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Dirichlet problem for quasi-linear elliptic equations *

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Abstract

We study the Dirichlet Problem associated to the quasilinear elliptic problem

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \mathcal{A}_i(x, u(x), \nabla u(x)) + \mathcal{B}(x, u(x), \nabla u(x)) = 0.$$

Then we define a potential theory related to this problem and we show that the sheaf of continuous solutions satisfies the Bauer axiomatic theory.

1 Introduction

The objective of this paper is to study the weak solutions of the following quasilinear elliptic equation in $\mathbb{R}^d$, $(d \geq 2)$:

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \mathcal{A}_i(x, u(x), \nabla u(x)) + \mathcal{B}(x, u(x), \nabla u(x)) = 0 \tag{1.1}$$

where $\mathcal{A}_i : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $\mathcal{B} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are given Carathéodory functions satisfying the conditions introduced in section 2.

An example of equation (1.1) is the perturbed p-Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) + \mathcal{B}(., u, \nabla u) = 0, \quad 1 < p < d. \tag{1.2}$$

When $p = 2$, equation (1.2) reduces to the perturbed Laplace equation

$$-\Delta u + \mathcal{B}(., u, \nabla u) = 0. \tag{1.3}$$

Another example included in this study is the linear equation

$$\mathcal{L}(u) = -\sum_j \Big(\sum_i a_{ij} \frac{\partial u}{\partial x_i} + d_j u \Big) + \sum_j b_j \frac{\partial u}{\partial x_j} + cu = 0,$$

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where $\mathcal{L}$ is assumed to satisfy conditions stated in [25] (see also [12]).

Equation (1.1) have been investigated in many interesting papers [24, 26, 11, 21, 2]. Several papers have introduced an axiomatic potential theory for the nonlinear equation (1.2) when $\mathcal{B} = 0$; see for example [11]. For equations of type (1.3), see [1, 2, 3, 4].

The existence of weak solutions of (1.1) in variational forms was treated by means of the sub-supersolution argument [7, 8]. Later on, Dancers/Sweers [6], Kura [15], Carl [5], Lakshmikantham [10], Papageorgiou [23], Le/Schmitt [19], and others treated the existence of weak extremal solutions of nonlinear equations of type (1.1) by means of the sub-supersolution method. Le [17] studied the existence of extremal solutions of the problem

$$\int_\Omega A(x,\nabla u(x))(\nabla v - \nabla u)dx \geq \int_\Omega \mathcal{B}(x,u(x))(v(x)-u(x))dx, \tag{1.4}$$

for all $v \in K$, $u \in K$, where K is a closed convex subset of $W_0^{1,p}(\Omega)$.

Note that the solutions of (1.4) correspond to the obstacle problem treated in section 5 of this paper. Remark that in the references cited above, often $\mathcal{B} = \mathcal{B}(x,u(x))$ and the growth of $\mathcal{B}$ in u is less then $p-1$ and when $\mathcal{B} = \mathcal{B}(x,u,\nabla u)$, the growth of $\mathcal{B}$ in u and ∇u is less then $p-1$, but in our case the growth of $\mathcal{B}$ in ∇u is is allowed to go until $p - 1 + \frac{p}{n}$ and there is no condition on the growth of $\mathcal{B}$ in u.

Our aim in this paper is to solve the Dirichlet problem for (1.1) with a continuous data boundary and to give an axiomatic of potential theory related to the associated problem.

This paper consists of four sections. First, we recall some definitions for the (weak) subsolutions, supersolutions and solutions of the equation (1.1). In particular, we prove that the supremum of two subsolutions is a subsolution and that the infimum of two supersolutions is also a supersolution. In section 3, we give some conditions that allow us to have the comparison principle for sub and supersolutions. After this preparation we are able in section 4 to solve the Dirichlet problem related to the equation (1.1). So at first we prove the existence of solutions to the associated variational problem, after what we solve the Dirichlet problem for continuous data boundary. In the last section, we define a potential theory related to the equation (1.1), so we obtain that the sheaf of continuous solutions of (1.1) satisfies the Bauer axiomatic theory [4]. We prove also that the set of all hyperharmonic functions and the set of all hypoharmonic functions are sheaves.

Notation Throughout this paper we will use the following notation: $\mathbb{R}^d$ is the real Euclidean d-space, $d \geq 2$. For an open set U of $\mathbb{R}^d$, we denote by $C^k(U)$ the set of functions which k-th derivative is continuous for k positive integer, $C^\infty(U) = \cap_{k\geq 1} C^k(U)$ and by $C_0^\infty(U)$ the set of all functions in $C^\infty(U)$ with compact support. $L^q(E)$ is the space of all q^{th}-power Lebesgue integrable functions defined on measurable set E. $W^{1,q}(U)$ is the $(1,q)$-Sobolev space on U. $W_0^{1,q}(U)$ is the closure of $C_0^\infty(U)$ in $W^{1,q}(U)$ relatively to its norm. $W^{-1,q'}(U)$

denotes the dual of $W_0^{1,q}(U)$, $q' = \frac{q}{q-1}$. For a Lebesgue measurable set E, $|E|$ denotes the Lebesgue measure of E. $u \vee v$ and $u \wedge v$ design respectively the supremum and the infinimum of u and v. $u^+ = u \vee 0$ and $u^- = u \wedge 0$. We write $\rightharpoonup$(resp. $\rightarrow$) to design the weak (resp. strong) convergence.

2 Supersolutions of (1.1)

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d \geq 2$) with smooth boundary $\partial\Omega$ and let $\mathcal{L}$ be a quasi-linear elliptic differential operator in divergence form

$$\mathcal{L}(u)(x) = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \mathcal{A}_i(x, u(x), \nabla u(x)) + \mathcal{B}(x, u(x), \nabla u(x)) \quad \text{a.e. } x \in \Omega$$

where $\mathcal{A}_i : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $\mathcal{B} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are given Carathéodory functions. Let $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_d)$ and $1 < p < d$. We suppose that the following conditions are fulfilled: for a.e. $x \in \Omega, \forall \zeta \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^d$:

(P1) $|\mathcal{A}(x, \zeta, \xi)| \leq k_0(x) + b_0(x)|\zeta|^{p-1} + a|\xi|^{p-1}$

(P2) $(\mathcal{A}(x, \zeta, \xi) - \mathcal{A}(x, \zeta, \xi'))(\xi - \xi') > 0$, if $\xi \neq \xi'$.

