*Electronic Journal of Differential Equations*, Vol. 2002(2002), No. 84, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

# CONSTRUCTING UNIVERSAL PATTERN FORMATION PROCESSES GOVERNED BY REACTION-DIFFUSION SYSTEMS

SEN-ZHONG HUANG

Abstract. For a given connected compact subset $K$ in $\mathbb{R}^n$ we construct a smooth map $F$ on $\mathbb{R}^{1+n}$ in such a way that the corresponding reaction-diffusion system $u_t = D\Delta u + F(u)$ of $n+1$ components $u = (u_0, u_1, \dots, u_n)$, accompanying with the homogeneous Neumann boundary condition, has an attractor which is isomorphic to $K$. This implies the following universality: The make-up of a pattern with arbitrary complexity (e.g., a fractal pattern) can be realized by a reaction-diffusion system once the vector supply term $F$ has been previously properly constructed.

## 1. Introduction

Reaction-diffusion systems of the form

$$u_t = D\Delta u + F(u), \quad x \in \Omega, t > 0,$$

model the diffusion through a domain $\Omega \subset \mathbb{R}^k$ of $m$ interacting species or chemicals, where the $i$-th component $u_i$ of $u = (u_1, \dots, u_m)$ represents the density or concentration of the $i$-th reactants and $D = (d_1, \dots, d_m)$ is the matrix of the diffusion constants $d_i > 0$ (cf. [7, 10, 1, 12, 13]). One hard mathematical task for the practical use of these models is to find appropriate vector supply terms $F$ in such a way that the pattern formation process governed by the corresponding reaction-diffusion system coincides with the phenomenon observed in the laboratory experiments or in nature. Famous examples in this direction include the Kolmogorov-Fischer equations modeling the two-species interactions, the Field-Noyes equations modeling the Belousov-Zhabotinsky reactions in chemical kinetics, the Hodgkin-Huxley equations or the FitzHugh-Nagumo equations modeling the nerve impulse transmission (cf. [7, 10, 1, 12, 13]), and the Murray equations ([8], [9]) modeling the mammalian coat markings. However, as commented by J.D. Murray [10, p.238], "for most practical models of real world situations it is premature, to say the least, to spend too much time on sophistical generalisations before the simpler versions have been shown to be inadequate when compared with experiments," and, "the generalisations seem endless."

---

2000 *Mathematics Subject Classification.* 35B40, 70G60, 35Q99.
*Key words and phrases.* Attractor, pattern formation.

Submitted August 28, 2002. Published October 4, 2002.

Murray's comment raises implicitly a very crucial theoretical problem, namely, the following *universality problem*: Can the reaction-diffusion systems model each phenomenon of *arbitrary complexity*?

Because the pattern formation process is the main subject in question, we can pose the above universality problem in a more restrictive way, i.e., we ask if the complexity of patterns modeled by the reaction-diffusion systems can be allowed to be arbitrary.

A pattern is the eventual result of a time evolution of a biological or chemical or physical process and thus has the following two main features: a) Long-time effect – it describes the long-time behavior of that process, and b) Great randomness of the initial conditions – the initial conditions which lead to the same pattern have a great randomness due to the circumstance fluctuation.

Based on this observation, we see that a pattern is a kind of attractor. Here, by an attractor for a reaction-diffusion system we mean the mathematical object which attracts an open set of initial data in such a way that the trajectories starting from this initial data set eventually end up on the attractor in question (this is just the long-time effect of an attractor). The openness of the set of initial data guarantees the required great randomness of initial data which lead to the same pattern (attractor). Moreover, this openness corresponds to the practical need (e.g., for the computer simulation) that there is a positive probability that computed trajectories will tend to the attractor. Using the concept of attractors, the above universality problem can be mathematically more exactly reformulated as follows:

*Can the complexity of attractors for the reaction-diffusion systems be allowed to be arbitrary?*

In this article we will tackle the "universality problem" mathematically posed in this way. What we will do is to give a method of constructing the vector supply term $F$ for the purpose that the corresponding reaction-diffusion system has an attractor whose complexity is allowed to be arbitrary in some sense.

In the following statement, $\Omega$ is any given smooth bounded domain in $\mathbb{R}^k$. Moreover, for every fixed $n \in \mathbb{N}$ we identify the Euclidean space $\mathbb{R}^n$ as a subspace of the product Banach space $C(\bar{\Omega})^n$ by identifying a point $(a_1, \dots, a_n) \in \mathbb{R}^n$ with the vector of constant functions $(a_1\mathbf{1}, \dots, a_n\mathbf{1}) \in C(\bar{\Omega})^n$. We recall the notion *$\omega$-limit set* $\omega(\psi, E)$ which is used to describe the asymptotic behavior of a map $\psi : [0, \infty) \to E$ in a Banach space $E$ and is defined as the set of all points $p \in E$ for which there exists a sequence $(t_k)_{k\in\mathbb{N}}$ (probably depending on the points $p$) such that $t_k \to \infty$ and $\|\psi(t_k) - p\| \to 0$ as $k \to \infty$. For our present situation, the notion $\omega(u, C)$ for a map $u : [0, \infty) \to C(\bar{\Omega})^{1+n}$ is used to denote the $\omega-$limit set of $u$ which is calculated in the Banach space $C(\bar{\Omega})^{1+n}$. Our main result reads as follows.

