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Existence of global solutions for systems of reaction-diffusion equations on unbounded domains *

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Abstract

We consider, an initial-value problem for the thermal-diffusive combustion system

$$
\begin{aligned}
u_t &= a\Delta u - uh(v) \\
v_t &= b\Delta u + d\Delta v + uh(v),
\end{aligned}
$$

where $a > 0$, $d > 0$, $b \neq 0$, $x \in \mathbb{R}^n$, $n \geq 1$, with $h(v) = v^m$, m is an even nonnegative integer, and the initial data u_0, v_0 are bounded uniformly continuous and nonnegative. It is known that by a simple comparison if $b = 0$, $a = 1$, $d \leq 1$ and $h(v) = v^m$ with $m \in \mathbb{N}^*$, the solutions are uniformly bounded in time. When $d > a = 1$, $b = 0$, $h(v) = v^m$ with $m \in \mathbb{N}^*$, Collet and Xin [2] proved the existence of global classical solutions and showed that the L^∞ norm of v can not grow faster than $O(\log \log t)$ for any space dimension. In our case, no comparison principle seems to apply. Nevertheless using techniques form [2], we essentially prove the existence of global classical solutions if $a < d$, $b < 0$, and $v_0 \geq \frac{b}{a-d} u_0$.

1 Introduction

In this paper, we are concerned with the existence of global solutions of the reaction-diffusion system

$$
\begin{aligned}
u_t &= a\Delta u - uv^m, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\
v_t &= b\Delta u + d\Delta v + uv^m, \quad (x,t) \in \mathbb{R}^n \times (0,\infty),
\end{aligned}
\tag{1.1}
$$

with initial data

$$
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}^n. \tag{1.2}
$$

In (1.1), the constants a, b, d are such that $a > 0$, $d > 0$, $b \neq 0$, m is an even nonnegative integer. Also $4ad \geq b^2$ which reflects the parabolicity of the system.

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Δ is the Laplace operator in x. In (1.2), the initial data u_0, v_0 are nonnegative and are in $C_{UB}(\mathbb{R}^n)$ the space of uniformly bounded continuous functions on $\mathbb{R}^n$.

One of the basic questions for (1.1) with L^∞ initial data is the Existence of global solutions and possibly bounds uniform in time. For $b = 0$ and the Arrhenius reaction, i.e. with $u \exp\{-E/v\}$, $E > 0$ replacing uv^m in (1.1), there are many works on global solutions, see Avrin [1], Larrouturou [7] for results in one space dimension, among others. Recently, Collet and Xin [2] has studied the system (1.1) but with $b = 0$; they proved the existence of global solutions and showed that the L^∞ norm of v can not grow faster than $O(\log\log t)$ for any space dimension.

The system

$$\begin{aligned} u_t &= a\Delta u - uh(v), \quad (x,t) \in \Omega \times (0,\infty) \\ v_t &= b\Delta u + d\Delta v + uh(v), \quad (x,t) \in \Omega \times (0,\infty), \end{aligned}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with Neumann boundary conditions, $b \geq 0$, $a > d$, $v_0 \geq \frac{b}{a-d} u_0 \geq 0$, and $h(s)$ is a differentiable nonnegative function on $\mathbb{R}$ has been studied by Kirane [6].

Such equations describe reaction-diffusion processus in physics, chemistry, biology and population dynamics. If we have two substances of concentrations $u = u(x,t)$ and $v = v(x,t)$ in interaction, the positive numbers a and d are the so-called diffusion coefficients and $b = \vartheta a$, where ϑ is an arbitrary real number which describes the drift of the mass of the substance of concentration $v(x,t)$ (cf. [3, 4, 9]).

Our work is a continuation of the work of Collet and Xin [2]. Here we Have a triangular diffusion matrix ($b \neq 0$). By the same idea we prove the existence of global solutions to system (1.1).