(P3) $\mathcal{A}(x, \zeta, \xi)\xi \geq \alpha|\xi|^p - d_0(x)|\zeta|^p - e(x)$

(P4) $|\mathcal{B}(x, \zeta, \xi)| \leq k(x) + b(x)|\zeta|^\alpha + c|\xi|^r$, $0 < r < \frac{p}{(p^*)'}$, $\alpha \geq 0$.

Here a, c and α are positive constants, $p' = \frac{p}{p-1}$, $p^* = \frac{dp}{d-p}$, while k_0, b_0, d_0, e, k and b are measurable functions on Ω satisfying: $k_0 \in L^{p'}$, $b_0 \in L^{\frac{d}{p-1}}$, $k \in L^q$, $(p^*)' < q < (\frac{d}{p-\epsilon} \wedge \frac{p}{r})$ and d_0, e, $b \in L^{\frac{d}{p-\epsilon}}$, $(0 < \epsilon < 1)$.

We can easily show that if $u \in W^{1,p}(\Omega)$, then $A(., u, \nabla u) \in L^{p'}$ and that $\mathcal{B}(., u, \nabla u) \in L^{(p^*)'}$ when $\alpha \leq p - 1$.

Definition We say that a function $u \in W^{1,p}_{\rm loc}(\Omega)$ is a (weak) solution of (1.1), if

$$\begin{gathered} \mathcal{B}(., u, \nabla u) \in L^{(p^*)'} \\ \int_\Omega \mathcal{A}(., u, \nabla u)\nabla\varphi + \int_\Omega \mathcal{B}(., u, \nabla u)\varphi = 0, \end{gathered} \tag{2.1}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

We say that $u \in W^{1,p}_{\rm loc}(\Omega)$ is a supersolution (resp. subsolution) of (1.1) if

$$\begin{gathered} \mathcal{B}(., u, \nabla u) \in L^{(p^*)'} \\ \int_\Omega \mathcal{A}(., u, \nabla u)\nabla\varphi + \int_\Omega \mathcal{B}(., u, \nabla u)\varphi \geq 0 \quad (\text{resp. } \leq 0) \end{gathered}$$

for every nonnegative function $\varphi \in W_0^{1,p}(\Omega)$.

Note that if u is a supersolution of (1.1) then $-u$ is a subsolution of the equation

$$-\operatorname{div}\widehat{\mathcal{A}}+\widehat{\mathcal{B}}=0$$

where $\widehat{\mathcal{A}}(x,\zeta,\xi)=-\mathcal{A}(x,-\zeta,-\xi)$ and $\widehat{\mathcal{B}}(x,\zeta,\xi)=-\mathcal{B}(x,-\zeta,-\xi)$. Furthermore, the structure of $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ are similar to that of $\mathcal{A}$ and $\mathcal{B}$.

We recall that if u is a bounded supersolution (resp. subsolution), then u is upper (resp. lower) semicontinuous in Ω [21, Corollary 4.10].

Proposition 2.1 *Let u and v be two subsolutions of* (1.1) *in Ω such that*

$$(\mathcal{A}(.,v,\nabla u)-\mathcal{A}(.,u,\nabla u))\nabla(v-u)\geq 0,\quad a.e. x\in\Omega.$$

Then, $\max(u,v)$ *is also a subsolution. A similar statement holds for the minimum of two supersolutions.*

Proof. Fix φ in $C_0^\infty(\Omega)$, $\varphi\geq 0$. Let $\Omega_1=\{x\in\Omega: u>v\}$, $\Omega_2=\{x\in\Omega: u\leq v\}$ and put $I=\int_\Omega \mathcal{A}(.,u\vee v,\nabla(u\vee v))\nabla\varphi=I_1+I_2$ where

$$I_1=\int_{\Omega_1}\mathcal{A}(.,u,\nabla u)\nabla\varphi \quad\text{and}\quad I_2=\int_{\Omega_2}\mathcal{A}(.,v,\nabla v)\nabla\varphi.$$

Let $\rho_n:\mathbb{R}\to\mathbb{R}$ be such that $\rho_n\in\mathcal{C}^1(\mathbb{R})$,

$$\rho_n(t)=\begin{cases}1 & \text{if } t\geq 1/n\\ 0 & \text{if } t\leq 0\end{cases}$$

and $\rho_n'>0$ on $]0,1/n[$. For each $x\in\Omega$ define $q_n(x)=\rho_n((u-v)(x))$. We see that $q_n\in W^{1,p}_{\rm loc}(\Omega)$, $q_n\to 1_{\Omega_1}$ and $\|q_n\|_\infty\leq 1$. It follows by Lebesgue's Theorem of dominated convergence that $I_1=\lim_{n\to\infty}\int_{\Omega_1}q_n\mathcal{A}(.,u,\nabla u).\nabla\varphi$ and $I_2=\lim_{n\to\infty}\int_{\Omega_2}(1-q_n)\mathcal{A}(.,v,\nabla v).\nabla\varphi$. Hence

$$\begin{aligned}\int_\Omega q_n\mathcal{A}(.,u,\nabla u).\nabla\varphi &= \int_\Omega \mathcal{A}(.,u,\nabla u)\nabla.(q_n\varphi)-\int_\Omega \mathcal{A}(.,u,\nabla u)\varphi.\nabla(q_n)\\ &\leq -\int_\Omega \mathcal{B}(.,u,\nabla u)(q_n\varphi)-\int_{\Omega_n}\mathcal{A}(.,u,\nabla u)\varphi.\nabla(q_n),\end{aligned}$$

where $\Omega_n=\{x\in\Omega: v<u<v+\frac{1}{n}\}$.

