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Regularity bounds on Zakharov system evolutions *

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Abstract

Spatial regularity properties of certain global-in-time solutions of the Zakharov system are established. In particular, the evolving solution $u(t)$ is shown to satisfy an estimate $\|u(t)\|_{H^s} \leq C|t|^{(s-1)+}$, where H^s is the standard spatial Sobolev norm. The proof is an adaptation of earlier work on the nonlinear Schrödinger equation which reduces matters to bilinear estimates.

1 Introduction

We consider the initial value problem for the Zakharov system [15] on $\mathbb{R}^2$

$$\begin{aligned} iu_t + \Delta u = nu, &\qquad u : \mathbb{R}^2 \times [-T_*, T^*] \to \mathbb{C}, \\ \Box n = \Delta |u|^2, &\qquad n : \mathbb{R}^2 \times [-T_*, T^*] \to \mathbb{R}, \\ (u, n, \dot{n})(0) = (\phi, a, b). & \end{aligned} \tag{1.1}$$

Suppose b is such that there exists $V : \mathbb{R}^2 \to \mathbb{R}^2$ with $b = \nabla \cdot V$. Then the Zakharov system may be rewritten in Hamiltonian form with Hamiltonian

$$H(u, \bar{u}, n, V) = \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{1}{2}(n^2 + |V|^2) + n|u|^2 dx. \tag{1.2}$$

For initial data ϕ small enough in L^2 we can conclude from conservation of (1.2) that

$$\|u(t), n(t), \dot{n}(t)\|_{H_1} = \|u(t), n(t), \dot{n}(t)\|_{H^1 \times L^2 \times \widehat{H}^{-1}} \leq C\|\phi, a, b\|_{H_1} \tag{1.3}$$

where $H_1 := H^1 \times L^2 \times \widehat{H}^{-1}$ and $\widehat{H}^{-1}$ is defined by $\|b\|_{\widehat{H}^{-1}} = \|V\|_{L^2}$. Local wellposedness of (1.1) for data $(\phi, a, b) \subset H_1$ was established in [6, 7], with the lifetime of existence satisfying

$$T > \|\phi, a, b\|_{H_1}^{-\alpha} \quad \text{for some } \alpha > 0. \tag{1.4}$$

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The regularity requirements for the local results in [6, 7] have subsequently been improved in [9]. Hence, for data implying a priori H_1 control (1.3), the local result may be iterated to prove the existence of global-in-time solutions of (1.1). In fact, global solutions of the initial value problem (1.1) had been shown to exist earlier [1] ($d = 1$) [14] ($d = 2$) using energy methods in spaces requiring more regularity than H_1.

Remark. The initial value problem (1.1) has solutions which blow up in finite time [11, 10]. At the present time, there is no criteria known which identifies those initial data leading to blow up and those leading to global-in-time solutions. This paper provides regularity bounds on those global solutions obtained by iterating the local wellposedness result using a priori H_1 control.

Let $\square^{-1}F$ denote the solution of the inhomogeneous wave equation with zero data,

$$\begin{gathered} \square n = F, \\ (n, \dot{n})(0) = (0, 0). \end{gathered} \tag{1.5}$$

Let $W(a, b)$ denote the solution of

$$\begin{gathered} \square n = 0 \\ (n, \dot{n})(0) = (a, b). \end{gathered} \tag{1.6}$$

Note that $\square^{-1}F$ and $W(a, b)$ may be explicitly represented using the Fourier transform. The (formal) solution of the second equation in (1.1) is

$$n = W(a, b) + \square^{-1}(\Delta|u|^2). \tag{1.7}$$

Substituting this expression for n into the first equation in (1.1) gives

$$\begin{gathered} u_t = i\Delta u - iW(a, b)u - i\,\square^{-1}\Delta(|u|^2)u, \\ u(0) = \phi. \end{gathered} \tag{1.8}$$

Note that the regularity properties of the data a, b and of u, inferred from solving (1.8), determine the regularity properties of n through (1.7).