In the sequel, we use the notation:

- $\|\cdot\|$ is the supremum norm on $\mathbb{R}^n$: $\|u\| = \sup_{x\in\mathbb{R}^n} |u(x)|$
- For any $\theta \in C_{UB}(\mathbb{R}^n)$, we write $\int \theta \equiv \int_{\mathbb{R}^n} \theta(x)dx$
- For $\theta \in L^p(\mathbb{R}^n)$ ($p \geq 1$), we denote $\|\theta\|_p^p = \int |\theta|^p$.

2 Existence of a Local Solutions and its Positivity

We convert the system (1.1)-(1.2) to an abstract first order system in the Banach space $X := C_{UB}(\mathbb{R}^n) \times C_{UB}(\mathbb{R}^n)$ of the form

$$\begin{gathered} w'(t) = Aw(t) + F(w(t)), \quad t > 0, \\ w(0) = w_0 \in X\,. \end{gathered} \tag{2.1}$$

Here $w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$; the operator A is defined as

$$Aw := \begin{pmatrix} a\Delta & 0 \\ b\Delta & d\Delta \end{pmatrix} w = (a\Delta u, b\Delta u + d\Delta v),$$

where $D(A) := \{w = \begin{pmatrix} u \\ v \end{pmatrix} \in X : \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} \in X\}$. The function F is defined as $F(w(t)) = \begin{pmatrix} -u(t)h(v(t)) \\ u(t)h(v(t)) \end{pmatrix}$.

It is known that for $\lambda > 0$ the operator $\lambda\Delta$ generates an analytic semigroup $G(t)$ in the space $C_{UB}(\mathbb{R}^n)$:

$$G(t)u = (4\pi\lambda t)^{-n/2} \int_{\mathbb{R}^n} \exp\big(-\frac{|x-y|^2}{4\lambda t}\big)u(y)dy. \tag{2.2}$$

Hence, the operator $A = \begin{pmatrix} a\Delta & 0 \\ b\Delta & d\Delta \end{pmatrix}$ generates an analytic semigroup defined by

$$S(t) = \begin{pmatrix} S_1(t) & 0 \\ \frac{b}{a-d}\left(S_1(t) - S_2(t)\right) & S_2(t) \end{pmatrix}, \tag{2.3}$$

where $S_1(t)$ is the semigroup generated by the operator $a\Delta$, and $S_2(t)$ is the semigroup generated by the operator $d\Delta$ (See [6]).

Since the map F is locally Lipschitz in w in the space X, then proving the existence of classical solutions on maximal existence interval $[0, T_0)$ is standard (cf. [5], [8]).

3 Existence of a Global solution and its Boundedness

For the existence of a global solution, we use the fact that the solutions are positive.

Proposition 3.1 *Let (u, v) be the solution of the problem* (1.1)-(1.2) *such that*

$$u_0(x) \geq 0, \quad x \in \mathbb{R}^n, \tag{3.1}$$

$$a > d, \quad b > 0, \quad v_0(x) \geq \frac{b}{a-d}u_0(x) \quad \forall x \in \mathbb{R}^n, \tag{3.2}$$

then

$$u(x,t) \geq 0, \quad v(x,t) \geq \frac{b}{a-d}u(x,t) \quad \forall (x,t) \in \mathbb{R}^n \times (0, T_0). \tag{3.3}$$

Moreover, the solution is global and uniformly bounded on $\mathbb{R}^n \times [0, \infty)$. In fact for any $t > 0$, we have the estimates:

$$\|u(t)\| \leq \|u_0\| \tag{3.4}$$

$$\|v(t)\| \leq \left(\frac{b}{a-d} + \sqrt{\frac{a}{d}} + \frac{b}{a-d}\sqrt{\frac{a}{d}}\right)\|u_0\| + \|v_0\|. \tag{3.5}$$

Proof. The nonnegativity of u is obtained by simple application of the comparison theorem. Then by the maximum principle we get (3.4). To prove $v \geq 0$ under the conditions (3.1)-(3.2) (see [6]).