Put $I_n=\int_\Omega q_n\mathcal{A}(.,u,\nabla u).\nabla\varphi$ and $J_n=\int_\Omega(1-q_n)\mathcal{A}(.,v,\nabla v).\nabla\varphi$. Then, similarly we have

$$\int_\Omega(1-q_n)\mathcal{A}(.,v,\nabla v).\nabla\varphi\leq-\int_\Omega(1-q_n)\mathcal{B}(.,v,\nabla v)\varphi+\int_{\Omega_n}\mathcal{A}(.,v,\nabla v)\varphi.\nabla(q_n).$$

So, we get

$$\begin{aligned}I_n+J_n &\leq -\int_\Omega \mathcal{B}(.,u,\nabla u)(q_n\varphi)-\int_\Omega(1-q_n)\mathcal{B}(.,v,\nabla v)\varphi\\ &\quad+\int_{\Omega_n}(\mathcal{A}(.,v,\nabla v)-\mathcal{A}(.,u,\nabla u))\varphi.\nabla(q_n).\end{aligned}$$

Using that $\nabla(q_n) = \rho_n'(u-v)\nabla(u-v)$, we get

$$\begin{aligned} I_n + J_n &\leq -\int_\Omega \mathcal{B}(.,u,\nabla u)(q_n\varphi) - \int_\Omega (1-q_n)\mathcal{B}(.,v,\nabla v)\varphi \\ &\quad -\int_{\Omega_n} \rho_n'(u-v)(\mathcal{A}(.,v,\nabla v) - \mathcal{A}(.,u,\nabla u))\varphi.\nabla(v-u) \\ &\leq -\int_\Omega \mathcal{B}(.,u,\nabla u)(q_n\varphi) - \int_\Omega (1-q_n)\mathcal{B}(.,v,\nabla v)\varphi. \end{aligned}$$

Finally, we have

$$\int_\Omega \mathcal{A}(.,u\vee v,\nabla(u\vee v)).\nabla\varphi + \int_\Omega \mathcal{B}(.,u\vee v,\nabla(u\vee v))\varphi \leq 0$$

which completes the proof. $\square$

We say that $\mathcal{L}$ satisfies the property $(\pm)$, if for every $k > 0$ and every supersolution (resp. subsolution) u of (1.1), the function $u + k$ (resp. $u - k$) is also a supersolution (resp. subsolution) of (1.1)

Remark 2.1 *1) Suppose that for each $u \in W^{1,p}_{\text{loc}}(\Omega)$ and each $k > 0$,*

$$\int (\mathcal{A}(.,u+k,\nabla u) - \mathcal{A}(.,u,\nabla u)).\nabla\varphi + \int (\mathcal{B}(.,u+k,\nabla u) - \mathcal{B}(.,u,\nabla u))\varphi \geq 0 \tag{2.2}$$

for every nonnegative function $\varphi \in W^{1,p}_0(\Omega)$. Then $\mathcal{L}$ satisfies the property $(\pm)$.
2) Note that if $\mathcal{L}(u) = -\sum_j \frac{\partial}{\partial x_j}(\sum_i a_{ij}\frac{\partial u}{\partial x_i} + d_j u) + (\sum_i b_i \frac{\partial u}{\partial x_i} + cu)$ is a linear elliptic operator of second order satisfying the conditions of [12], then (2.2) is equivalent to $(-\sum_j(d_j) + c) \geq 0$ in the distributional sense.
3) Suppose that $\mathcal{A}(x,\zeta,\xi) = \mathcal{A}(x,\xi)$ and for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^d$ the map: $\zeta \to \mathcal{B}(x,\zeta,\xi)$ is increasing. Then the property $(\pm)$ holds.

3 Comparison principle

In this section, we will give some conditions needed for the comparison principle. This principle makes it possible to solve the Dirichlet problem and to develop a potential theory in our case.

We say that the *comparison principle* holds for $\mathcal{L}$, if for every supersolution u and every subsolution v of (1.1) on Ω, such that

$$\limsup_{x\to y} v(x) \leq \liminf_{x\to y} u(x)$$

for all $y \in \partial\Omega$ and both sides of the inequality are not simultaneously $+\infty$ or $-\infty$, we have $v \leq u$ a.e. in Ω.

Theorem 3.1 *Suppose that the operator $\mathcal{L}$ satisfies either one of the property $(\pm)$ and the following strict monotony condition (see [22]):*

$$(\mathcal{A}(x,\zeta,\xi) - \mathcal{A}(x,\zeta',\xi')).(\xi-\xi') + (\mathcal{B}(x,\zeta,\xi) - \mathcal{B}(x,\zeta',\xi'))(\zeta-\zeta') > 0$$

for $(\zeta,\xi)\neq(\zeta',\xi')$. Let u be a supersolution and v be a subsolution of (1.1), on Ω, such that

$$\limsup_{x\to y} v(x)\leq \liminf_{x\to y} u(x)$$

for all $y\in\partial\Omega$ and both sides of the inequality are not simultaneously $+\infty$ or $-\infty$, then $v\leq u$ a.e. in Ω.

Proof. Let $\varepsilon>0$ and K be a compact subset of Ω such that $v-u\leq\varepsilon$ on $\Omega\backslash K$, then the function $\varphi=(v-u-\varepsilon)^+\in W_0^{1,p}(\Omega)$. Testing by φ, we obtain that

$$\begin{aligned} 0 \;\leq\; & \int_{v>u+\varepsilon} (\mathcal{A}(.,u+\varepsilon,\nabla u)-\mathcal{A}(.,v,\nabla v))\nabla(v-u-\varepsilon) \\ & + \int_{v>u+\varepsilon} (\mathcal{B}(.,u+\varepsilon,\nabla u)-\mathcal{B}(.,v,\nabla v))(v-u-\varepsilon)\leq 0\,. \end{aligned}$$

Hence $\nabla(v-u-\varepsilon)^+=0$ and $(v-u-\varepsilon)^+=0$ a.e. in Ω. It follows that $v\leq u+\varepsilon$ a.e. in Ω and therefore $v\leq u$ a.e. in Ω □

Corollary 3.2 *we suppose that $\mathcal{A}(x,\zeta,\xi)=\mathcal{A}(x,\xi)$ and $\mathcal{B}(x,\zeta,\xi)=\mathcal{B}(\zeta)$ such that the map $\zeta\to\mathcal{B}(x,\zeta)$ is increasing for a.e. x in Ω. Then, the comparison principle holds.*

Theorem 3.3 *Suppose that*

i) $[\mathcal{A}(x,\zeta,\xi)-\mathcal{A}(x,\zeta',\xi')].(\xi-\xi')\geq\gamma|\xi-\xi'|^p$ *for all ζ,ζ' in $\mathbb{R}$, for all $\xi,\xi'\in\mathbb{R}^d$, a.e. x in Ω and for some $\gamma>0$.*

ii) *For a.e. $x\in\Omega$ and for all $\xi\in\mathbb{R}^d$, the map $\zeta\to\mathcal{B}(x,\zeta,\xi)$ is increasing,*

iii) $|(\mathcal{B}(x,\zeta,\xi)-\mathcal{B}(x,\zeta,\xi')|\leq b(x,\zeta)|\xi-\xi'|^{p-1}$ *for a.e. $x\in\Omega$, for all $\zeta\in\mathbb{R}$ and for all $\xi,\xi'\in\mathbb{R}^d$. Where $\sup_{|\zeta|\leq M}b(.,\zeta)\in L^s_{\rm loc}(\Omega)$, $s>d$, for all $M>0$.*

Then the comparison principle holds.