Let $\mathcal{S}$ denote the Schwarz class. Consider initial data $\phi, a \in \mathcal{S},\ b \in \mathcal{S} \bigcap \widehat{H}^{-1}$ implying a priori H_1 control (1.3). How do the regularity properties of the global solution $(u(t), n(t))$ behave as $t \to \infty$? In particular, can we describe, or at least bound from above and below, $\|u(t), n(t), \dot{n}(t)\|_{H_s}$ for $s \gg 1$ as $t \to \infty$? These estimates quantify the shift of the conserved L^2 mass in frequency space. In particular, the upper bounds we obtain in this paper limit the rate of transfer from low frequencies to high frequencies. By the note following (1.8), it suffices to understand $\|u(t)\|_{H^s}$.

The local result for (1.8) implies

$$\sup_{t \in [0,T]} \|u(t)\|_{H^s} \leq \|\phi\|_{H^s} + C\|\phi\|_{H^s} \tag{1.9}$$

which iterates to give an exponential bound $\|u(t)\|_{H^s} \leq C^{|t|}$. Bourgain observed that a slight improvement of (1.9)

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \leq \|\phi\|_{H^s} + C\|\phi\|_{H^s}^{1-\delta},\ 0<\delta<1, \tag{1.10}$$

implies the polynomial bound $\|u(t)\|_{H^s} \leq C|t|^{1/\delta}$. This observation was exploited in [3] to prove polynomial bounds on high Sobolev norms for solutions of the nonlinear Schrödinger equation and certain nonlinear wave equations.

Staffilani [12, 13] improved the degree of the polynomial upper bound using a different approach to prove (1.10) in the case of the nonlinear Schrödinger equation. The crucial bilinear estimate used in this approach has recently been improved [8] giving a slightly better polynomial estimate. This paper adapts the arguments from [13] for the nonlinear Schrödinger equation to prove similar polynomial bounds on high Sobolev norms for the global solutions of (1.1) constructed in [6, 7].

Theorem 1.1 *Assume* $(\phi, a, b) \in \mathcal{S}\times\mathcal{S}\times(\mathcal{S}\cap\widehat{H}^{-1})$. *Global solutions of* (1.1) *satisfying* (1.3) *also satisfy*

$$\|u(t)\|_{H^s} \leq C|t|^{(s-1)+}. \tag{1.11}$$

The question of lower bounds showing growth of high Sobolev norms remains a fascinating open question. For a more thorough discussion, including model equations other than the Zakharov system, see the book of Bourgain [5].

2 Reduction to bilinear estimate

Our goal is to bound $\|u(t)\|_{H^s}$ for u, the solution of (1.8), with $t\in[0,T]$ and T as in (1.4). Since $\|u(t)\|_{L^2} = \|\phi\|_{L^2}$ for all t, it suffices to bound $\|B^s u(t)\|_{L^2}$ where $B=\sqrt{-\Delta}$. Let's assume $s=2m$ for $1\ll m\in\mathbb{N}$ to avoid certain technical issues involving fractional derivatives below. Let $\langle\cdot,\cdot\rangle$ denote the standard L^2_x inner product, $\langle f,g\rangle = \int_{\mathbb{R}^2} f\bar{g}dx$. By the fundamental theorem of calculus,

$$\|B^s u(t)\|_{L^2}^2 = \|B^s u(0)\|_{L^2}^2 + \int_0^t \frac{d}{d\sigma}\langle B^s u(\sigma), B^s u(\sigma)\rangle d\sigma. \tag{2.1}$$

We calculate

$$I = 2\Re \int_0^t \langle B^s \dot{u}(\sigma), B^s u(\sigma)\rangle d\sigma. \tag{2.2}$$

Now, using the equation (1.8), we find

$$\begin{aligned} I = &-2\Im \int_0^t \langle B^s \Delta u(\sigma), B^s u(\sigma)\rangle d\sigma \\ &+ 2\Im \int_0^t \langle B^s[W(a,b)(\sigma)u(\sigma)], B^s u(\sigma)\rangle d\sigma \\ &+ 2\Im \int_0^t \langle B^s[\Box^{-1}\Delta(|u|^2)u(\sigma)], B^s u(\sigma)\rangle d\sigma. \end{aligned} \tag{2.3}$$