The solution (u, v) satisfies the integral equations

$$u(x,t) = S_1(t)u_0 - \int_0^t S_1(t-\tau)u(\tau)v^m(\tau)dt, \tag{3.6}$$

$$\begin{aligned} v(x,t) =& \frac{b}{a-d}S_1(t)u_0 + S_2(t)\big(v_0 - \frac{b}{a-d}u_0\big) - \frac{b}{a-d}\int_0^t S_1(t-\tau)u(\tau)v^m(\tau) \\ &+ \int_0^t S_2(t-\tau)\big(u(\tau)v^m(\tau) + \frac{b}{a-d}u(\tau)v^m(\tau)\big)d\tau\,. \end{aligned} \tag{3.7}$$

Here $S_1(t)$ and $S_2(t)$ are the semigroups generated by the operators $a\Delta$ and $d\Delta$ in the space $C_{UB}(\mathbb{R}^n)$ respectively. Since $a > d$, using the explicit expression of $S_1(t-\tau)\,[u(\tau)v^m(\tau)]$ and $S_2(t-\tau)\,[u(\tau)v^m(\tau)]$, one can observe that

$$\int_0^t S_2(t-\tau)\,[u(\tau)v^m(\tau)]\,d\tau \leq \sqrt{\frac{a}{d}}\int_0^t S_1(t-\tau)\,[u(\tau)v^m(\tau)]\,d\tau. \tag{3.8}$$

On the other hand, using (3.7), (3.8) and the positivity of the function u given in (3.6), we deduce the estimate (3.5). Thus, from (3.4) and (3.5), we deduce that the solution (u, v) is global and uniformly bounded on $\mathbb{R}^n \times [0, \infty)$.

In the case where $d > a$, no comparison principle seems to apply. Nevertheless, we prove the existence of global classical solutions but with $b < 0$.

Theorem 3.2 *Let* (u, v) *be the solution of the problem* (1.1)-(1.2) *satisfying* (3.1) *and*

$$a < d, \quad b < 0, \quad v_0(x) \geq \frac{b}{a-d}u_0(x) \quad \forall x \in \mathbb{R}^n, \tag{3.9}$$

then we have (3.3) *and* (3.4)*. Moreover,* (u, v) *is a global solution.*

Proof. By the same method given in [6], we can show that under the conditions (3.1) and (3.9) the solution satisfies (3.3) and (3.4). However, in this case, it is not easy to prove global existence.

To derive estimates of solutions independent of T_0, so as to continue the classical solutions forever in time, we need to some lemmas.

Lemma 3.3 *Let* (u, v) *be the solution of the system* (1.1)-(1.2)*. Define the functional* $L(u, v) = (\alpha + 2u - \ln(1 + u)) \exp(\varepsilon v)$ *with* $\alpha, \varepsilon > 0$*. Then, for any* $\varphi = \varphi(x, t)$ $(x \in \mathbb{R}^n)$ *a smooth nonnegative function with exponential spacial* *decay at infinity, we have*

$$\begin{aligned}\frac{d}{dt}\int \varphi L & \\ &\leq \int (\varphi_t + d\Delta\varphi) L + \int \varphi(L_2 - L_1)uv^m + \int ((d-a)L_1 + bL_2)\nabla\varphi\nabla u \\ &- \int \varphi\left[(aL_{11} + bL_{12})|\nabla u|^2 + ((a+d)L_{12} + bL_{22})\nabla u\nabla v + dL_{22}|\nabla v|^2\right],\end{aligned} \tag{3.10}$$

where $L_1 = \frac{\partial L}{\partial u}$, $L_2 = \frac{\partial L}{\partial v}$, $L_{11} = \frac{\partial^2 L}{\partial u^2}$, $L_{12} = \frac{\partial^2 L}{\partial u \partial v}$, $L_{22} = \frac{\partial^2 L}{\partial v^2}$.