Proof. The main idea in this proof comes from Professor J. Maly'. Let $\rho>0$, $M=\sup(v-u)$ and put $w=v-u-\rho$. Take w^+ as test function . Then, we get

$$\int_\Omega [\mathcal{A}(.,u,\nabla u)-\mathcal{A}(.,v,\nabla v)]\,.\nabla(w^+)+\int_\Omega [\mathcal{B}(.,u,\nabla u)-\mathcal{B}(.,v,\nabla v)]\,(w^+)\geq 0$$

and by consequence

$$\begin{aligned} \gamma\int_\Omega|\nabla w^+|^p \;&\leq\; \int_\Omega b(x,v)|\nabla w^+|^{p-1}w^+ \\ &\leq\; C\Big[\int_\Omega|\nabla w^+|^p\Big]^{\frac{p-1}{p}}\Big[\int_\Omega (w^+)^{p^*}\Big]^{\frac{1}{p^*}}|A_\rho|^{\frac{s-d}{sd}} \\ &\leq\; C\|\nabla w^+\|_p^p\,|A_\rho|^{\frac{s-d}{sd}}\,. \end{aligned}$$

where $A_\rho = \{\rho < v - u < M\}$. Hence we get $|A_\rho| \to 0$ when $\rho \to M$, which is impossible if $M > 0$. Thus, $v \le u$ on Ω $\square$

4 Dirichlet Problem

Existence of solutions for $0 \le \alpha \le p-1$ and $0 \le r \le p-1$

Definition Let $g \in W^{1,p}(\Omega)$. We say that u is a solution of problem (P) if

$$u - g \in W_0^{1,p}(\Omega),$$
$$\int_\Omega \mathcal{A}(.,u,\nabla u).\nabla\varphi + \int_\Omega \mathcal{B}(.,u,\nabla u)\varphi = 0 \quad \forall\varphi \in W_0^{1,p}(\Omega).$$

Remark 4.1 *Put $v = u - g$, then u is a solution of the above problem (P) if and only if v is a solution of*

$$\begin{gathered} u \in W_0^{1,p}(\Omega) \\ \int_\Omega \mathcal{A}_g(.,u,\nabla u)\nabla\varphi + \int_\Omega \mathcal{B}_g(.,u,\nabla u)\varphi = 0, \quad \forall\varphi \in W_0^{1,p}(\Omega), \end{gathered} \tag{4.1}$$

where $\mathcal{A}_g(.,u,\nabla u) = \mathcal{A}(.,u+g,\nabla(u+g))$ and $\mathcal{B}_g(.,u,\nabla u) = \mathcal{B}(.,u+g,\nabla(u+g))$.

Let $T : W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$ be the operator defined by

$$\langle T(u), v\rangle = \int \mathcal{A}_g(.,u,\nabla u)\nabla v + \int \mathcal{B}_g(.,u,\nabla u)v \quad \forall v \in W_0^{1,p}(\Omega).$$

Next we will establish the existence of solution of (4.1) when $0 \le \alpha \le p-1$ and $0 \le r \le p-1$. Let $C = C(d,p)$ be a constant such that $\|u\|_{p*} \le C\|u\|_p$ for every $u \in W_0^{1,p}(\Omega)$. Then, we get the following result.

Proposition 4.1 *Suppose that $0 \le \alpha \le p-1$ and $0 \le r \le p-1$. If Ω is small (i.e $\alpha > C(\|d_0\|_{n/p} + \|b\|_{n/p})$), then the operator T is coercive.*

Proof. We have

$$\begin{aligned} \langle T(u), u\rangle &= \int \mathcal{A}(u+g,\nabla(u+g))\nabla u + \int \mathcal{B}(u+g,\nabla(u+g))u \\ &\ge \big(\alpha - C\|d_0\|_{d/p} - C\|b\|_{d/p}\big)\|\nabla u\|_p^p - H_1(\|u\|,\|\nabla u\|,\|g\|,\|\nabla g\|) \end{aligned}$$

where $C = C(d,p)$ and the growth of H_1 in $\|u\|$ and $\|\nabla u\|$ is less then $p-1$. So, let Ω be small enough such that $\alpha > C(\|d_0\|_{n/p}+\|b\|_{n/p})$. Hence, $\frac{\langle T(u),u\rangle}{\|\nabla u\|_p} \to +\infty$ as $\|\nabla u\|_p \to +\infty$ and therefore the operator T is coercive. $\square$

Proposition 4.2 *Suppose that $0 \le \alpha \le p-1$ and $0 \le r \le p-1$. Then, the operator T is pseudomonotone and satisfies the well known property (S_+): If $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle T(u_n) - T(u), u_n - u\rangle \le 0$, then $u_n \to u$.*

The proof of this proposition is found in [21].

Theorem 4.3 *Suppose that T satisfies the coercive condition on Ω. Then (4.1) has at least one weak solution in $W_0^{1,p}(\Omega)$.*

Proof. The operator T is pseudomonotone, bounded continuous and coercive. Hence, by [22] T is surjective. $\square$

Existence of solutions for $\alpha \geq 0$ and $p-1 < r < \frac{p}{(p^*)'}$

Definition Let g be an element of $W^{1-\frac{1}{p}}(\partial\Omega)$.

We say that a function u is a solution of (4.2) with boundary value g if

$$\begin{gathered} u \in W^{1,p}(\Omega), \mathcal{B}(.,u,\nabla u) \in L^{p*'}_{\rm loc}\Omega \\ u = g \text{ in } W^{1-\frac{1}{p}}(\partial\Omega), \\ \int_\Omega \mathcal{A}(.,u,\nabla u)\nabla\varphi + \int_\Omega \mathcal{B}(.,u,\nabla u)\varphi = 0 \quad \forall\varphi \in W_0^{1,p}(\Omega). \end{gathered} \tag{4.2}$$

(For the definition and properties of the space $W^{1-\frac{1}{p}}(\partial\Omega)$ see e.g. [20]).