Denote the three terms on the right-side of (2.3) by $I_1 + I_2 + I_3$. Upon writing $-\Delta = B^2$ and integrating by parts, the first term I_1 is seen to have a real integrand so this term is zero. The term I_2 involves $B^s(W(a,b)u)$. Various terms arise from the Leibniz rule for differentiating a product. The most dangerous of these is $W(a,b)B^s u$ but, since $W(a,b)$ is a real-valued function, this term leads to a purely real integrand in (2.3) and so disappears. Hence, the term I_2 leads to a sum of terms of the form

$$C\Im\int_0^t \langle [B^{s_1}W(a,b)(\sigma)][B^{s_2}u(\sigma)], B^s u(\sigma)\rangle d\sigma, \tag{2.4}$$

where $s = s_1 + s_2,\ 1 \le s_1 \le s,\ 0 \le s_2 \le s-1,\ s_1, s_2 \in \mathbb{N}$,

We can multiply by a smooth cutoff function in time $\psi_T \sim \chi_{[0,T]}$ and estimate these terms via the Hölder inequality by

$$\|B^{s_1}W(a,b)\|_{L^2_{x,t\in[0,T]}}\|B^{s_2}u\|_{L^4_{xt}}\|B^s u\|_{L^4_{xt}}. \tag{2.5}$$

The Strichartz estimate for the paraboloid and properties of $X_{s,b}$ spaces [2] imply for $b = \frac{1}{2}+$,

$$\|B^{\tilde{s}}u\|_{L^4_{xt}} \le C_T\|u\|_{X_{\tilde{s},b}}. \tag{2.6}$$

Here the space $X_{s,b}$ is defined using the norm

$$\|u\|_{X_{s,b}} = \Big(\int (1+|k|)^{2s}(1+|\lambda+|k|^2|)^{2b}|\widehat{u}(k,\lambda)|^2 dk d\lambda\Big)^{1/2}.$$

The local wellposedness result [6, 9] gives

$$\|u\|_{X_{\tilde{s},b}} \le C\|u(0)\|_{H^{\tilde{s}}}. \tag{2.7}$$

Therefore, the second term I_2 in (2.3) is estimated by a sum of terms of the form

$$\|a,b\|_{H^{s_1}\times H^{s_1-1}}\|u(0)\|_{H^{s_2}}\|u(0)\|_{H^s}. \tag{2.8}$$

The first factor is bounded by a constant which depends upon the initial data a, b. The second factor may be interpolated between $\|\phi\|_{H^1}$ and $\|\phi\|_{H^s}$ which leads to the bound

$$|I_2| \le C\sum_{0\le s_2\le s-1}\|u(0)\|_{H^s}^{1+\frac{s_2-1}{s-1}} \le C\|u(0)\|_{H^s}^{2-\frac{1}{s-1}}. \tag{2.9}$$

It remains to bound I_3. Since differentiation in x commutes with $\Box^{-1}\Delta$, the Leibniz rule shows

$$I_3 = 2\Im\sum_{\substack{0\le\sigma_1,\sigma_2,\sigma_3\le s\\ \sigma_1+\sigma_2+\sigma_3=s}} c_{\sigma_1,\sigma_2,\sigma_3}\int_0^t \langle \Box^{-1}\Delta(B^{\sigma_1}uB^{\sigma_2}\bar{u})B^{\sigma_3}u, B^s u\rangle d\tau. \tag{2.10}$$

In the case $\sigma_3 = s$, the resulting integrand is purely real so this term disappears. Consider first those terms with $\sigma_3 \le s-2$ and after treating these we will consider the terms with $\sigma_3 = s-1$. Since we are interested in proving a local-in-time estimate, we can insert a smooth cutoff $\psi_T \sim \chi_{[0,T]}$ and wish to bound

$$|\psi_T(t)\int_0^t \Box^{-1}\Delta(B^{\sigma_1}uB^{\sigma_2}\bar{u})B^{\sigma_3}uB^s\bar{u}d\tau|. \tag{2.11}$$

