\in\mathbb{R}} \int_{-\infty}^t |C_{ij}(t,s)|ds < +\infty$.

Then equation (3.5) has an ω-periodic solution.

Proof. For $x = (x_1, x_2, \ldots, x_n)^T$, define

$$f_i(t, x_t) = -x_i(t)\sum_{j=1}^n b_{ij}(t)x_j(t) - x_i(t)\sum_{j=1}^n \int_{-\infty}^t C_{ij}(t,s) g_{ij}(x_j(s)) ds$$

for $i = 1, 2, \ldots, n$ and set $f = (f_1, f_2, \ldots, f_n)^T$. Then (3.5) can be written in the form of (1.1) with (H_1)− (H4) satisfied. Define

$$b^* = \max\{\|b_{ij}\| : i, j = 1, 2, \cdot, \cdot, \cdot, n\}$$

$$C^* = \max\Big\{\sup_{t\in\mathbb{R}} \sum_{j=1}^n \int_{-\infty}^t |C_{ij}(t,s)|ds : i = 1, 2, \cdot, \cdot, \cdot, n\Big\}$$

$$g^*(u) = \max\{g_{ij}(u) : i, j = 1, 2, \cdot, \cdot, \cdot, n\}$$

Let $x \in K$, where K is defined in (2.4). Then

$$\begin{aligned} |f_i(t,x_t)| &\le |x_i(t)|\Big[b^*\|x\| + \sum_{j=1}^{n}\int_{-\infty}^{t}|C_{ij}(t,s)|g_{ij}(\|x_j\|)ds\Big] \\ &\le |x_i(t)|[b^*\|x\| + C^*g^*(\|x\|)] \end{aligned}$$

and

$$\int_0^\omega |f(t,x_s)|ds \le \omega\|x\|[b^*\|x\| + C^*g^*(\|x\|)].$$

This implies

$$\frac{\int_0^\omega |f(t,x_s)|ds}{\|x\|} \le \omega[b^*\|x\| + C^*g^*(\|x\|)] \to 0$$

as $\|x\| \to 0$. Since $x_i(t) \ge \sigma\|x_i\|$ for all $t \in \mathbb{R}$ whenever $x \in K$ and $b_{ij}(t), C_{ij}(t,s)$ have the same sign, we obtain

$$\begin{aligned} &\int_0^\omega |f_i(t,x_s)|ds \\ &= \sum_{j=1}^{n}\int_0^\omega x_i(t)|b_{ij}(t)|x_j(t)dt + \sum_{j=1}^{n}\int_0^\omega x_i(t)\int_{-\infty}^{t}|C_{ij}(t,s)|g_{ij}(x_j(s))dsdt \\ &\ge \int_0^\omega |b_{ii}(t)|x_i(t)|^2dt \ge \sigma^2\|x_i\|^2\int_0^\omega |b_{ii}(t)|dt \end{aligned}$$

and

$$\int_0^\omega |f(t,x_s)|ds \ge \sigma^2\sum_{i=1}^{n}\|x_i\|^2\min_{1\le i\le n}\int_0^\omega |b_{ii}(t)|dt \ge \frac{\sigma^2}{n}\|x\|^2\min_{1\le i\le n}\int_0^\omega |b_{ii}(t)|dt.$$

Here we have applied the inequality $\Big(\sum_{i=1}^n \|x_i\|\Big)^2 \le n\sum_{i=1}^n \|x_i\|^2$. Thus,

$$\frac{\int_0^\omega |f(t,x_s)|ds}{\|x\|} \to +\infty \text{ as } \|x\| \to +\infty.$$

By Corollary 2.5, equation (3.5) has an ω-periodic solution.

Remark. It is clear from the proof of Theorem 3.3 that condition (iv) can be replaced by

(iv*) $\displaystyle\int_0^\omega\int_{-\infty}^{t}|C_{ii}(t,s)|dsdt \ne 0$ and $g_{ii}(u) \to +\infty$ as $u \to +\infty$.

We conclude this paper by investigating the following scalar equation of advanced and delay type which is highly nonlinear and takes a quite general form.

$$\dot{x}(t) = a(t)x(t) + \int_{-\infty}^{+\infty} C(s)g(t, x(t-\tau_0(t)), x(t-\tau_1(t)), x(t+s))ds \tag{3.6}$$

where $a, \tau_0, \tau_1 \in C(\mathbb{R}, \mathbb{R})$ are ω-periodic, $g \in C(\mathbb{R}^4, \mathbb{R}^+)$, and $C \in L(\mathbb{R}_-, \mathbb{R})$ is piecewise continuous.