Proof. Note that $L \geq 0$, $L_1 \geq 0$, $L_2 \geq 0$, $L_{11} \geq 0$, $L_{12} \geq 0$ and $L_{22} \geq 0$. We can differentiate under the integral symbol

$$\frac{d}{dt}\int \varphi L = \int \varphi_t L + \int \varphi(L_2 - L_1)uv^m + a\int \varphi L_1\Delta u + b\int \varphi L_1 \Delta u + d\int \varphi L_2 \Delta v\,. \tag{3.11}$$

Using integration by parts, we get

$$\begin{aligned}\int \varphi L_1\Delta u &= -\int L_1\nabla\varphi\nabla u - \int \varphi L_{11}|\nabla u|^2 - \int \varphi L_{12}\nabla u\nabla v,\\ \int \varphi L_2\Delta u &= -\int L_2\nabla\varphi\nabla u - \int \varphi L_{12}|\nabla u|^2 - \int \varphi L_{22}\nabla u\nabla v,\\ \int \varphi L_2\Delta v &= -\int L_2\nabla\varphi\nabla v - \int \varphi L_{22}|\nabla v|^2 - \int \varphi L_{12}\nabla u\nabla v.\end{aligned} \tag{3.12}$$

Since $-\int L_2\nabla\varphi\nabla v = \int L\Delta\varphi + \int L_1\nabla\varphi\nabla u$, using (3.12) in (3.11) our basic identity (3.10) follows. ♢

Lemma 3.4 *In lemma 3.3, there exist two positive numbers* $\alpha = \alpha(a, b, d, \|u_0\|)$ *and* $\varepsilon = \varepsilon(a, b, d, \|u_0\|)$ *depending only on the coefficients* a, b, d *and the datum* $\|u_0\|$*, such that*

$$\begin{aligned}\frac{d}{dt}\int \varphi L \leq \int (\varphi_t + d\Delta\varphi) L - \frac{1}{2}\int \varphi L_1 uv^m & \\ + \int ((d-a)L_1 + bL_2)\nabla\varphi\nabla u - \frac{1}{2}\int \varphi\left[\frac{a}{2}L_{11}|\nabla u|^2 + dL_{22}|\nabla v|^2\right].&\end{aligned} \tag{3.13}$$

Proof. We choose α and ε in lemma 3.3 such that for any $(u,v) \in [0, \|u_0\|] \times \mathbb{R}^+$,

$$L_2 \leq \frac{1}{2} L_1, \tag{3.14}$$

$$(a+d)^2 L_{12}^2 + b^2 L_{22}^2 + b(2a+d) L_{12} L_{22} - ad L_{11} L_{22} \leq 0 \tag{3.15}$$

$$\frac{L_1^2}{L_{11}} \leq L. \tag{3.16}$$

$$L_{12} \leq \frac{a}{2\,|b|} L_{11}. \tag{3.17}$$

We verify these conditions as follows. Let $L_1 = (2 - \frac{1}{1+u}) e^{\varepsilon v}$, and $L_2 = \varepsilon\,(\alpha + 2u - \ln(1+u))\, e^{\varepsilon v}$; so (3.14) is satisfied if

$$\varepsilon \leq \frac{1 + 2\|u_0\|}{2(\alpha + 2\|u_0\|)(1 + \|u_0\|)}. \tag{3.18}$$

We have $L_{11} = e^{\varepsilon v}/(1+u)^2$, $L_{12} = \varepsilon(2 - \frac{1}{1+u}) e^{\varepsilon v}$ and $L_{22} = \varepsilon^2(\alpha + 2u - \ln(1+u)) e^{\varepsilon v}$.

