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REACTION-DIFFUSION SYSTEMS WITH 1-HOMOGENEOUS NON-LINEARITY

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ABSTRACT. We describe the dynamics of a system of two reaction-diffusion equations with 1-homogeneous non-linearity. We show that either an order-preserving property holds and can be used in order to determine the limiting behaviour in some (invariant) sets or the long time behaviour of all solutions can be described by looking at one scalar reaction-diffusion equation only.

1. INTRODUCTION

The dynamics of solutions of one reaction-diffusion equation

$$\frac{d}{dt}y = y_{xx} + g(t,x,y) \tag{1.1}$$

with Dirichlet or other boundary conditions on an interval $(0,L)$, $L>0$, has been examined by many authors [10, 11, 12, 15, 18]. Far less is known for a system of two reaction-diffusion equations

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + f\big(t,x,\begin{pmatrix} u \\ v \end{pmatrix}\big), \quad x \in (0,L),\ t>0. \tag{1.2}$$

In this paper we examine system (1.2) with a 1-homogeneous non-linearity $f(u,v) = h_1(u,v)$, which means that $h_1(\mu u, \mu v) = \mu h_1(u,v)$ for all real μ. Furthermore, we assume that h_1 is C^1 in $\mathbb{R}^2 \setminus \{(0,0)\}$.
It turns out that there are two different cases, depending on whether the set

$$E := \Big\{\phi_0 \in \mathbb{R} : \begin{pmatrix} -\sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \cdot h_1\begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix} = 0\Big\}$$

is empty or not.

Case 1: $E \neq \emptyset$. When $E=\mathbb{R}$ the dynamics has already been discussed in [2], so we concentrate on the case $E \neq \mathbb{R}$. Since $E \neq \emptyset$ implies that E has infinitely many elements, we can take $\alpha, \beta \in E$, $\alpha < \beta$, such that $\gamma \notin E$ for all $\alpha < \gamma < \beta$. We write the initial data (u_0, v_0) in polar coordinates

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = r_0 \begin{pmatrix} \cos\varphi_0 \\ \sin\varphi_0 \end{pmatrix}. \tag{1.3}$$

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If r_0 is non-negative and φ_0 has its values in the interval (α, β), then we show that the solution (u, v) starting at (u_0, v_0) is flattened out and tends to a planar solution

$$\bar{r}_0 \begin{pmatrix} \cos \bar{\varphi}_0 \\ \sin \bar{\varphi}_0 \end{pmatrix} \tag{1.4}$$

where $\bar{\varphi}_0$ is a constant and equals either α or β. We call a state $(\bar{u}_0, \bar{v}_0)$ *planar*, if it can be written in the form (1.4) with a constant, i.e. x-independent angle $\bar{\varphi}_0$. We note that $(\bar{u}_0, \bar{v}_0)$ is planar if and only if $\bar{u}_0$ and $\bar{v}_0$ are linearly dependent. A solution $(\bar{u}, \bar{v})$ is called planar if $(\bar{u}, \bar{v})(t)$ is planar for all t. This means that the angle function $\bar{\varphi}$ might depend on t but not on the space variable x. It is easy to observe that $(\bar{u}, \bar{v})(t)$ is a planar solution if and only if the initial value is planar. This notation is motivated by the fact that for a planar solution $(\bar{u}, \bar{v})$ the curve

$$\gamma_t : [0, L] \ni x \mapsto (x, \bar{u}(t, x), \bar{v}(t, x)) \in [0, L] \times \mathbb{R}^2$$

lies in a plane P_t (which depends on t), as one can see in Figure 1.

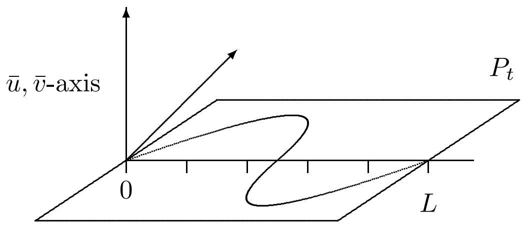

FIGURE 1. Curve γ_t for a planar solution $(\bar{u}, \bar{v})$

The proof of the fact that the angular function $\varphi(t, \cdot)$ associated with $(u, v)(t)$ converges either to α or to β is based on the following order-preserving property:

If $(\bar{u}_-, \bar{v}_-)(t)$ is a planar solution with $\bar{\varphi}_-(0) \leq \varphi(0, \cdot)$, then we get $\bar{\varphi}_-(t) \leq \varphi(t, \cdot)$ for all $t \geq 0$. Similarly, we get a function $\bar{\varphi}_+$ satisfying $\bar{\varphi}_+(t) \geq \varphi(t, \cdot)$ for all $t \geq 0$. Hence, the solution (u, v) lies between a lower and an upper planar solution.

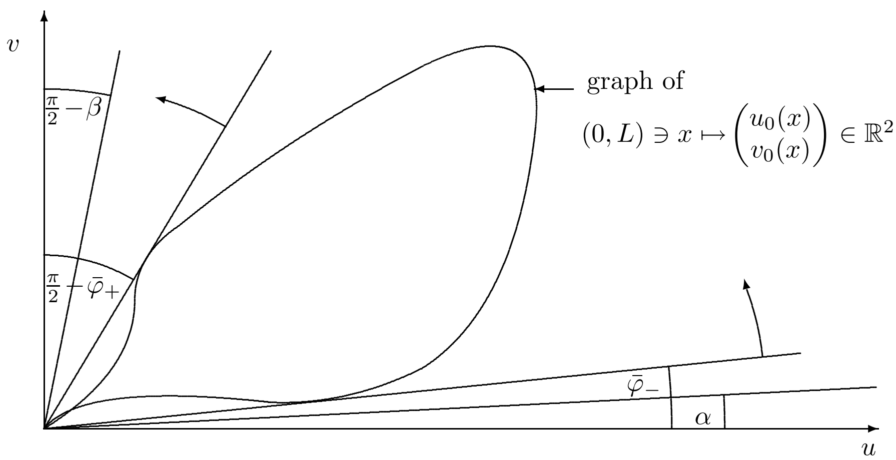

FIGURE 2. A situation in which $\bar{\varphi}_-, \bar{\varphi}_+$ and φ converge to β

Then it only remains to show that the angle functions $\bar{\varphi}_{\pm}(t)$ of the planar solutions both converge either to α or to β.