We say that u is an upper supersolution of (4.2) with boundary value g if

$$\begin{gathered} u \in W^{1,p}(\Omega), \mathcal{B}(.,u,\nabla u) \in L^{p*'}_{Loc}\Omega \\ u \geq g \text{ in } W^{1-\frac{1}{p}}(\partial\Omega), \\ \int_\Omega \mathcal{A}(.,u,\nabla u)\nabla\varphi + \int_\Omega \mathcal{B}(.,u,\nabla u)\varphi \geq 0 \end{gathered}$$

for all $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$.

Similarly, a lower subsolution is characterized by the reverse inequality signs in the above definition.

We recall the following result given in [18, Theorem 2.2].

Theorem 4.4 *Suppose that there exists an ordered pair $\varphi \leq \psi$ of subsolution and supersolution of* (4.2) *satisfying the following condition: There exists $k \in L^q(\Omega)$, $q > p^{*\prime}$ such that for all $\xi \in \mathbb{R}^d$ and all ζ with $\varphi(x) \leq \zeta \leq \psi(x)$, $|\mathcal{B}(x,\zeta,\xi)| \leq k(x) + c|\xi|^r$ a.e.$x \in \Omega$. Then,* (4.2) *has at least one solution $u \in W_0^{1,p}(\Omega)$ such that $\varphi \leq u \leq \psi$.*

Proposition 4.5 *Suppose that* (4.2) *admits a pair of bounded lower subsolution u and upper supersolution v such that $u \leq v$, then there exists a solution w of* (4.2) *such that $u \leq w \leq v$.*

Proof. Let M be a positive real such that $\|u\|_\infty, \|v\|_\infty, \|g\|_\infty \leq M$. Then, for each ζ such that $u(x) - g(x) \leq \zeta \leq v(x) - g(x)$, we have $|\mathcal{B}(x,\zeta,\xi)| \leq k(x) + b(x)M^\alpha + 2^r c|\nabla g|^r + c|\xi|^r$ for a.e. $x \in \Omega$. In addition, u (resp. v) is a lower subsolution (resp. upper supersolution) of (4.2). Hence by the last Theorem, there exists a solution w of (4.2) such that $u \leq w \leq v$. $\square$

Corollary 4.6 *Suppose that all positive constants are supersolutions and all negative constants are subsolutions. Then for each $g \in W^{1,p}(\overline{\Omega}) \cap L^\infty(\Omega)$, there exists a bounded solution w of (4.2) such that $\|w\|_\infty \leq \|g\|_\infty$.*

Proof. We see that $v = \|g\|_\infty$ is an upper supersolution and $u = -\|g\|_\infty$ is a lower subsolution. Hence by the Proposition given above, we get a solution $u \leq w \leq v$ $\square$

4.1 Dirichlet Problem

In this section, we assume that $\mathcal{A}(.,0,0) = 0$ and $\mathcal{B}(.,0,0) = 0$ a.e. in Ω, that the property $(\pm)$ is satisfied, and that the comparison principle holds.

Suppose that the open set Ω is regular ($p-$regular) [21, 11]. Then it is known that if u is a solution of (1.1) in Ω satisfying $u - f \in W_0^{1,p}(\Omega)$ with $f \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, then

$$\lim_{x \to z} u(x) = f(z) \quad \forall z \in \partial\Omega.$$

Definition Let f be a continuous function on $\partial\Omega$. We say that $u \in C(\overline{\Omega}) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ solves the Dirichlet problem with boundary value f if u is a solution of (1.1) such that $\lim_{x\to z} u(x) = f(z)$, for all $z \in \partial\Omega$.

Theorem 4.7 *For each $f \in C(\partial\Omega)$, there exists u in $C(\overline{\Omega}) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ solving the Dirichlet problem with boundary value f.*

Proof By the Tieze's extension Theorem, we can assume that $f \in C_c^\infty(\mathbb{R}^d)$. Let $(f_n)_n$ be a sequence of mollifiers of f such that $\|f_n - f\| \leq 1/2^n$ on $\overline{\Omega}$.

let u_n denote the continuous solution of

$$\begin{gathered} u_n - f_n \in W_0^{1,p}(\Omega), \\ \int_\Omega \mathcal{A}(.,u_n,\nabla u_n)\nabla\varphi + \int_\Omega \mathcal{B}(.,u_n,\nabla u_n)\varphi = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega). \end{gathered} \tag{4.3}$$

So, by the comparison principle, $|u_n - u_m| \leq \frac{1}{2^n} + \frac{1}{2^m}$. Hence, the sequence $(u_n)_n$ converges uniformly on $\overline{\Omega}$ to a continuous function u. Let M be a positive real such that for all n: $|f_n| + |f| \leq M$ and $|u_n| + |u| \leq M$ on Ω.

Let $G \subset \overline{G} \subset \Omega$, take φ as a test function in (4.3) such that $\varphi = \eta^p u_n, \eta \in C_c^\infty(\Omega), 0 \leq \eta \leq 1$ and $\eta = 1$ on G. Then

$$\begin{aligned} &\int_\Omega \mathcal{A}(.,u_n,\nabla u_n)\eta^p\nabla(u_n) \\ &\qquad = -p\int_\Omega \mathcal{A}(.,u_n,\nabla u_n)u_n\eta^{p-1}\nabla(\eta) - \int_\Omega \mathcal{B}(.,u_n,\nabla u_n)u_n\eta^p \end{aligned}$$

Using the assumptions on $\mathcal{A}$ and $\mathcal{B}$, we get

$$\begin{aligned}
&\alpha\int_\Omega \eta^p|\nabla(u_n)|^p \\
&\le pM\int_\Omega k_0|\nabla\eta| + pM^p\int_\Omega b_0|\nabla\eta| + pM\int_\Omega a|\nabla u_n|^{p-1}\eta^{p-1}|\nabla\eta| \\
&\quad +cM\int_\Omega |\nabla u_n|^r\eta^p + \int_\Omega (M^p d_0 + Mk + M^{\alpha+1}b + e) \\
&\le a(p-1)^{-1}M\varepsilon^{\frac{p}{p-1}}(\int_\Omega |\nabla u_n|^p\eta^p) + crp^{-1}M\varepsilon^{\frac{p}{r}}(\int_\Omega |\nabla u_n|^p\eta^p) \\
&\quad +C(M,\Omega,\eta,\nabla\eta).
\end{aligned}$$

Thus, for ε small enough, we obtain

$$\int_G |\nabla(u_n)|^p \le C(M,\Omega,\eta,\nabla\eta,\varepsilon).$$

So $(\nabla u_n)_n$ is bounded in $L^p(G)$ and therefore $(\nabla u_n)_n$ converges weakly to ∇u in $(L^p(G))^d$.