The condition (3.15) is satisfied if

$$\begin{aligned} 4(a+d)^2 + b^2\varepsilon^2 \left(\alpha + 2\|u_0\|\right)^2 + b(2a+d)\varepsilon \left(\alpha + 2\|u_0\|\right) \\ - \frac{ad}{\left(\alpha + 2\|u_0\|\right)^2} \left(\alpha - \ln(1 + \|u_0\|\right) \leq 0. \end{aligned} \tag{3.19}$$

This equation is verified if $b^2\varepsilon^2 \left(\alpha + 2\|u_0\|\right)^2 \leq 1$ and

$$4(a+d)^2 + \frac{ad}{\left(1 + \|u_0\|\right)^2} \left(\alpha - \ln(1 + \|u_0\|\right) + 1 \leq \alpha \frac{ad}{\left(1 + \|u_0\|\right)^2}.$$

Hence we get from these equations that

$$\alpha \geq \ln(1 + \|u_0\|) + \frac{(1 + \|u_0\|)^2}{ad} \left(1 + 4(a+d)^2\right), \tag{3.20}$$

$$\varepsilon \leq \frac{1}{|b| \left(\alpha + 2\|u_0\|\right)}. \tag{3.21}$$

Now, to verify (3.16), It suffices to take

$$\alpha \geq \ln(1 + \|u_0\|) + \left(1 + 2\|u_0\|\right)^2. \tag{3.22}$$

To verify (3.17), it suffices to take

$$\varepsilon \leq \frac{a}{2\,|b| \left(1 + \|u_0\|\right)^2}. \tag{3.23}$$

Thus, from (3.18), (3.20)-(3.23), the real positive constants α and ε cited in the lemma are defined by

$$\alpha \geq \ln(1+\|u_0\|) + \max\Big\{\frac{(1+\|u_0\|)^2}{ad}\left(1+4(a+d)^2\right), (1+2\|u_0\|)^2 .\Big\}, \quad (3.24)$$

$$\varepsilon \leq \min\Big\{\frac{1+2\|u_0\|}{2(\alpha+2\|u_0\|)(1+\|u_0\|)}, \frac{1}{|b|\,(\alpha+2\|u_0\|)}, \frac{a}{2\,|b|\,(1+\|u_0\|)^2}\Big\}. \quad (3.25)$$

From (3.14) we get

$$\int \varphi(L_2-L_1)uv^m \leq -\frac{1}{2}\int \varphi L_1 uv^m \quad (3.26)$$

and from (3.15), (3.17) we get

$$\begin{aligned} &(aL_{11}+bL_{12})\,|\nabla u|^2 + ((a+d)L_{12}+bL_{22})\nabla u\nabla v + dL_{22}\,|\nabla v|^2 \\ &\geq \frac{1}{2}\left[(aL_{11}+bL_{12})\,|\nabla u|^2 + dL_{22}\,|\nabla v|^2\right] \\ &\geq \frac{1}{2}\left[\frac{a}{2}L_{11}\,|\nabla u|^2 + dL_{22}\,|\nabla v|^2\right]. \end{aligned} \quad (3.27)$$

From (3.26) and (3.27) into (3.10) we get our desired inequality (3.13). As a consequence of the expressions of α, ε and the functional L, they must be such that $\alpha > 16$, $\varepsilon < 1/16$. ♢

Lemma 3.5 *With the value of α given in* (3.24) *and of ε given in* (3.25)*, there exist a test function φ and two real positive constants β and σ such that*

$$\int \varphi L \leq \beta e^{\sigma t}, \quad \forall t > 0. \quad (3.28)$$

Proof. As in [2], we define the test function

$$\varphi(x) = \frac{1}{(1+|x-x_0|)^2}, \quad (3.29)$$

where x_0 is arbitrary in $\mathbb{R}^n$. It is clear that φ is a smooth function with exponential decay at infinity and satisfies

$$|\Delta\varphi| \leq K\varphi, \quad |\nabla\varphi| \leq K\varphi, \quad (3.30)$$

for some positive constant K.