We note that in the case $E = \frac{\pi}{2}\mathbb{Z}$ the non-linearity h_1 can be written in the form

$$h_1(u,v) = \begin{pmatrix} u\zeta_1(u,v) \\ v\zeta_2(u,v) \end{pmatrix}.$$

Then the maximum principle yields that solutions (u,v) with initial value (u_0,v_0), $u_0 > 0, v_0 > 0$, stay positive for all $t > 0$, and our result follows from well known order-preserving properties [9, 14]. We note that in this paper we use a different ordering relation which has the advantage that it does also work if $E \neq \frac{\pi}{2}\mathbb{Z}$. Therefore, our result in case 1 is a kind of generalization of the well known order-preserving results for 1-homogeneous non-linearities.

Case 2: $E = \emptyset$**.** Another concept, however, is needed in the case $E = \emptyset$. In this case, loosely speaking, the part of the non-linearity which induces a rotation on $\mathbb{R}^2$ around the origin does never vanish. Hence, we cannot expect the existence of any stationary solution besides zero, but we should be able to observe periodic (or more difficult) motion whenever zero is unstable.

The description of the dynamics in this case seems to be really complicated, but we show that a major part can be done by looking at planar solutions that are obtained from a scalar (time-periodic) reaction-diffusion equation instead of a system of two equations. This method can be looked at as a generalization of the approach used in [3, 4, 5] for the model system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + \begin{pmatrix} -v \\ u \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} F \begin{pmatrix} u \\ v \end{pmatrix} \tag{1.5}$$

As a main result we obtain that each bounded solution of (1.2) with 1-homogeneous non-linearity h_1 tends either to a stationary or periodic (planar) solution, i.e. we get a Poincaré-Bendixson result.

Remarks on both cases. We note that the fact that the diffusion rates coincide in both equations is a crucial point. Numerical studies show that the dynamics can be totally different from the cases mentioned above for different diffusion rates, in general. The restrictions on the non-linearity, however, can be weakened. It turns out that all of our results hold for a non-linearity f of the form

$$f(t,x,u,v) = h_1(u,v) + \begin{pmatrix} u \\ v \end{pmatrix} F(t,x,u,v)$$

where $F : [0,\infty) \times [0,L] \times \mathbb{R}^2 \to \mathbb{R}$ is C^1. We will work with the more general system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + h_1(u,v) + \begin{pmatrix} u \\ v \end{pmatrix} F(t,x,u,v) \tag{1.6}$$

with either Dirichlet or Neumann boundary conditions for the rest of this paper.

2. Notation and preliminaries

Let $\mathbb{R}^2_* := \mathbb{R}^2 \setminus \{(0,0)\}$. For every $(\xi,\eta) \in \mathbb{R}^2_*$ the set $\{(\xi,\eta),(-\eta,\xi)\}$ is a base of $\mathbb{R}^2$ and, thus, $h_1(\xi,\eta) \in \mathbb{R}^2$ can be written in the form

$$h_1\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} f_0 \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} -\eta \\ \xi \end{pmatrix} g_0 \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{2.1}$$

where $f_0(\xi,\eta), g_0(\xi,\eta) \in \mathbb{R}$ are uniquely determined. Since $h_1 \in C^1(\mathbb{R}^2_*, \mathbb{R}^2)$ and (2.1) implies that

$$\begin{aligned} f_0\begin{pmatrix}\xi\\ \eta\end{pmatrix} &= \frac{1}{\xi^2+\eta^2}\begin{pmatrix}\xi\\ \eta\end{pmatrix} h_1\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \\ g_0\begin{pmatrix}\xi\\ \eta\end{pmatrix} &= \frac{1}{\xi^2+\eta^2}\begin{pmatrix}-\eta\\ \xi\end{pmatrix} h_1\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \end{aligned}$$

we get $f_0, g_0 \in C^1(\mathbb{R}^2_*, \mathbb{R})$. Since for all $\mu \in \mathbb{R}$

$$\begin{aligned} &\mu\begin{pmatrix}\xi\\ \eta\end{pmatrix} f_0\begin{pmatrix}\xi\\ \eta\end{pmatrix} + \mu\begin{pmatrix}-\eta\\ \xi\end{pmatrix} g_0\begin{pmatrix}\xi\\ \eta\end{pmatrix} \\ &= \mu h_1\begin{pmatrix}\xi\\ \eta\end{pmatrix} = h_1\begin{pmatrix}\mu\xi\\ \mu\eta\end{pmatrix} \\ &= \mu\begin{pmatrix}\xi\\ \eta\end{pmatrix} f_0\begin{pmatrix}\mu\xi\\ \mu\eta\end{pmatrix} + \mu\begin{pmatrix}-\eta\\ \xi\end{pmatrix} g_0\begin{pmatrix}\mu\xi\\ \mu\eta\end{pmatrix}, \end{aligned}$$

we obtain that

$$f_0\begin{pmatrix}\mu\xi\\ \mu\eta\end{pmatrix} = f_0\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \quad g_0\begin{pmatrix}\mu\xi\\ \mu\eta\end{pmatrix} = g_0\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

Therefore, it suffices to define f_0, g_0 on S^1 where S^1 is defined by

$$S^1 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

In particular, this implies that f_0, g_0 and their derivatives are bounded (on S^1 and, hence, on $\mathbb{R}^2_*$).

For $\phi_0 \in \mathbb{R}$ we introduce the solution $\phi(\cdot;\phi_0) : \mathbb{R} \to \mathbb{R}$ of the ODE

$$\begin{gathered} \frac{d}{dt}\phi = g_0(\cos\phi, \sin\phi), \\ \phi(0) = \phi_0. \end{gathered} \tag{2.2}$$

Since g_0 is C^1 on S^1 (and has, thus, bounded derivatives), ϕ is well defined.