Fix D an open subset of G and let $\eta \in C_0^\infty(G)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on D. Take $\psi = \eta(u_n - u)$ as test function, then

$$\begin{aligned}
&-\int_\Omega \eta\mathcal{A}(.,u_n,\nabla u_n).\nabla(u_n-u) \\
&= \int_\Omega (u_n-u)\mathcal{A}(.,u_n,\nabla u_n).\nabla\eta + \int_\Omega \mathcal{B}(.,u_n,\nabla u_n)(u_n-u)\eta
\end{aligned}$$

Since $\mathcal{A}(.,u_n,\nabla u_n)$ is bounded in $L^{p'}(G)$ and $\mathcal{B}(.,u_n,\nabla u_n)$ is bounded in $L^q(G)$,

$$\lim_{n\to\infty}\int_G \mathcal{A}(.,u_n,\nabla u_n)(u_n-u)\nabla\eta = 0,$$
$$\lim_{n\to\infty}\int_G \mathcal{B}(.,u_n,\nabla u_n)(u_n-u)\eta = 0.$$

Consequently, $\lim_{n\to\infty}\int_G \mathcal{A}(.,u_n,\nabla u_n)\eta\nabla(u_n-u) = 0$ and

$$\lim_{n\to\infty}\int_G (\mathcal{A}(.,u_n,\nabla u_n) - \mathcal{A}(.,u_n,\nabla u))\nabla(u_n-u) = 0.$$

To complete the proof, we need to prove that $(\nabla u_n)_n$ converges to ∇u a.e. in Ω. That is the aim of the following lemma.

Lemma 4.8 *Let $G \subset \Omega$ and suppose that the sequence $(\nabla u_n)_n$ is bounded in $L^p(G)$ and*

$$\lim_{n\to\infty}\int_G \left[\mathcal{A}(.,u_n,\nabla u_n) - \mathcal{A}(.,u,\nabla u)\right].\nabla(u_n-u) = 0.$$

Then $\mathcal{A}(.,u_n,\nabla u_n) \to \mathcal{A}(.,u,\nabla u)$ weakly in $L^{p'}(G)$.

Proof. Put $v_n = [\mathcal{A}(.,u_n,\nabla u_n) - \mathcal{A}(.,u_n,\nabla u)]\,.\nabla(u_n-u)$. Since

$$\begin{aligned}\int_G v_n &= \int_G [\mathcal{A}(.,u_n,\nabla u_n) - \mathcal{A}(.,u,\nabla u)]\,.\nabla(u_n-u)\\ &\quad - \int_G [\mathcal{A}(.,u_n,\nabla u) - \mathcal{A}(.,u,\nabla u)]\,.\nabla(u_n-u),\end{aligned}$$

for a subsequence we get

$$\lim_{n\to\infty} [\mathcal{A}(.,u_n,\nabla u_n) - \mathcal{A}(.,u_n,\nabla u)]\,.\nabla(u_n-u) = 0$$

a.e. $x \in G\setminus N$ with $|N| = 0$. Let $x \in G\setminus N$. By the assumptions on $\mathcal{A}$ we have

$$v_n(x) \ge \alpha|\nabla u_n(x)|^p - F(|\nabla u_n(x)|^{p-1}, |\nabla u(x)|^{p-1}).$$

Consequently, $(\nabla u_n(x))_n$ is bounded and converges to some $\xi \in \mathbb{R}^d$. It follows that $[\mathcal{A}(.,u,\xi) - \mathcal{A}(.,u,\nabla u)].(\xi - \nabla u) = 0$ and hence $\xi = \nabla u$. Finally we conclude that $\mathcal{A}(.,u_n,\nabla u_n) \to \mathcal{A}(.,u,\nabla u)$ a.e. in G and $\mathcal{A}(.,u_n,\nabla u_n)$ converge weakly to $\mathcal{A}(.,u,\nabla u)$ in $L^{p'}(G)$. $\square$

Now we go back to the proof of Theorem 4.7. Using Lemma 4.8, we conclude that $\nabla u_n \to \nabla u$ a.e. in Ω and $\mathcal{A}(.,u_n,\nabla u_n) \rightharpoonup \mathcal{A}(.,u,\nabla u)$ in $L^{p'}(D)$. Hence,

$$\int_D \mathcal{A}(.,u,\nabla u)\nabla\varphi + \int_D \mathcal{B}(.,u,\nabla u)\varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega).$$

Moreover, using the fact that

$$-\frac{1}{2^n} - \frac{1}{2^m} \le u_m - u_n \le \frac{1}{2^n} + \frac{1}{2^m} \quad \forall n,m$$

we obtain

$$-\frac{1}{2^n} + u_n \le u \le \frac{1}{2^n} + u_n, \quad \forall n.$$

So, we deduce that for all n and all $z \in \partial\Omega$,

$$-\frac{1}{2^n} + f_n(z) \le \liminf_{x\in\Omega, x\to z} u(z) \le \limsup_{x\in\Omega,x\to z} u(z) \le \frac{1}{2^n} + f_n(z)$$

which implies $\lim_{x\to z} u(x) = f(z)$ and completes the proof of Theorem 4.7. $\square$

Remark 4.2 Using the same techniques as in the proof of Theorem 4.7 we can show that every increasing and locally bounded sequence $(u_n)_n$ of supersolutions of (1.1) in Ω is locally bounded in $W^{1,p}(\Omega)$ and that $u = \lim_n u_n$ is a supersolution of (1.1) in Ω.