From (3.13) with the test function given in (3.29) and taking in consideration (3.14), (3.17), (3.30), we obtain

$$\frac{d}{dt}\int \varphi L \leq Kd\int \varphi L + K\left[(d-a)+\frac{1}{2}\,|b|\right]\int L_1\varphi\,|\nabla u| - \frac{1}{4}a\int \varphi L_{11}\,|\nabla u|^2. \quad (3.31)$$

We can easily deduce that

$$
\begin{aligned}
K\big[(d-a)+\frac{1}{2}\,|b|\,\big]\int\varphi L_1\,|\nabla u| - \frac{1}{4}a\int\varphi L_{11}\,|\nabla u|^2 \\
\le \frac{K^2}{a}\big((d-a)+\frac{1}{2}\,|b|\,\big)^2\int\varphi\frac{L_1^2}{L_{11}}. \quad (3.32)
\end{aligned}
$$

Because of (3.16), $\frac{L_1^2}{L_{11}} \le L$; hence, from (3.32) and (3.31), we obtain

$$
\frac{d}{dt}\int\varphi L \le \big[Kd+\frac{K^2}{a}\big((d-a)+\frac{1}{2}\,|b|\,\big)^2\big]\int\varphi L.
$$

Thus, we obtain the relation (3.28). More precisely,

$$
\beta \le (\alpha+2\|u_0\|)\,e^{\varepsilon\|v_0\|}\|\varphi\|_1 \quad \text{and} \quad \sigma = Kd+\frac{K^2}{a}\big(\,|a-d|+\frac{1}{2}\,|b|\,\big)^2.
$$

♢

We use our bound on the nonlinear functional (3.28) to control the L^p, $1<p<\infty$, norms of solutions over any unit cube in space. Here we rely on the fact that the integrand L is exponential in v, and so bounds any powers of v from above.

Lemma 3.6 *For any unit cube Q and any finite $p \ge 1$,*

$$
\int_Q |v|^p\,dx \le 2^n\frac{\beta}{\alpha\varepsilon^p}e^{\sigma t}(p+1)^{p+1}. \quad (3.33)
$$

Proof. For any nonnegative integer p, we have

$$
\beta e^{\sigma t} \ge \int\varphi L \ge \alpha\int\varphi e^{\varepsilon v} \ge \frac{\alpha\varepsilon^p}{p!}\int_Q v^p\,dx\,. \quad (3.34)
$$

By taking x_0 as defined in (3.29) at the center of Q, we get (3.33). ♢

As consequence of the lemma above, we can prove that there exist two constants $c=c(n,\lambda)$ and $\omega=\omega(n,\lambda,\|u_0\|,\|v_0\|)$ such that

$$
\begin{aligned}
G(t)u(t)v^m(t) \equiv & \big(4\pi\lambda t\big)^{-n/2}\int_{\mathbb{R}^n}\exp\big(-\frac{|x-y|^2}{4\lambda t}\big)u(y,t)v^m(y,t)dy \\
\le & c\omega e^{\frac{\sigma}{p}t}\varepsilon^{-m}(mp+1)^{m+\frac{1}{p}}\left(t^{\frac{n}{2q}}+t^{-\frac{n}{2p}}\right),
\end{aligned} \quad (3.35)
$$

for any $p>\max\left\{1,\frac{n}{2}\right\}$, $1/P+1/q=1$. The inequality follows. Here $G(t)$ is the semigroup generated by the operator $\lambda\Delta(\lambda>0)$ on the space $C_{UB}(\mathbb{R}^n)$ (See [2]). ♢

Existence of a Global Solution

From (3.7) we have

$$\begin{aligned} v(x,t) =& \frac{b}{a-d} S_1(t) u_0 + S_2(t)\big(v_0 - \frac{b}{a-d} u_0\big) - \frac{b}{a-d}\int_0^t S_1(t-\tau) u(\tau) v^m(\tau) d\tau \\ & + \big(1 + \frac{b}{a-d}\big)\int_0^t S_2(t-\tau) u(\tau) v^m(\tau) d\tau. \end{aligned} \tag{3.36}$$