Lemma 2.1. *The function $\phi(\cdot;\phi_0)$ is either constant or strictly monotone.*

Proof. If $g_0(\cos\phi_0, \sin\phi_0) = 0$, then $\phi(t;\phi_0) = \phi_0$ for all t. We assume that $g_0(\cos\phi_0, \sin\phi_0) \neq 0$, without loss of generality, we deal only with $g_0(\cos\phi_0, \sin\phi_0) > 0$. If ϕ was not strictly increasing, then there would be some $t_0 \in \mathbb{R}$ with $\frac{d}{dt}\phi(t_0;\phi_0) = 0$ which would imply that $g_0(\cos\phi(t_0;\phi_0), \sin\phi(t_0;\phi_0)) = 0$ and, thus, ϕ would be constant, contradicting the assumption.

Lemma 2.2. *If $g_0(\cos\phi_*, \sin\phi_*) = 0$ for some $\phi_* \in [0, 2\pi)$, i.e. $\phi_* \in E$, then $\phi(\cdot;\phi_0)$ is bounded for all ϕ_0; if there is no such ϕ_*, i.e. $E = \emptyset$, then $\phi(\cdot;\phi_0)$ is unbounded.*

Proof. 1. If there is ϕ_* as above, then every non-constant function $\phi(\cdot;\phi_0)$ satisfies

$$\phi(t;\phi_0) \neq \phi_* \mod 2\pi \quad \text{for all } t.$$

Since $\phi(\cdot;\phi_0)$ is continuous, $\phi(\mathbb{R};\phi_0)$ is contained in an interval of length 2π; in particular, $\phi(\cdot;\phi_0)$ is bounded.

2. If $\phi(\cdot;\phi_0)$ is bounded, then the limit

$$\phi_* := \lim_{t\to\infty} \phi(t;\phi_0) \quad \in \mathbb{R}$$

exists by Lemma 2.1. Thus, there is a sequence $t_n \nearrow \infty$ such that $\frac{d}{dt}\phi(t_n;\phi_0) \to 0 \quad (n \to \infty)$. This implies that $g_0(\cos\phi_*, \sin\phi_*) = 0$ which completes the proof.

3. Main results

The case $E = \emptyset$. An element (u_0, v_0) of $C^1([0, L], \mathbb{R}^2)$ is called *planar*, if u_0 and v_0 are linearly dependent over $\mathbb{R}$, i.e. if there are $r_0 \in C^1([0, L], \mathbb{R})$ and $\phi_0 \in \mathbb{R}$ such that

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix} r_0 \,.$$

The set of all planar elements of $C^1([0, L], \mathbb{R}^2)$ is denoted by $\mathcal{P}$.

Theorem 3.1. *Let $(u, v) \in C^1([0, \infty) \times [0, L], \mathbb{R}^2)$ be a (classical) solution of system (1.6) with Dirichlet or Neumann boundary conditions. If $E = \emptyset$, then we get*

$$\mathrm{dist}_{C^1}\{(u, v)(t, \cdot), \mathcal{P}\} \to 0 \quad (t \to \infty)\,.$$

Since the ω-limit set $\omega = \omega(u, v)$ associated with (u, v) consists of all states at which the orbit $\{(u, v)(t) : t \geq 0\}$ accumulates, this result means that in the case $E = \emptyset$ the ω-limit set of any (bounded) solution (u, v) of (1.6) is contained in $\mathcal{P}$.

By Theorem 3.1, the crucial part of the dynamics of (1.6) takes place in the set $\mathcal{P}$ of planar elements. The following theorems deal with the dynamics on $\mathcal{P}$.

Theorem 3.2. *The set $\mathcal{P}$ is positively invariant, i.e. every solution (u, v) of (1.6) which starts in $\mathcal{P}$ is a planar solution, i.e. $(u, v)(t, \cdot) \in \mathcal{P}$ for all $t \geq 0$. If $(u_0, v_0) = (\cos\phi_0, \sin\phi_0)r_0$ with $r_0 \in C^1([0, L], \mathbb{R})$, $\phi_0 \in \mathbb{R}$, then (u, v) has the form*

$$\begin{pmatrix} u \\ v \end{pmatrix}(t, \cdot) = \begin{pmatrix} \cos\phi(t; \phi_0) \\ \sin\phi(t; \phi_0) \end{pmatrix} r(t, \cdot) \tag{3.1}$$

where $r : [0, \infty) \times [0, L] \to \mathbb{R}$ satisfies the scalar equation

$$\frac{d}{dt} r = r_{xx} + r\left[f_0\begin{pmatrix} \cos\phi(t; \phi_0) \\ \sin\phi(t; \phi_0) \end{pmatrix} + F\big(t, x, \begin{pmatrix} \cos\phi(t; \phi_0) \\ \sin\phi(t; \phi_0) \end{pmatrix} r\big)\right] \tag{3.2}$$

with initial value r_0.

We note that the assertion of Theorem 3.2 is also true in the case $E \neq \emptyset$, but the dynamics on $\mathcal{P}$ is important only if we have the result of Theorem 3.1 which ensures that all solutions finally tend to planar ones.

Theorem 3.3. *Let (u, v) be a (bounded) planar solution and choose ϕ_0, r_0, r as in Theorem 3.2. If $E = \emptyset$, then $\phi(\cdot; \phi_0) \mod 2\pi$ is periodic and the period p_ϕ is given by*

$$p_\phi = \inf\{\tau > 0 : |\phi(\tau; 0)| = 2\pi\}\,.$$

In particular, p_ϕ depends only on g_0. If, in addition, (1.6) is autonomous or F is periodic in t with a period p_F such that p_F/p_ϕ is rational, then the ω-limit set associated with (u, v) consists of periodic planar elements only, i.e. if we take an initial value $(\bar{u}_0, \bar{v}_0) \in \omega(u, v)$ in the ω-limit set associated with (u, v), then $(\bar{u}, \bar{v})$ is a periodic planar solution of (1.6).

The case $E \neq \emptyset$. Given $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, we call

$$\begin{aligned} I_{\alpha,\beta} : \quad = \quad & \Big\{(u_0, v_0) \in C^1([0, L], \mathbb{R}^2) : (u_0, v_0) = r_0(\cos\varphi_0, \sin\varphi_0) \text{ with} \\ & r_0, \varphi_0 \in C([0, L], \mathbb{R}) \text{such that } r_0 > 0 \text{ and } \alpha < \varphi_0 < \beta \text{ in } (0, L)\Big\} \end{aligned}$$

the angular space between α and β. If, in addition,

- $\alpha, \beta \in E$ and
- $\gamma \notin E$ for all $\alpha < \gamma < \beta$,

then we call $I_{\alpha,\beta}$ an angular space associated with E. We note that $E \neq \emptyset$ implies that E contains infinitely many elements. We show that an angular space associated with E is positively invariant under the semiflow generated by (1.6) and all solutions with initial value in this angular space tend to planar solutions with (constant) angle α or β.