A formal "integration by parts" (which is justified rigorously in the next section when we define $(\Box^{-1}\Delta)^{1/2}$) allows us to bound by

$$|\psi_T(t)\int_0^t (\Box^{-1}\Delta)^{1/2}(B^{\sigma_1}uB^{\sigma_2}\bar{u})(\Box^{-1}\Delta)^{1/2}(B^{\sigma_3}uB^s\bar{u})d\tau| \tag{2.12}$$

and Cauchy-Schwarz reduces matters to controlling

$$\|(\Box^{-1}\Delta)^{1/2}(B^{\sigma_1}uB^{\sigma_2}\bar{u})\|_{L^2_{xT}}\|(\Box^{-1}\Delta)^{1/2}(B^{\sigma_3}uB^s\bar{u})\|_{L^2_{xT}}. \tag{2.13}$$

Proposition 2.1 *Let $0 \le s_1 \in \mathbb{N}$ and $1 \le s_2 \in \mathbb{N}$, $s_1 \le s_2$. For $b > 1/2$,*

$$\|(\Box^{-1}\Delta)^{1/2}([B^{s_1}u_1][B^{s_2}\overline{u_2}])\|_{L^2_{xT}} \le C\|u_1\|_{X_{s_1+1,b}}\|u_2\|_{X_{s_2-\frac{1}{2},b}}. \tag{2.14}$$

The estimate is also valid if the complex conjugation is moved from u_2 to u_1 on the left-side of (2.14).

Suppose the proposition is true. The bilinear expressions in (2.13) are estimated by

$$\|u\|_{X_{\sigma_1+1,b}}\|u\|_{X_{\sigma_2-\frac{1}{2},b}}\|u\|_{X_{\sigma_3+1,b}}\|u\|_{X_{s-\frac{1}{2},b}}. \tag{2.15}$$

Using the local result we know $\|u\|_{X_{\tilde{s},b}} \le C\|u(0)\|_{H^{\tilde{s}}}$ and upon interpolating the various $H^{\tilde{s}}$ norms between H^1 and H^s (using (1.3)) bounds (2.15) by

$$C\|u(0)\|_{H^s}^{\frac{\sigma_1+\sigma_2-\frac{1}{2}-1+\sigma_3+s-\frac{1}{2}-1}{s-1}}. \tag{2.16}$$

Recalling that $\sigma_1+\sigma_2+\sigma_3 = s$, the exponent simplifies to $2-\frac{1}{s-1}$, just as in (2.9).

Now, consider a term in (2.10) with $\sigma_3 = s-1$. Evidently, $\sigma_1 = 1,\ \sigma_2 = 0$ or $\sigma_1 = 0,\ \sigma_2 = 1$. In this case, we apply Cauchy-Schwarz directly to the term as it appears in (2.10) to bound by

$$\|\Box^{-1}\Delta(B^{\sigma_1}uB^{\sigma_2}\bar{u})\|_{L^2_{xT}}\|B^{\sigma_3}uB^s\bar{u}\|_{L^2_{xT}}. \tag{2.17}$$

The second factor is readily estimated using the Bourgain's refinement of the Strichartz inequality [4] to give

$$\|u\|_{X_{\sigma_3+\frac{1}{2}+,b}}\|u\|_{X_{s-\frac{1}{2},b}} \le C\|u\|^2_{X_{s-\frac{1}{2}+,b}}. \tag{2.18}$$

The first factor in (2.17) is bounded using a variant of Proposition 2.1.