5 Sheaf property for Superharmonic functions

The obstacle Problem

Definition Let f, $h \in W^{1,p}(\Omega)$ and let

$$K_{f,h} = \{u \in W^{1,p}(\Omega) : h \le u \text{ a.e. in } \Omega, u - f \in W_0^{1,p}(\Omega)\}.$$

If $f = h$, we denote $K_{f,h} = K_f$.

We say that a function $u \in K_{f,h}$ is a solution to the obstacle problem in $K_{f,h}$ if

$$\int_\Omega \mathcal{A}(.,u,\nabla u).\nabla(v-u) + \int_\Omega \mathcal{B}(.,u,\nabla u)(v-u) \ge 0$$

whenever $v \in K_{f,h}$. This function u is called solution of the problem with obstacle h and boundary value f.

Remark 5.1 Since $u + \varphi \in K_{f,h}$ for all nonnegative $\varphi \in W_0^{1,p}(\Omega)$, the solution u to the obstacle problem is always a supersolution of *(1.1)* in Ω. Conversely, a supersolution of *(1.1)* is always a solution to the obstacle problem in $K_u(D)$ for all open $D \subset \overline{D} \subset \Omega$.

Theorem 5.1 *Let h and f be in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$. If v is an upper bounded supersolution of* (4.2) *with boundary value f such that $v \ge h$, then there exists a solution u to the obstacle problem in $K_{f,h}$ with $u \le v$.*

Proof. As in [18], we introduce the function

$$g(x,\zeta,\xi) = \begin{cases} \widetilde{\mathcal{B}}(x,\zeta,\xi) & \text{if } \zeta \le v(x) \\ \widetilde{\mathcal{B}}(x,v,\nabla v) & \text{if } \zeta > v(x). \end{cases}$$

As in [13], we define the function

$$\mathbf{a}(x,\zeta,\xi) = \begin{cases} \mathcal{A}(x,\zeta,\xi) & \text{if } \zeta \le v(x) \\ \mathcal{A}(x,v,\nabla v) & \text{if } \zeta > v(x). \end{cases}$$

Note that $\mathbf{a}$ satisfies the conditions (P1), (P2), and (P3).

A Lemma in [7, p.52] proves that the map $u \to g(x,u,\nabla u)$ from $W^{1,p}(\Omega)$ to $L^{p'}(\Omega)$ is bounded and continuous. Without loss of generality we can assume that $r \ge p-1$. Let $l = \max\{q', \frac{p}{p-r}\} - 1$, and define the following penalty term

$$\gamma(x,s) = [(s - v(x))^+]^l \quad \forall x \in \Omega, s \in \mathbb{R}.$$

Let $M > 0$ and consider the map $T : K_{0,h} \to W^{-1,p'}(\Omega)$ defined by

$$\langle T(u), w\rangle = \int_\Omega \mathbf{a}(.,u,\nabla u)\nabla w + \int_\Omega g(.,u,\nabla u)w + M\int_\Omega \gamma(.,u)w.$$

Then for any $u, w \in K_{0,h}$, we have

$$| \int_\Omega g(x,u,\nabla u)w| \le c_1\|w\|_{l+1} + c_2\|\nabla u\|_p^r\|w\|_{l+1},$$
$$| \int_\Omega \gamma(x,u)w| \le c_3\|w\|_{l+1} + c_4\|u\|_{l+1}^l\|w\|_{l+1},$$

and for each $u \in K_{f,h} - f$, we have

$$\int_\Omega \gamma(.,u)u \ge c_5\|u\|_{l+1}^{l+1} - c_6.$$

An easy computation shows that for $\varepsilon > 0$,

$$\begin{aligned}(T(u),u) \quad &\ge \quad (\alpha - c_2\varepsilon)\|\nabla u\|_p^p - (c\|u\|_p^p + c_1\|u\|_{l+1}^{l+1} + c_2c(\varepsilon)\|u\|_{l+1}^{l+1}) \\ &\quad\quad +Mc_5\|u\|_{l+1}^{l+1} - Mc_6 - c_1c_7.\end{aligned}$$

where $c(\varepsilon)$ is a constant which depends on ε and $c > 0$. Now, we choose M large to get the operator T coercive. Since T is bounded , pseudomonotone and continuous, then by a Theorem in [22], there exists $w \in K_{0,h}$ such that $(T(w), u - w) \ge 0$ for all $u \in K_{0,h}$.

Next, we show that $w \le v$. Since $w - ((w - v) \vee 0) \in K_{0,h}$ and since v is a supersolution of (4.2), it follows that

$$\int_{\{w>v\}} [\mathcal{A}(.,w,\nabla w) - \mathcal{A}(.,v,\nabla v)]\nabla(w-v) \le M \int_{\{w>v\}} \gamma(.,w)(v-w).$$

Thus by (P2), $(w - v)^+ = 0$ a.e. in Ω and hence, $w \le v$ on Ω. Finally, if we take $w_1 = w + f$, we obtain a supersolution of the obstacle problem $K_{f,h}$. $\square$

Nonlinear Harmonic Space

Definition Let V be a regular set. For every $f \in C(\partial V)$, we denote by $H_V f$ the solution of the Dirichlet problem with the boundary data f.

Proposition 5.2 *Let f and g in $C(\partial V)$ be such that $f \le g$. Then*

i) $H_V f \le H_V g$

ii) *For every $k \ge 0$, we have $H_V(k+f) \le H_V(f)+k$ and $H_V(f)-k \le H_V(f-k)$.*

Definition Let U be an open set. We denote by $\mathcal{U}(U)$ the set of all open, regular subsets of U which are relatively compact in U.

We say that a function u is harmonic on U, if $u \in C(U)$ and u is a solution of (1.1). We denote by $\mathcal{H}(U)$ the set of all harmonic functions on U. Then,

$$\mathcal{H}(U) = \big\{u \in C(U) : H_V u = u \text{ for every } V \in \mathcal{U}(U)\big\}.$$

A lower semicontinuous function u is said to be hyperharmonic on U, if

- $-\infty < u$
- $u \neq \infty$ in each component of U
- For each regular set $V \subset \overline{V} \subset \Omega$ and for every $f \in \mathcal{H}(V) \cap C(\overline{V})$, the inequality $f \leq u$ on ∂V implies $f \leq u$ in V .

We denote by ${}^*\mathcal{H}(U)$ the set of all hyperharmonic functions on U.