Theorem 3.4. *Let $I_{\alpha,\beta}$ be an angular space associated with E and (u,v) a (bounded) solution with initial value $(u,v)(0,\cdot) \in I_{\alpha,\beta}$. Then we get:*

(a) *$(u,v)(t,\cdot) \in I_{\alpha,\beta}$ for all $t \geq 0$.*
(b) *There are continuous functions $\varphi : (0,\infty) \to C([0,L],\mathbb{R})$ and $r : (0,\infty) \to C^1([0,L],\mathbb{R})$ such that*
$$\begin{pmatrix} u \\ v \end{pmatrix} = r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} .$$
(c) *If $g_0(\cos\gamma, \sin\gamma) > 0$ for $\alpha < \gamma < \beta$, then $\varphi(t,\cdot)$ converges to β in $C([0,L],\mathbb{R})$, in case $g_0(\cos\gamma,\sin\gamma) < 0$ to α.*

4. Interpretation of the results

If we have $E = \emptyset$, then the rotation on the values of (u,v) driven by g_0 does never stop — formally this means that $\phi(\cdot;\phi_0)$ is strictly monotone and unbounded. In this case, our main result, Theorem 3.1, ensures that a major part of the dynamics of (1.6) can be described by looking only at the dynamics on $\mathcal{P}$, because we know that all solutions of (1.6) tend to $\mathcal{P}$ for $t \to \infty$. Then Theorems 2 and 3 show that the dynamics on $\mathcal{P}$ can be described by solving first the ODE (2.2) and then looking at the scalar reaction-diffusion equation (3.2) which has been studied in [6, 7, 8]. This is obviously much easier than dealing with the full system (1.6); in particular we can use oscillation number results in order to attack equation (3.2) which are, in general, not available for systems.

Nevertheless, we note that the global attractor of (1.6) (if it exists) contains, in general, more elements than just the union of $\omega(u_0,v_0)$ for all points (u_0,v_0). We can even show that in some cases the global attractor will contain non-planar elements. The results [5] on unstable directions for periodic solutions of the model system (1.5) could, for example, be used in order to prove the existence of non-planar heteroclinic solutions.

If we have $E \neq \emptyset$ and $\alpha \in E$, then $\phi(t;\alpha) = \alpha$ for all t and the set $\{(u_0,v_0) := r_0(\cos\alpha,\sin\alpha) : r_0 \in C^1([0,L],\mathbb{R})\}$ is invariant under the semiflow induced by (1.6). In this case Theorem 3.4 describes the dynamics of all solutions starting in an angular space $I_{\alpha,\beta}$ between two consecutive elements of E, i.e. in a space which is bounded by invariant sets.

Theorem 3.4 says that all solutions which start in the angular space $I_{\alpha,\beta}$ will be flattened out and converge to the boundary of this angular space (see Figure 2). In the proof of Theorem 3.4 we will see that the crucial point is that $\alpha \leq \phi_- \leq \varphi(t_0,\cdot) \leq \phi_+ \leq \beta$ for some t_0 yields
$$\phi(t-t_0;\phi_-) \leq \varphi(t,\cdot) \leq \phi(t-t_0;\phi_+) \quad \text{for all } t \geq t_0.$$
This (weak) order-preserving property together with the fact that $\phi(\cdot;\phi_0)$ converges for any ϕ_0 is the main idea of the proof of Theorem 3.4.

We note that outside the angular spaces even non-planar equilibria may exist. We can, for example, take $L = \pi$, $h_1(u, v) = (u, 4v)$, and $F(t, x, u, v) = 0$ which has the non-planar equilibrium $(\sin x, \sin(2x))$ (and $E = \frac{\pi}{2}\mathbb{Z}$).

5. Construction of an appropriate transformation

Let $(u, v) \in C^1([0,\infty) \times [0, L], \mathbb{R}^2)$ be a solution of (1.6) with Dirichlet or Neumann boundary conditions. For $\varphi \in \mathbb{R}$ we consider

$$M(\varphi) := \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} .$$

Given $\phi_0 \in \mathbb{R}$ we introduce

$$\begin{pmatrix} w \\ z \end{pmatrix} (t, \cdot) := M(-\phi(t; \phi_0)) \begin{pmatrix} u \\ v \end{pmatrix} (t, \cdot) .$$

An elementary computation shows that (w, z) satisfies

$$\begin{aligned} &\frac{d}{dt}\begin{pmatrix} w \\ z \end{pmatrix} \\ &= \begin{pmatrix} w_{xx} \\ z_{xx} \end{pmatrix} + \begin{pmatrix} -z \\ w \end{pmatrix} \Big[g_0\big(M(\phi(t;\phi_0)) \begin{pmatrix} w \\ z \end{pmatrix}\big) - \frac{d}{dt}\phi(t;\phi_0)\Big] \\ &\quad + \begin{pmatrix} w \\ z \end{pmatrix} \Big[F\big(t, x, M(\phi(t;\phi_0)) \begin{pmatrix} w \\ z \end{pmatrix}\big) + f_0\big(M(\phi(t;\phi_0)) \begin{pmatrix} w \\ z \end{pmatrix}\big)\Big] . \end{aligned} \tag{5.1}$$

6. Oscillation number results

The concept of oscillation numbers [1, 13, 16] (sometimes also called zero number, lap number or Matano number) deals with solutions $y : [0,\infty) \times [0, L] \to \mathbb{R}$ of a linear equation

$$\frac{d}{dt} y = y_{xx} + q(t, x)y \tag{6.1}$$

where $q : [0,\infty) \times [0, L] \to \mathbb{R}$ is bounded on $[0, T] \times [0, L]$ for all $T > 0$. We define the number of sign changes (the oscillation number) by

$$\begin{aligned} Z(t) \quad := \quad & \sup\big\{k \in \mathbb{N} : \exists\, 0 < x_1 < \cdots < x_k < L : \\ & y(t, x_j)y(t, x_{j+1}) < 0 \;\forall\, 1 \le j \le k-1 .\big\} \end{aligned}$$

It is remarkable that $Z(t)$ is finite for all $t > 0$, no matter how many zeros the initial state $y(0, \cdot)$ has, as Angenent [1] proved. The most important result about oscillation numbers is that $Z : (0,\infty) \to \mathbb{N}$ is a non-increasing function. Precisely, the value of $Z(t)$ decreases whenever $y(t, \cdot)$ has a multiple zero, i.e. if $y(t_0, \cdot)$ has a multiple zero, then $Z(t_1) > Z(t_2)$ for all $t_1 < t_0 < t_2$.