Proposition 2.2 *Let $0 \le s_1 \in \mathbb{N}$, $1 \le s_2 \in \mathbb{N}$, $s_1 \le s_2$. For $b > 1/2$,*

$$\|\Box^{-1}\Delta([B^{s_1}u_1][B^{s_2}\overline{u_2}])\|_{L^2_{xT}} \le C\|u_1\|_{X_{s_1+1,b}}\|u\|_{X_{s_2,b}}. \tag{2.19}$$

The estimate is also valid if the complex conjugation is moved from u_2 to u_1 on the left-side of (2.19).

Combining (2.18), (2.19) shows (2.17) may be bounded using $\sigma_1 = 1, \sigma_2 = 0$ or $\sigma_1 = 0, \sigma_2 = 1$ and $\sigma_3 = s-1$,

$$\|u\|_{X_{1,b}}\|u\|_{X_{1,b}}\|u\|^2_{X_{s-\frac{1}{2}+,b}} \le C\|u\|^2_{X_{s-\frac{1}{2}+,b}}.$$

The local result and interpolation bounds this by

$$C\|u(0)\|_{H^s}^{2(\frac{s-\frac{1}{2}-1}{s-1})+} = C\|u(0)\|_{H^s}^{2-\frac{1}{s-1}+}$$

which (up to the +) is the same as in (2.9), (2.16).

Summarizing, the two Propositions above show that the integral term in (2.1) is bounded by

$$C\|u(0)\|_{H^s}^{2-\frac{1}{s-1}+}.$$

We may assume that $\|B^s u(t)\|_{L^2} \ge \|B^s u(0)\|_{L^2}$ for otherwise (1.10) is automatic. Therefore, we can divide (2.1) through by $\|B^s u(t)\|_{L^2}$ and with L^2 conservation observe (1.10) holds with $\delta = \frac{1}{s-1}-$ proving Theorem 1.1.

The next section establishes the Propositions and defines $(\Box^{-1}\Delta)^{1/2}$ used in the treatment of $\sigma_3 \le s-2$ terms above.

3 Bilinear Estimates

In this section, we present a proof of Proposition 2.1. Along the way we will observe explicit properties of the operator $\Box^{-1}\Delta$ which allow us to justify step (2.12) in the previous section. Proposition 2.2 will follow from modifications of the proof of Proposition 2.1.

The operator $\Box^{-1}$ was defined as the mapping taking the inhomogeneity F to the solution of the linear initial value problem (1.5). It can be explicitly represented using the Fourier transform as

$$\begin{aligned}&\Box^{-1}F(x,t)\\ &= -\iint e^{ik\cdot x}\big[e^{i\lambda t} - \frac{1}{2}(1+\frac{\lambda}{|k|})e^{i|k|t} - \frac{1}{2}(1-\frac{\lambda}{|k|})e^{-i|k|t}\big]\frac{\widehat{F}(k,\lambda)}{(\lambda-|k|)(\lambda+|k|)}dk\,d\lambda\end{aligned} \tag{3.1}$$

where $\widehat{F}$ denotes the space-time Fourier transform of F. A Taylor series argument show that the apparent singularities along $\lambda \pm |k| = 0$ do not occur and that

$$|\widehat{\Box^{-1}\Delta}F(k,\lambda)| \le C|\widehat{F}(k,\lambda)|\frac{|k|}{(1+|\lambda \pm |k||)} + |\widehat{F}(k,\lambda)|_{\{\lambda=\mp|k|\}}|\frac{|k|}{(1+|\lambda \pm |k||)}. \tag{3.2}$$

From an L^2 point-of-view, it is therefore natural to define for real numbers α,

$$\begin{aligned}&(\widehat{\Box^{-1}\Delta})^{\alpha}(k,\lambda)\\ &\quad = |\widehat{F}(k,\lambda)|\big(\frac{|k|}{(1+|\lambda \pm |k||)}\big)^{\alpha} + \big|\widehat{F}(k,\lambda)|_{\{\lambda=\mp|k|\}}\big|\big(\frac{|k|}{(1+|\lambda \pm |k||)}\big)^{\alpha}.\end{aligned} \tag{3.3}$$