An upper semicontinuous function u is said to be hypoharmonic on U, if

- $u < +\infty$
- $u \neq \infty$ in each component of U
- For each regular set $V \subset \overline{V} \subset \Omega$ and each $f \in \mathcal{H}(\overline{V}) \cap C(\overline{V})$, the inequality $f \geq u$ on ∂V implies $f \geq u$ in V .

We denote by $\mathcal{H}_*(U)$ the set of all hypoharmonic functions on U.

Proposition 5.3 *Let $u \in {}^*\mathcal{H}(U)$ and $v \in \mathcal{H}_*(U)$, then for each $k \geq 0$ we have $u + k \in {}^*\mathcal{H}(U)$ and $v - k \in \mathcal{H}_*(U)$.*

Proposition 5.4 *Let u be a superharmonic function and v be a subharmonic function on U such that*

$$\lim\sup_{x \to z} v(x) \leq \lim\inf_{x \to z} u(x)$$

for all $z \in \partial U$, and both sides of the previous inequality are not simultaneously $+\infty$ or $-\infty$, then $v \leq u$ in U.

Proof. Let $x \in U$ and $\varepsilon > 0$. Choose a regular open set $V \subset \overline{V} \subset U$ such that $x \in V$ and $v < u + \varepsilon$ on ∂V. Let $(\varphi_i) \in C^\infty(\Omega)$ be a decreasing sequence converging to v in $\overline{V}$. Then $\varphi_i \leq u + \varepsilon$ on ∂V for i large. Let $h = H_V(\varphi_i)$, then $v \leq h \leq u + \varepsilon$ on V. By letting $\varepsilon \to 0$, we get $v(x) \leq u(x)$. $\square$

Theorem 5.5 *The space $(\mathbb{R}^d, \mathcal{H})$ satisfies the Bauer convergence property.*

Proof. Let $(u_n)_n$ be an increasing sequence in $\mathcal{H}(U)$ locally bounded. By Theorem 4.11 in [21], for every $V \subset \overline{V} \subset U$, the set $\{u_n(x), x \in \overline{V}, n \in \mathbb{N}\}$ is equicontinuous . Then the sequence converges locally and uniformly in U to a continuous function u. Take $\varepsilon > 0$, since $u - \varepsilon \leq u_n \leq u + \varepsilon$, we get $H_V(u) - \varepsilon \leq u_n \leq H_V(u) + \varepsilon$ and therefore $H_V(u) = u$ $\square$

Theorem 5.6 *Suppose that the conditions in subsection 4.1 are satisfied, $k_0 = e = k = 0$ and $\alpha \geq p - 1$. Then $(\mathbb{R}^d, \mathcal{H})$ is a nonlinear Bauer harmonic space.*

Proof. It is clear that $\mathcal{H}$ is a sheaf of continuous functions and by Theorem 4.7 there exists a basis of regular sets stable by intersection. The Bauer convergence property is fulfilled by Theorem 5.5. Since $k_0 = e = k = 0$ and $\alpha \geq p-1$, we have the following form of the Harnack inequality (e.g. [21],[26] or [24]): For every non empty open set U in $\mathbb{R}^d$, for every constant $M > 0$ and every compact K in U, there exists a constant $C = C(K, M)$ such hat

$$\sup_K u \leq C \inf_K u$$

for every $u \in \mathcal{H}^+(U)$ with $u \leq M$. It follows that the sheaf $\mathcal{H}$ is non degenerate. □

Theorem 5.7 *Suppose that the condition of strict monotony holds. Let $u \in \mathcal{H}^*(\Omega) \cap L^\infty(\Omega)$. Then u is a supersolution on U.*

Proof. Let $V \subset \overline{V} \subset \Omega$. Let $(\varphi_i)_i$ be an increasing sequence in $C_c^\infty(\Omega)$ such that $u = \sup_i \varphi_i$ on $\overline{V}$. Let

$$K_{\varphi_i} = \big\{w \in W^{1,p}_{\rm loc}(\Omega) : \varphi_i \leq w,\ \ w - \varphi_i \in W_0^{1,p}(V)\big\}.$$

We know by Theorem 5.1 that there exists a solution u_i to the obstacle problem K_{φ_i} such that $\|u_i\|_\infty \leq \|\varphi_i\|_\infty$. We claim that $(u_i)_i$ is increasing. In fact $u_i \wedge u_{i+1} \in K_{\varphi_i}$, then

$$\begin{aligned} &\int_{\{u_i > u_{i+1}\}} (\mathcal{A}(., u_i, \nabla u_i) - \mathcal{A}(., u_{i+1}, \nabla u_{i+1}))\nabla(u_{i+1} - u_i) \\ &+ \int_{\{u_i > u_{i+1}\}} (\mathcal{B}(., u_i, \nabla u_i) - \mathcal{B}(., u_{i+1}, \nabla u_{i+1}))(u_{i+1} - u_i) \quad \geq 0. \end{aligned}$$

Hence $\nabla(u_{i+1} - u_i)^+ = 0$ a.e. which yields that $u_i \leq u_{i+1}$ a.e. in V .

On the other hand, for each i the function u_i is a solution of (1.1) in $D_i := \{\varphi_i < u_i\}$. Indeed, let $\psi \in C_c^\infty(W)$, $W \subset \overline{W} \subset D_i$, and $\varepsilon > 0$ such that $\varepsilon\|\psi\| \leq \inf_{\overline{W}}(u_i - \varphi_i)$. Then, we get $u_i + \varepsilon\psi \in K_{\varphi_i}$ and

$$\int_W \mathcal{A}(., u_i, \nabla u_i).\nabla\psi + \int_W \mathcal{B}(., u_i, \nabla u_i)\psi = 0.$$

Since

$$\liminf_{x \to y} u(x) \geq u(y) \geq \varphi_i(y) = \lim_{x \to y} u_i(x)$$

for all $y \in \partial D_i$, it yields, by the comparison principle, that $u \geq u_i$ in D_i. Hence $u \geq u_i$ in D. Thus $u = \lim_{i\to\infty}\varphi_i \leq \lim_{i\to\infty} u_i \leq u$. Finally, using Remark 4.2 we complete the proof. □

Theorem 5.8 *Suppose that the condition of strict monotonicity holds. Then ${}^*\mathcal{H}$ is a sheaf.*

The proof of this theorem is the same as in [2, Theorem 4.2].

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