These oscillation number results play an important role in the examination of a scalar reaction-diffusion equation but can in general not be applied to systems of reaction-diffusion equations. Whenever z is not the zero function, we want to have results on the oscillation number of z like those mentioned in the last section. Thus, we have to show that z satisfies some equation of the form (6.1). Since z

solves

$$\begin{aligned} \frac{d}{dt}z &= z_{xx} + w\Big[g_0\big(M(\phi(t;\phi_0))\begin{pmatrix} w \\ z \end{pmatrix}\big) - \frac{d}{dt}\phi(t;\phi_0)\Big] \\ &\quad + z\Big[F\big(t,x,M(\phi(t;\phi_0))\begin{pmatrix} w \\ z \end{pmatrix}\big) + f_0\big(M(\phi(t;\phi_0))\begin{pmatrix} w \\ z \end{pmatrix}\big)\Big] \end{aligned}$$

by (5.1), it only remains to show that

$$q_1 := \frac{w}{z}\Big[g_0\big(M(\phi(t;\phi_0))\begin{pmatrix} w \\ z \end{pmatrix}\big) - \frac{d}{dt}\phi(t;\phi_0)\Big]$$

is bounded on $[0,T]\times[0,L]$ for all $T>0$. If q_1 is bounded, then z will solve equation (6.1) with

$$q := q_1 + F\big(t,x,M(\phi(t;\phi_0))\begin{pmatrix} w \\ z \end{pmatrix}\big) + f_0\big(M(\phi(t;\phi_0))\begin{pmatrix} w \\ z \end{pmatrix}\big)$$

and we will be able to apply oscillation number results to z.

We assume that q_1 is not bounded. Then there is $T>0$ and a sequence (t_n,x_n) in $[0,T]\times[0,L]$ such that $|q_1(t_n,x_n)| \to \infty \quad (n\to\infty)$ and $z(t_n,x_n)\neq 0$ for all $n\in\mathbb{N}$. We may assume that (t_n,x_n) converges to $(t_0,x_0)\in[0,T]\times[0,L]$. Since g_0 and, by definition of ϕ, also $\frac{d}{dt}\phi$ are bounded, this implies that

$$\frac{z(t_n,x_n)}{w(t_n,x_n)} \to 0 \quad (n\to\infty)\,.$$

Hence, (ω_n,ζ_n) defined by

$$\begin{pmatrix} \omega_n \\ \zeta_n \end{pmatrix} := \frac{1}{\sqrt{w^2(t_n,x_n)+z^2(t_n,z_n)}}\begin{pmatrix} w(t_n,x_n) \\ z(t_n,x_n) \end{pmatrix}$$

fulfill $\zeta_n \to 0 \quad (n\to\infty)$. This means that $|\omega_n|\to 1 \quad (n\to\infty)$ and without loss of generality, we may assume that $\omega_n \to 1 \quad (n\to\infty)$. Since the map

$$[0,2\pi]\times S^1 \ni (\varphi,(\omega,\zeta)) \mapsto g_0\big(M(\varphi)\begin{pmatrix} \omega \\ \zeta \end{pmatrix}\big) \in \mathbb{R}$$

is (uniformly) continuously differentiable, it is Lipschitz-continuous in $(\omega,\zeta)\in S^1$ uniformly for all $\varphi\in[0,2\pi]$ with Lipschitz-constant L_g. Hence, we get

$$\begin{aligned} &\Big|g_0\big(M(\phi(t_n;\phi_0))\begin{pmatrix} w(t_n,x_n) \\ z(t_n,x_n) \end{pmatrix}\big) - \frac{d}{dt}\phi(t_n;\phi_0)\Big| \\ &= \Big|g_0\big(M(\phi(t_n;\phi_0))\begin{pmatrix} \omega_n \\ \zeta_n \end{pmatrix}\big) - g_0\big(M(\phi(t_n;\phi_0))\begin{pmatrix} 1 \\ 0 \end{pmatrix}\big)\Big| \\ &\le L_g\Big|\begin{pmatrix} \omega_n \\ \zeta_n \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big| \\ &= L_g\sqrt{(\omega_n-1)^2+\zeta_n^2}\,. \end{aligned}$$

We define $\alpha_n\in\mathbb{R}$ by $(\omega_n,\zeta_n) = (\cos\alpha_n,\sin\alpha_n)$. Then $\tan\alpha_n = z(t_n,x_n)/w(t_n,x_n)$ and $\alpha_n \to 0 \ (\mod \pi)$ $(n\to\infty)$. Furthermore, it follows that

$$\sqrt{(\omega_n-1)^2+\zeta_n^2} = 2\big|sin(\frac{\alpha_n}{2})\big| = O(\tan\alpha_n) = O\big(\frac{z(t_n,x_n)}{w(t_n,x_n)}\big) \quad (n\to\infty)$$

and, thus, $q_1(t_n,x_n)$ is bounded for $n\to\infty$ contradicting our assumption.

7. Proof of Theorem 3.1

Step 1: Assume that the assertion is false and construct a non-planar element in the ω-limit set under this assumption.