In particular, we have defined the operator $\big(\Box^{-1}\Delta\big)^{1/2}$ which appears in the statement of Proposition 2.1. For two functions of space-time, F, G, which are cutoff to $t \in [0,T]$, consider the expression (analagous to (2.11))

$$\iint (\Box^{-1}\Delta)(F)G\,dx\,dt = \iint (\widehat{\Box^{-1}\Delta)(F})\widehat{G}\,dk\,d\lambda. \tag{3.4}$$

We insert (3.2) and take the absolute value under the integral sign. Then, upon writing

$$\frac{|k|}{(1+|\lambda \pm |k|\,|)} = \frac{|k|^{1/2}}{(1+|\lambda \pm |k|\,|)^{1/2}}\frac{|k|^{1/2}}{(1+|\lambda \pm |k|\,|)^{1/2}}$$

we observe that

$$\Big|\iint (\Box^{-1}\Delta)(F)G\,dx\,dt\Big| \le \Big|\iint (\Box^{-1}\Delta)^{1/2}\widetilde{F}\cdot(\Box^{-1}\Delta)^{1/2}\widetilde{G}\,dx\,dt\Big|. \tag{3.5}$$

where $\widetilde{F}(x,t) = \int e^{i(kx+\lambda t)}|\widehat{F}(k,\lambda)|dkd\lambda$ and $\widetilde{G}$ is similarly defined. For proving L^2-type estimates, the distinction between F and $\widetilde{F}$ is unimportant. In particular, the "integration by parts" step (2.12) is validated.

Now that $(\Box^{-1}\Delta)^{1/2}$ has been given a precise meaning, we turn our attention to proving the inequality (2.14)

Proof of Proposition 2.1 Since $\widehat{\Box^{-1}\Delta}(k,\lambda) \sim \frac{|k|}{\lambda \pm |k|}$, we see that $\Box^{-1}\Delta$ can be as bad as one derivative in x. Therefore, the number of derivatives on both the left-side and right-side of (2.14) is $s_1 + s_2 + \frac{1}{2}$. The desired estimate (2.14) may be reexpressed using duality and certain renormalizations as

$$\begin{aligned}\int_* d(k,\lambda)\big(\frac{|k|}{(1+|\lambda \pm |k||)}\big)^{1/2}\frac{(1+|k_1|)^{-1}c(k_1,\lambda_1)}{(1+|\lambda_1 \pm |k_1|^2|)^b}\frac{(1+|k_2|)^{\frac{1}{2}}c(k_2,\lambda_2)}{(1+|\lambda_2 \pm |k_2|^2|)^b}&\\ \le \|d\|_{L^2}\|c_1\|_{L^2}\|c_2\|_{L^2}&\end{aligned} \tag{3.6}$$

where $\int_*$ is shorthand for $\int_{\substack{k=k_1+k_2\\ \lambda=\lambda_1+\lambda_2}}$ and without loss of generality we assume $d, c_1, c_2 \ge 0$. The choices of $\pm$ in (3.6) are assumed to be independent in the following analysis. In fact, this is only the first contribution arising from the right-side of (3.3). The other "on-light-cone" piece may be similarly estimated. We analyze (3.6) in cases depending upon the size of $|k_1|, |k_2|$.

Case 1. $|k_1|, |k_2| \leq 10$. We may ignore $|k|^{\frac{1}{2}}(1+|k_1|)^{-1}(1+|k_2|)^{\frac{1}{2}}$ and then drop the (potentially helpful) wave remnant $(1+|\lambda \pm |k||)^{\frac{1}{2}}$ to bound the left-side of (3.6) by

$$\int_* d(k,\lambda)\frac{c_1(k_1,\lambda_1)}{(1+|\lambda_1 \pm |k_1|^2|)^b}\frac{c_2(k_2,\lambda_2)}{(1+|\lambda_2 \pm |k_2|^2|)^b}. \tag{3.7}$$