Since the semigroups $S_1(t)$ and $S_2(t)$ are of contractions and $\frac{b}{a-d} > 0$, we can deduce

$$\frac{b}{a-d} S_1(t) u_0 + S_2(t)\big(v_0 - \frac{b}{a-d} u_0\big) \le \frac{b}{a-d}\|u_0\| + \|v_0\|. \tag{3.37}$$

Integrating on $\tau \in [0,t]$ and using (3.35), we obtain

$$\begin{aligned} \int_0^t S_1(t-\tau) u(\tau) v^m(\tau) d\tau \le & c_1\omega_1 \int_0^t \Big((t-\tau)^{\frac{n}{2q}} + (t-\tau)^{\frac{n}{2q}}\Big) d\tau \\ \le & c_1\omega_1 \big[\frac{2q}{2q+n} t^{\frac{n}{2q}+1} + \frac{2p}{2p-n} t^{1-\frac{n}{2p}}\big], \end{aligned} \tag{3.38}$$

and

$$\int_0^t S_1(t-\tau) u(\tau) v^m(\tau) d\tau \le c_2\omega_2 \big[\frac{2q}{2q+n} t^{\frac{n}{2q}+1} + \frac{2p}{2p-n} t^{1-\frac{n}{2p}}\big], \tag{3.39}$$

where $c_1 = c(n,a)$, $\omega_1 = \omega(n,a,\|u_0\|, \|v_0\|)$, $c_2 = c(n,d)$, $\omega_2 = \omega(n,d,\|u_0\|, \|v_0\|)$. Using (3.37)-(3.39) in (3.36), we get

$$\|v(t)\| \le \frac{b}{a-d}\|u_0\| + \|v_0\| + \big(1 + \frac{b}{a-d}\big) c_2\omega_2 \big[\frac{2q}{2q+n} t^{\frac{n}{2q}+1} + \frac{2p}{2p-n} t^{1-\frac{n}{2p}}\big], \tag{3.40}$$

for all $t \ge 0$, where $p > \frac{n}{2}$. Thus the estimates (3.4) and (3.40) and the standard parabolic regularity theory implies the existence of a global classical solution $(u,v) \in (C([0,+\infty); C_{UB}) \cap C^1((0,+\infty); C_{UB}))^2$. ♢

Corollary 3.7 *Under the assumptions in Theorem 3.2, there exists a classical global solution when the nonlinear reaction term has the form $uf(v)$, where $f(v)$ is nonnegative continuous in $v \in \mathbb{R}$ and nondecreasing for $v \ge 0$, $f(0) = \lim_{v \searrow 0^+} f(v) = 0$, and $\lim_{v \nearrow \infty} f(v) > 0$, $\lim_{v \nearrow \infty} \frac{1}{v}\log[f(v)] = 0$. In particular, this form includes the Arrhenius reaction $uv^m \exp[-\frac{E}{v}]$, for m an even nonnegative integer and $E > 0$.*

Proof. The estimates in Theorem 3.2 remain true for $f(v)$, since it is bounded from above by the exponential function $e^{\varepsilon v}$ in the nonlinear functional L, thanks to the subexponential growth condition on f. In fact, inequality (3.34), now simply reads

$$\beta e^{\sigma t} \ge \int \varphi L \ge \alpha \int \varphi e^{\varepsilon v} \ge \frac{\alpha}{2^n c_p}\int_Q [f(v)]^p \, dx,$$

for some constant c_p depending on p and f. The remaining estimates go through as before. Elsewhere we replace v^m by $f(v)$. $\diamondsuit$

4 Remarks

(a) In the system (1.1) we have assumed that m is an even integer number in order the maximum principle will be applicable, and consequently the second component v will be positive. The positivity of v is used to prove the global existence of solutions. In the other cases of m the existence of global solution is unknown.

(b) In the case where $a < d$ and $b > 0$; the possibility of the existence of a global solution is an open question.

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