By standard arguments [12, 17] we know that the solutions of (1.6) build a (local) semiflow on some Sobolev space; in particular, the orbit

$$\Gamma := \{(u,v)(t) : t \geq 0\}$$

of the bounded solution (u,v) is a precompact subset of the space $X^\alpha := D(\Delta^\alpha)$, $\alpha \in (0,1)$ (in the notation of Henry [12]. Taking $\alpha < 1$ sufficiently large and using the Sobolev embedding theorem, we obtain that Γ lies precompact in $C^1([0,L],\mathbb{R}^2)$. This implies that, if the assertion of our theorem does not hold, then there is $(u_0', v_0') \in C^1([0,L],\mathbb{R}^2) \setminus \mathcal{P}$ and a sequence (t_n), $t_n \nearrow \infty$, such that

$$\|(u,v)(t_n,\cdot) - (u_0', v_0')\|_{C^1} \to 0 \quad (n \to \infty)\,.$$

This means that (u_0', v_0') is contained in the ω-limit set associated with (u,v). Let (u', v') be the solution of (1.6) with initial value (u_0', v_0'). We note that (u', v') is defined for all $t \in \mathbb{R}$ since (u_0', v_0') is an element of the ω-limit set.

Step 2: Definition of ϕ_0. We want to take $\phi_0 \in \mathbb{R}$ such that $z'(0,\cdot)$ defined using the formula

$$\begin{pmatrix} w' \\ z' \end{pmatrix}(t,\cdot) := M(-\phi(t;\phi_0)) \begin{pmatrix} u' \\ v' \end{pmatrix}(t,\cdot)$$

has a multiple zero. If u_0' or v_0' have a multiple zero, then we can take $\phi_0 = \pi/2$ or $\phi_0 = 0$. If neither u_0' nor v_0' have multiple zeros, then we proceed as follows: Using the result of the last section, we can apply oscillation number results to z' — for every ϕ_0. Since (u', v') are also defined for negative t, z' solves an equation of the form (6.1) for all $t \in \mathbb{R}$. Then Angenent's result [1] ensures that the number of zeros of $z'(t,\cdot)$ is finite for all t; in particular for $t = 0$. Applying this result for $\phi_0 = \pi/2$ and $\phi_0 = 0$, we obtain that u_0' as well as v_0' have only finitely many zeros (which are all simple by assumption). Hence, there is a continuous function $\varphi_0' : [0,L] \to \mathbb{R}$ which satisfies

$$\tan \varphi_0'(x) = \frac{v_0'(x)}{u_0'(x)} \quad \text{for all } x \in [0,L] \text{ with } u_0'(x) \neq 0.$$

This result is an easy consequence of de l'Hospital's formula, and it can also be found in [2, p.696,Lemma 2]. We set

$$\phi_0 := \max\left\{\varphi_0'(x) : x \in [0,L]\right\}$$

and take $x_0 \in [0,L]$ such that $\varphi_0'(x_0) = \phi_0$. Then we get $z'(0,x_0) = 0$ using the definition of z'. It remains to show that $\frac{\partial}{\partial x} z'(0,x_0) = 0$.

Case 1: $w'(0,x_0) = 0$. Then $z'(0,x_0) = 0$ implies that $u_0'(x_0) = v_0'(x_0) = 0$ and, thus,

$$\tan \phi_0 = \frac{\frac{\partial}{\partial x} v_0'(x_0)}{\frac{\partial}{\partial x} u_0'(x_0)}\,.$$

Hence, it follows that

$$\frac{\frac{\partial}{\partial x} z'(0,x_0)}{\frac{\partial}{\partial x} w'(0,x_0)} = \tan(\varphi_0'(x_0) - \phi_0) = 0\,.$$

In particular, we have $\frac{\partial}{\partial x} z'(0,x_0) = 0$.

Case 2: $w'(0,x_0) \neq 0$. Let U_0 be a neighborhood of x_0 in $[0,L]$ in which $w'(0,\cdot)$ has no zeros. Since $\varphi_0'(x) \le \phi_0$ for all $x \in [0,L]$, we get

$$0 \ge \tan(\varphi_0'(x) - \phi_0) = \frac{z'(0,x)}{w'(0,x)} \quad \text{for all } x \in U_0.$$

Since $w'(0,\cdot)$ has no sign change in U_0, it follows that $z'(0,\cdot)$ must not have a sign change in U_0, too. Hence x_0 must be a multiple zero of $z'(0,\cdot)$.

Step 3: Applying oscillation number results to z'. As shown in Section 6, we can apply oscillation number results to z'. Let $Z' : \mathbb{R} \to \mathbb{N}$ be the corresponding oscillation number. We note that we can define Z' on the whole real axis because z' is defined for all $t \in \mathbb{R}$ (and solves a linear equation of the form (6.1) on each interval $[-\tau,\infty)$, $\tau \ge 0$). Since $z'(0,\cdot)$ has a multiple zero by Step 2, we get

$$Z'(t_-) > Z'(t_+) \quad \text{for all } t_- < 0 < t_+.$$

We note that $z'(t,\cdot)$ can only have multiple zeros for countably many $t \in \mathbb{R}$ since Z' has its values in $\mathbb{N}$, is non-increasing and decreases each time $z'(t,\cdot)$ has a multiple zero. Thus, we can choose $t_- < 0 < t_+$ in a way that $z'(t_\pm,\cdot)$ have no multiple zeros.

Step 4: Definition of (w,z). By assumption, E is empty and, thus, $\phi(\cdot;\phi_0)$ is unbounded by Lemma 2.2. Therefore, $\phi(\cdot;\phi_0)$ has to be periodic; we denote its period by p. Without loss of generality, we assume that $t_n \mod p \in [0,p)$ has the same value τ for all n; otherwise we take a subsequence (t_{n_k}) such that $m_k := t_{n_k} \mod p$ converges in $[0,p]$, denote its limit by τ and set $\tau_k := t_{n_k} - (m_k - \tau)$. Then we get $(u,v)(\tau_k) \to (u_0', v_0')$, and we can use (τ_k) instead of (t_n). Without loss of generality, we assume that $\tau = 0$; otherwise replace (u,v) by $(\hat{u},\hat{v}) := (u,v)(\cdot - \tau)$ which does also solve an equation of the form (1.6). Then we introduce (w,z) by

$$\begin{pmatrix} w \\ z \end{pmatrix}(t,\cdot) := M(-\phi(t;\phi_0)) \begin{pmatrix} u \\ v \end{pmatrix}(t,\cdot) \quad \text{for } t \ge 0.$$