Fourier transform properties show this equals $\langle \widehat{\mathcal{D}}, \widehat{\mathcal{C}_1} * \widehat{\mathcal{C}_2}\rangle = \langle \mathcal{D}, \mathcal{C}_1\mathcal{C}_2\rangle$ where $\mathcal{D}$, $\mathcal{C}_1$, $\mathcal{C}_2$ are functions of space and time whose Fourier transforms are d, $\frac{c_1(k_1,\lambda_1)}{(1+|\lambda_1 \pm |k_1|^2|)^b}$, $\frac{c_2(k_2,\lambda_2)}{(1+|\lambda_2 \pm |k_2|^2|)^b}$, respectively. By Hölder's inequality, we can estimate by $\|\mathcal{D}\|_{L^2_{xT}}\|\mathcal{C}_1\|_{L^4_{xT}}\|\mathcal{C}_2\|_{L^4_{xT}}$ and obtain (3.6) in this case using Plancherel and the Strichartz inequality for the paraboloid as written in [2],

$$\Big\| \int \frac{a(k,\lambda)}{(1+|\lambda + |k|^2|)^b} dk d\lambda \Big\|_{L^4(\mathbb{R}^2_x \times \mathbb{R}_t)} \leq \|a\|_{L^2_{k\lambda}}, \quad b > \frac{1}{2}. \tag{3.8}$$

The standard steps going from(3.7) through $L^2L^4L^4$ to (3.6) will be omitted from the discussion below.

Case 2. $|k_1| \gtrsim |k_2|$, $|k_1| \gtrsim 10$. The case defining conditions imply $|k| \lesssim |k_1|$. We again ignore the wave remnant and use $(1+|k_1|)^{-1}$ to cancel away $|k|^{\frac{1}{2}}$ and $(1+|k_2|)^{\frac{1}{2}}$. We again encounter (3.7) and complete this case with the $L^2L^4L^4$ Hölder argument using (3.8).

Case 3. $|k_1| \ll |k_2|$, $|k_2| \gtrsim 10 \implies |k_2| \sim |k|$. The numerator $(1+|k_1|)^{-1}$ is not helpful in this case so we exploit the denominators in (3.6) to cancel $|k|^{\frac{1}{2}}$ and $(1+|k_2|)^{\frac{1}{2}}$. Since $k = k_1 + k_2$, $\lambda = \lambda_1 + \lambda_2$, the triangle inequality implies

$$\max(|\lambda \pm |k||, |\lambda_2 \pm |k_2|^2|, |\lambda_1 \pm |k_1|^2|) \geq |\pm|k| + |k_2|^2 - |k_1|^2| \sim |k_2|^2 \sim |k|^2. \tag{3.9}$$

Case 3.A. $|\lambda \pm |k||$ is the max in (3.9). We use the large denominator to cancel $|k|^{\frac{1}{2}}$ and $(1+|k_2|)^{\frac{1}{2}}$ and proceed as with (3.7).

Case 3.B. $|\lambda_2 \pm |k_2|^2|$ is the max in (3.9). Most of the large denominator is used to cancel away $|k|^{\frac{1}{2}}$ and $(1+|k_2|)^{\frac{1}{2}}$ and we need to control

$$\int_* \frac{d(k,\lambda)}{(1+|\lambda \pm |k||)^{\frac{1}{2}}}\frac{c_1(k_1,\lambda_1)}{(1+|\lambda_1 \pm |k_1|^2|)^b}\frac{c_2(k_2,\lambda_2)}{(1+|\lambda_2 \pm |k_2|^2|)^{b-\frac{1}{2}}}.$$

Since $b > \frac{1}{2}$, so that $b - \frac{1}{2} > 2\delta > 0$, and $|\lambda_2 \pm |k_2|^2| \gtrsim |\lambda \pm |k||$, we can write

$$\int_* \frac{(1+|k|)^{-\delta} d(k,\lambda)}{(1+|\lambda \pm |k||)^{\frac{1}{2}+}}\frac{(1+|k_1|)^{-\delta} c(k_1,\lambda_1)}{(1+|\lambda_1 \pm |k_1|^2|)^b} c_2(k_2,\lambda_2). \tag{3.10}$$