Let $u = (u_0, u_1, u_2)^T \in \mathbb{R}^3$ and define $g(t, u) = g(t, u_0, u_1, u_2)$. We assume that $g(t + \omega, u) = g(t, u)$ for all $(t, y) \in \mathbb{R}^4$ and

(P_1^*) $\displaystyle\int_0^\omega a(s)ds \neq 0,$

(P_2^*) $\displaystyle C(t)\int_0^\omega a(s)ds \leq 0$ for all $t \in \mathbb{R}$ and $\displaystyle\int_{-\infty}^{+\infty} |C(u)|du = 1.$

Theorem 3.4 *Under assumptions (P_1^*) and (P_2^*), equation (3.6) has at least one positive ω-periodic solution, provided one of the following conditions hold*

(P_3^*)

$$\lim_{|u|\uparrow 0} \max_{t\in[0,\omega]} \frac{g(t,u)}{|u|} = 0 \quad \text{and} \quad \lim_{|u|\uparrow\infty,\ u_j\geq\sigma|u|,\ 0\leq j\leq 2} \min_{t\in[0,\omega]} \frac{g(t,u)}{|u|} = \infty,$$

(P_4^*)

$$\lim_{|u|\uparrow 0,\ u_j\geq\sigma|u|,\ 0\leq j\leq 2} \min_{t\in[0,\omega]} \frac{g(t,u)}{|u|} = \infty \quad \text{and} \quad \lim_{|u|\uparrow+\infty} \max_{t\in[0,\omega]} \frac{g(t,u)}{|u|} = 0$$

where σ is given in (2.2) and $u = (u_0, u_1, u_2)^T$.

Proof . We first show that under (P_3^*), conditions (2.11) and (2.12) of Corollary 2.5 hold. Let

$$f(t, x_t) = \int_{-\infty}^{+\infty} C(s)g(t, x(t - \tau_0(t)), x(t - \tau_1(t)), x(t + s))ds.$$

By (P_3^*), for any $\varepsilon > 0$, there exists $R_1 > 0$ such that $|u| \leq R_1$ implies $|g(t, u)| \leq \varepsilon|u|$ for all $t \in [0, \omega]$. Now let $x \in C_\omega$ with $\|x\| < R_1/3$. Then

$$|x(t - \tau_0(t))| + |x(t - \tau_1(t))| + |x(t + s)| \leq 3\|x\| < R_1.$$

This yields

$$\begin{aligned} |f(t, x_t)| &\leq \int_{-\infty}^{+\infty} |C(s)|g(t, x(t - \tau_0(t)), x(t - \tau_1(t)), x(t + s))ds \\ &\leq \int_{-\infty}^{+\infty} |C(s)|[\varepsilon(|x(t - \tau_0(t))| + |x(t - \tau_1(t))| + |x(t + s)|)]ds \\ &\leq \int_{-\infty}^{+\infty} |C(s)|ds(3\varepsilon\|x\|) = 3\varepsilon\|x\| \end{aligned}$$

and

$$\frac{\int_0^\omega |f(t, x_t)|dt}{\|x\|} \leq 3\omega\varepsilon.$$

Thus, (2.11) holds. Next, it follows from the second part of (P_3^*) that for any $M > 0$ there exists $R_2 > 0$ such that $[|u| \geq R_2,\ u_j \geq \sigma|u|,\ 0 \leq j \leq 2]$ implies

$$g(t,u) \geq M|u|. \tag{3.7}$$

Let K be given in (2.4) and observe that if $x \in K$, then $x(t-\tau_j(t)) \geq \sigma\|x\|$, $|x(t+s)| \geq \sigma\|x\|$ for all $t \in \mathbb{R}$ and $j = 0, 1$, and

$$|x(t-\tau_0(t))| + |x(t-\tau_1(t))| + |x(t+s)| \geq 3\sigma\|x\| \geq 3\sigma \cdot R_2/(3\sigma) = R_2$$

whenever $\|x\| \geq R_2/(3\sigma)$. By (3.7), we have

$$\begin{aligned} & g(t, x(t-\tau_0(t)), x(t-\tau_1(t)), x(t+s)) \\ & \quad \geq \quad M(|x(t-\tau_0(t))| + |x(t-\tau_1(t))| + |x(t+s)|) \geq M(3\sigma\|x\|). \end{aligned}$$

Using the above inequality, we get

$$\begin{aligned} \int_0^\omega |f(t,x_t)|dt &= \int_0^\omega \int_{-\infty}^{+\infty} |C(s)| g(t, x(t-\tau_0(t)), x(t-\tau_1(t)), x(t+s)) ds\, dt \\ &\geq \int_0^\omega \int_{-\infty}^{+\infty} |C(s)|(3\sigma M\|x\|) ds\, dt = 3\sigma\omega M\|x\|. \end{aligned}$$

Thus,

$$\frac{\int_0^\omega |f(t,x_t)|dt}{\|x\|} \geq 3\sigma\omega M.$$

This proves (2.12) holds. By Corollary 2.5, equation (3.6) has an ω-periodic solution. We omit the case when (P_4^*) holds. This completes the proof.

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