We note that (t_n) was chosen in a way that t_n is an integer multiple of p for all n. Thus, we get $\phi(t_n;\phi_0) = \phi_0$ and

$$\begin{aligned} \begin{pmatrix} w \\ z \end{pmatrix}(t_n,\cdot) = \underbrace{M(-\phi(t_n;\phi_0))}_{=M(-\phi_0)} \begin{pmatrix} u \\ v \end{pmatrix}(t_n,\cdot) \quad &\to \quad M(-\phi_0) \begin{pmatrix} u' \\ v' \end{pmatrix}(0,\cdot) \\ &= \begin{pmatrix} w' \\ z' \end{pmatrix}(0,\cdot). \end{aligned} \tag{7.1}$$

Furthermore, an analogous computation shows that

$$\begin{pmatrix} w \\ z \end{pmatrix}(t_n + t_\pm,\cdot) \to \begin{pmatrix} w' \\ z' \end{pmatrix}(t_\pm,\cdot) \quad \text{in } C^1([0,L],\mathbb{R}).$$

Since $z'(t_+,\cdot)$ and $z'(t_-,\cdot)$ have no multiple zeros, it follows that there are positive integers n_+, n_- such that the number of zeros of $z(t_n+t_\pm,\cdot)$ and $z'(t_\pm,\cdot)$ coincide for all $n \ge n_\pm$. Since $Z'(t_-) > Z'(t_+)$, the number $Z(t)$ of zeros of $z(t,\cdot)$ accumulates at two different values $Z'(t_\pm)$. Since oscillation number results hold for z by Section 6, Z has to be non-increasing which is a contradiction.

Remark. We note that the assumption $E = \emptyset$ is used in Step 4 only. It ensures that $\phi(\cdot;\phi_0)$ is unbounded and, thus, periodic. Otherwise (7.1) does not hold and we cannot conclude that $(w',z')(0)$ is a limit point of some function (w,z) defined as above.

8. Proof of Theorems 3.2 and 3.3

Suppose that (u_0, v_0), ϕ_0 and r_0 are given as in Theorem 3.2. First define r by (3.2) and then (u, v) by (3.1). An elementary computation shows that (u, v) is a solution of (1.6) with initial value (u_0, v_0). Using the uniqueness of the solutions of (1.6) (see, for example, [13]), the assertion of Theorem 3.2 follows.

By Lemma 2.2, $\phi(\cdot; \phi_0)$ is unbounded. In particular, $\phi(\cdot; \phi_0)$ is not constant and Lemma 2.1 implies that it is strictly monotone. Without loss of generality assume that $\phi(\cdot; \phi_0)$ is strictly increasing. Since $\phi(t; \phi_0)$ is not bounded for $t \to \infty$, there is a uniquely determined $p_\phi > 0$ such that

$$\phi(p_\phi; \phi_0) = \phi_0 + 2\pi\,.$$

Thus, $\phi(\cdot; \phi_0)$ is periodic with period p_ϕ. Furthermore, there is $t_0 \in [\phi_0, \phi_0 + p_\phi)$ such that $\phi(t_0; \phi_0) = 2\pi k$ with some integer k. Then it follows that

$$\phi(t_0 + p_\phi j; \phi_0) = 2\pi(k + j) \quad \text{for all integers } j.$$

In particular, we get $\phi(t_0 - p_\phi k; \phi_0) = 0$ and, thus,

$$\phi(p_\phi; 0) = \phi(p_\phi; \phi(t_0 - p_\phi k; \phi_0)) = \phi(t_0 - p_\phi k + p_\phi; \phi_0) = \phi(t_0 - p_\phi(k-1); \phi_0) = 2\pi$$

which shows that p_ϕ is given by the expression mentioned in Theorem 3.3.

We assume that F is t-periodic with period p_F where p_ϕ/p_F is rational. This means that there is $p > 0$ such that p/p_ϕ as well as p/p_F are positive integers (in the autonomous case, one can just take $p = p_\phi$). In particular, equation (3.2) depends periodically on t (with period p).

Let $(\bar{u}_0, \bar{v}_0)$ be an element of $\omega(u, v)$ and $(\bar{u}, \bar{v})$ be the corresponding solution of (1.6). Then $(\bar{u}, \bar{v})$ is planar by Theorem 3.1. By Theorem 3.2, there is $\bar{\phi}_0 \in \mathbb{R}$ and a function $\bar{r}$ such that (3.1) and (3.2) hold. Since $(\bar{u}_0, \bar{v}_0) = \bar{r}(0)(\cos\bar{\phi}_0, \sin\bar{\phi}_0)$ is an element of $\omega(u, v)$, it is a chain-recurrent point. Then an elementary analysis like it was done in [5, Theorem 2] shows that $\bar{r}(0)$ is a also chain-recurrent point for equation (3.2). Then [8] ensures that $\bar{r}$ is a p-periodic solution of (3.2) and, since $\phi(\cdot; \phi_0)$ is p-periodic, $(\bar{u}, \bar{v})$ is a p-periodic solution of (1.6).

9. Proof of Theorem 3.4

1. We will prove the assertion only in the case $\alpha = -\pi/2$. The general case can easily be reduced to this situation setting

$$\begin{pmatrix} u' \\ v' \end{pmatrix}(t) := M(-\alpha - \pi/2)\begin{pmatrix} u \\ v \end{pmatrix}(t)\,.$$

Then (u', v') solves an equation similar to (1.6) replacing F, f_0, g_0 by

$$f_0'\begin{pmatrix} \xi \\ \eta \end{pmatrix} := f_0\big(M(\alpha + \pi/2)\begin{pmatrix} \xi \\ \eta \end{pmatrix}\big)\,,$$

$$g_0'\begin{pmatrix} \xi \\ \eta \end{pmatrix} := g_0\big(M(\alpha + \pi/2)\begin{pmatrix} \xi \\ \eta \end{pmatrix}\big)\,,$$

$$F'\left(t, x, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) := F\big(t, x, M(\alpha + \pi/2)\begin{pmatrix} \xi \\ \eta \end{pmatrix}\big)\,.$$

Hence, we have $\alpha' = -\pi/2$, $\beta' = \beta - \alpha - \pi/2$ and the assertion follows using $\varphi = \varphi' + \alpha + \pi/2$.