Let $\mathcal{D}(k,\lambda) = \frac{(1+|k|)^{-\delta} d(k,\lambda)}{(1+|\lambda \pm |k||)^{\frac{1}{2}+}}$ and $\mathcal{C} = \frac{(1+|k_1|)^{-\delta} c(k_1,\lambda_1)}{(1+|\lambda_1 \pm |k_1|^2|)^b}$. Then (3.10) may be expressed as $\langle \mathcal{D} * \mathcal{C}, c_2\rangle$ and Cauchy-Schwarz reduces matters to controlling $\|\mathcal{D} * \mathcal{C}\|_{L^2_{xT}}$. This is accomplished in the following lemma.

Lemma 3.1 *For $b = \frac{1}{2}+$ and a fixed small $\delta > 0$*

$$\int_{*,|k_2|,|k_1|\geq 10} f(k,\lambda)\frac{(1+|k_2|)^{-\delta}d(k_2,\lambda_2)}{(1+|\lambda_2 \pm |k_2||)^{\frac{1}{2}+}}\frac{(1+|k_1|)^{-\delta}c(k_1,\lambda_1)}{(1+|\lambda_1 \pm |k_1|^2|)^{b}} \leq \|f\|_{L^2}\|d\|_{L^2}\|c_1\|_{L^2}. \quad (3.11)$$

Proof. Since $\frac{1}{2}+$ and b exceed $\frac{1}{2}$, the estimate (3.11) may be reduced to the "on-curve" setting using parabolic (for c_1) and light-cone (for d) level set decompositions (see, for example, [8]). This reduces considerations to showing that

$$\int_{|k_2|\geq|k_1|\geq 1} f(x_1+k_2, \pm|k_1|^2 \pm |k_2|)|k_2|^{-\delta}\psi(k_2)|k_1|^{-\delta}\phi(k_1)dk_1dk_2 \leq \|f\|_{L^2_{k\lambda}}\|\psi\|_{L^2_k}\|\phi\|_{L^2_k}. \quad (3.12)$$

Consider the piece of the integration on the left-side of (3.12) arising from $\{k_2 : |k_2| \sim K_2(\text{dyadic})\} \times \{k_1 : |k_1| \sim K_1(\text{dyadic})\}$. We make a change of variables, where superscripts refer to vector components, $u^1 = k_1^1 + k_2^1,\ u^2 = k_1^2 + k_2^2,\ v = \pm|k_1|^2 \pm |k_2|$ and we assume that the component k_1^2 of k_1 satisfies $k_1^2 \sim |k_1| \sim K_1$. (This may be accomplished by cutting in pie slices and making a rotation of coordinates if necessary.) This change of variables followed by Cauchy-Schwarz shows (3.12) is bounded by

$$\|f\|_{L^2_{k\lambda}} K_1^{-\delta}K_2^{-\delta}\int_{|k_1^1|\leq K_1}\Big(\int_{|k_i|\sim K_i}|\psi(k_2)|^2|\phi(k_1)|^2\frac{1}{|J|}dk_1^2dk_2^1dk_2^2\Big)^{1/2}dk_1^1 \quad (3.13)$$

where the Jacobian is

$$|J| = |2k_1^2 \pm 1| \sim K_1. \quad (3.14)$$

We apply Cauchy-Schwarz in k_1^1 and pick up an extra factor of $K_1^{\frac{1}{2}}$ which is cancelled away by the Jacobian factor. The small prefactors $K_i^{-\delta}$ allow us to sum over large dyadic scales thereby proving (3.12) and the lemma.

The lemma shows that (3.10) is bounded as claimed in (3.6).

Case 3.C. $|\lambda_1 \pm |k_1|^2|$ is the max in (3.9). This case follows with a modification of the argument for Case 3.B.

The proof of Proposition 2.2 follows the same case structure as the proof of Proposition 2.1. The only difference is in the accounting of the extra $\frac{1}{2}$ derivative in both sides of (2.19) in comparison with (2.14).

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