2. We note that $\alpha = -\pi/2$ (Step 1) implies $\pi/2 \in E$ using the fact that $g_0(0, 1) = g_0(0, -1)$ (0-homogeneity). This gives $\beta \in (-\pi/2, \pi/2]$. Hence, $(u, v)(0) \in I_{\alpha,\beta}$

implies $u_0 = r_0 \cos\varphi_0$ with $r_0 \geq 0$ and $-\pi/2 \leq \alpha \leq \varphi_0 \leq \beta \leq \pi/2$ and, thus, $u_0 \geq 0$. Furthermore, we get $g_0(0,-1) = g_0(\cos\alpha, \sin\alpha) = 0$. Since the restriction of g_0 on S^1 is continuously differentiable, then function $\hat{g}_0$ given by

$$\hat{g}_0 : S^1 \ni (\cos\phi, \sin\phi) \mapsto \frac{g_0(\cos\phi, \sin\phi)}{\cos\phi} \sin\phi \in \mathbb{R}$$

is well defined and continuous. Setting $\hat{g}_0(\xi,\eta) := \frac{g_0(\xi,\eta)}{\xi}\eta$ for all $(\xi,\eta) \in \mathbb{R}^2_*$, $\hat{g}_0$ is 0-homogeneous, continuous and bounded; in particular, it is contained in $L^\infty(\mathbb{R}^2)$. Since u satisfies

$$\frac{d}{dt}u = u_{xx} + u(F + f_0) - vg_0 = u_{xx} + u(F + f_0 - \hat{g}_0)$$

and $u(0,\cdot) = u_0 \geq 0$, maximum principle arguments imply that, for all $t > 0$, $u(t,x) > 0$ for $x \in (0,L)$ and $u_x(t,0), u_x(t,L) \neq 0$ in case of Dirichlet boundary conditions and $u(t,x) > 0$ for $x \in [0,L]$ in case of Neumann boundary conditions. Thus, there is a continuous function $q : (0,\infty) \times [0,L] \to \mathbb{R}$ such that

$$q(t,x) = \frac{v(t,x)}{u(t,x)}, \quad x \in (0,L).$$

(Note that $q(t,0), q(t,L)$ can be defined using de'l Hospital's formula.) We define

$$\varphi : (0,\infty) \times [0,L] \ni (t,x) \mapsto \arctan q(t,x) \in (-\pi/2, \pi/2),$$

$\varphi(t,\cdot) \in C([0,L],\mathbb{R})$, and r by $r(t,\cdot) := u(t,\cdot)/\cos\varphi(t,\cdot)$. Then, (b) follows. We note that the positivity of u and the fact that $\varphi(t,\cdot)$ has its values in $(-\pi/2,\pi/2)$ implies that $r(t,\cdot)$ has only positive values in $(0,L)$ for all $t > 0$.

3. We take $t_0 > 0$ and set $\phi_- := \min_{[0,L]} \varphi(t_0,\cdot)$, $\phi_+ := \max_{[0,L]} \varphi(t_0,\cdot)$. Note that φ is continuous by Step 2. Let $z = z_\pm$ be defined as in Section 5 where the constant ϕ_0 should have the value $\phi(-t_0;\phi_\pm)$. Then we get

$$z_\pm(t_0,\cdot) = r(t_0,\cdot)\sin(\varphi(t_0,\cdot) - \phi_\pm).$$

Using the fact that $r(t_0,\cdot)$ is positive in $(0,L)$ by Step 2 and $\varphi(t_0,\cdot) - \phi_- \in [0,\pi)$, it follows that $z_-(t_0,\cdot) \geq 0$. We take $x_- \in [0,L]$ such that $\varphi(t_0,x_-) = \phi_-$. Then $z_-(t_0,\cdot)$ has a multiple zero at $x = x_-$. By Section 5, we can apply oscillation number results to z_-. Hence, $z_-(t,\cdot)$ is either the zero function or strictly positive in $(0,L)$ for all $t > t_0$. In particular, we get

$$\min_{[0,L]} \varphi(t,\cdot) \geq \phi(t - t_0;\phi_-) \quad \forall t > t_0.$$

Analogously, we get

$$\max_{[0,L]} \varphi(t,\cdot) \leq \phi(t - t_0;\phi_+) \quad \forall t > t_0.$$

4. The solutions of (1.6) form a (local) semiflow on the space $C^1([0,L],\mathbb{R}^2)$. Hence, $(u,v)(0) \in I_{\alpha,\beta}$ implies that there is $\delta > 0$ such that

$$\alpha < \varphi(t,\cdot) < \beta \quad \forall 0 \leq t < \delta.$$

Thus, we have $(u,v)(t) \in I_{\alpha,\beta}$ for $t \in [0,\delta)$. Take $t_0 := \delta/2$ and define $\phi_\pm$ as in Step 3. Then we get $\alpha < \phi_- \leq \phi_+ < \beta$ and, by definition of the function $\phi(\cdot;\phi_\pm)$,

$$\alpha < \phi(t;\phi_-) \leq \phi(t;\phi_+) < \beta \quad \forall t \geq 0.$$

Thus, Step 3 implies that $(u,v)(t) \in I_{\alpha,\beta}$ for all $t \geq \delta/2$ and, using the result mentioned above, for all $t \geq 0$. This proves (a).

5. Take $t_0 > 0$ and define $\phi_\pm$ as in Step 3. Then we get $(u, v)(t_0) \in I_{\alpha,\beta}$ by Step 4 and, thus,

$$\alpha < \phi_- \le \phi_+ < \beta \,.$$

If $g_0(\cos\phi_0, \sin\phi_0) > 0$ for $\alpha < \phi_0 < \beta$, then Lemmata 2.1 and 2.2 yield $\phi(t; \phi_\pm) \to \beta \quad (t \to \infty)$. Since

$$\phi(t - t_0; \phi_-) \le \min_{[0,L]} \varphi(t, \cdot) \le \max_{[0,L]} \varphi(t, \cdot) \le \phi(t - t_0; \phi_+) \quad \forall t > t_0$$

by Step 3, it follows that $\varphi(t, \cdot)$ converges to β in $C([0, L], \mathbb{R})$. Analogously, we get convergence to α in the case $g_0(\cos\phi_0, \sin\phi_0) < 0$ for $\alpha < \phi_0 < \beta$. This proves (c).

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