**Theorem 1.1.** *Let $n \in \mathbb{N}$ and $K \subset \mathbb{R}^n$ a given connected compact subset of arbitrary complexity. Then there exists a vector supply term $F_K$ of the form*

$$F_K(\phi, v) = (A(\phi), A'(\phi)v + f(\phi)) \quad for \quad \phi \in \mathbb{R}, v \in \mathbb{R}^n, \tag{1.1a}$$

*where $A$ is a smooth function on $\mathbb{R}$ and $f$ a smooth map on $\mathbb{R}^n$, such that the corresponding reaction-diffusion system*

$$\begin{aligned} u_t &= D\Delta u + F_K(u), \quad x \in \Omega, t > 0, \\ \frac{du}{dn} &= 0 \quad on \quad \partial\Omega \end{aligned} \tag{1.1b}$$

*of $n+1$ components $u = (\phi, v)$, accompanying with the zero flux boundary condition, generates a dynamical system in the state space $C(\bar{\Omega})^{1+n}$ with the following properties (i)-(ii):*
*(i) For each initial value $u_0 \in C(\bar{\Omega})^{1+n}$ the reaction-diffusion system (1.1a-b) has a global unique solution $u$, $u(0) = u_0$, such that $u$ is continuous in $\bar{\Omega} \times [0, \infty)$, $u_t, \Delta u$ as well as all partial derivatives $\frac{\partial u}{\partial x_i}$ are continuous in $\bar{\Omega} \times (0, \infty)$.*
*(ii) Each solution $u$ of the reaction-diffusion system (1.1a-b) starting from an initial value $u_0 = (\phi_0, v_0) \in C(\bar{\Omega})^{1+n}$ such that either $\phi_0 > 0$ or $\phi_0 < 0$ is asymptotically stable and converges to the set $\tilde{K} := \{0\} \times K$ in the sense that $\omega(u, C) = \tilde{K}$. As result, the connected compact set $\tilde{K}$ is an attractor for the reaction-diffusion system (1.1a-b) settled in the state space $C(\bar{\Omega})^{1+n}$.*

Here we choose the zero flux boundary condition (i.e., the homogeneous Neumann boundary condition) and the positive initial condition, since they are probably the most interested boundary and initial conditions in the biological or chemical situation. Namely, the zero flux boundary condition reflects the self-organization mechanism of pattern while the positive initial condition restricts the pattern formation process to such a beginning circumstance that each of the reactants has a positive distribution all over the reaction domain. Thus, our Theorem 1.1 should be considered as a partial but affirmative answer to the universality problem stated above and implies that any pattern (here $\tilde{K}$) which is isomorphic to a connected compact subset (here $K$) of the Euclidean space $\mathbb{R}^n$ can be seen as the final result of the pattern formation process governed by some appropriate reaction-diffusion system of $n+1$ components. Moreover, it implies the following universality: The make-up of a pattern $\tilde{K} \cong K \subset \mathbb{R}^n$ with arbitrary complexity (e.g., a fractal pattern [6]) can be realized by a reaction-diffusion system of the form (1.1a-b) once the vector supply term $F_K$ has been previously properly constructed.

Although the above construction is the product of theoretic thoughts, we are also interested in whether it is possible to derive such reaction-diffusion systems from any sequence of reasonable biochemical or physical situations. To have a close look of the mechanism of the system (1.1a-b), we write $D = \operatorname{diag}(d_0, \tilde{D})$ and then decompose the reaction-diffusion system (1.1a-b) into the reaction-diffusion equation for the first component $\phi$

$$\begin{gathered} \phi_t = d_0 \Delta \phi + A(\phi), \quad x \in \Omega, t > 0, \\ \frac{d\phi}{dn} = 0 \quad \text{on} \quad \partial\Omega \end{gathered} \tag{1.2a}$$

and the linear inhomogeneous reaction-diffusion system for the rest $n$ components $v$

$$\begin{gathered} v_t = \tilde{D} \Delta v + A'(\phi) v + f(\phi), \quad x \in \Omega, t > 0, \\ \frac{dv}{dn} = 0 \quad \text{on} \quad \partial\Omega. \end{gathered} \tag{1.2b}$$

In this way, it becomes clear that the first component $\phi$ of the vector $u = (\phi, v)$ works as the *activator* and the rest components $v$ as the *inhibitor* of the reaction-diffusion system (1.1a-b). It is worthy noting that the activator system (1.2a) is self-contained and the inhibitor system (1.2b) is an inhomogeneous nonautonomous *linear* system with an external force $f(\phi)$ and an internal friction force $A'(\phi)v$ supplied by the activator. This mechanism simplicity throws light on the possibility of deriving reaction-diffusion systems of the form (1.1a-b) from real world situations.

To close this introduction, we give a remark echoing with the referee's comments on the previous version of the present article. Starting from the pioneering work of A.M. Turing [15], there is a continuous interest in the mathematical modelings by the so-called "parameter space method", i.e., one looks for a vector supply term $F$ of the form $F(u) = G(u, \alpha)$ which depends on a set of parameters $\alpha$, and the choice of the form of such vector supply terms is in general based on the following philosophy: the form should be as simple as possible. Therefore, the usual candidate for such forms is the class of *elementary functions* like polynomials or rational functions. It is just very surprising that with such simple choice one has had a very deep look at the secrets of the developmental ground plan of the nature. A lot of such examples are prepared in the book [10] of J.D. Murray and in the article [1] of M.C. Cross and P.C. Hohenberg. One of them is the very fascinating ones posed by J.D. Murray [9] for modeling the mammalian coat markings. Murray's model is a reaction-diffusion system of the form

$$v_t = \Delta v + \gamma[a - v - g(v, w)], \quad w_t = \beta\Delta w + \gamma[\alpha(b - w) - g(v, w)],$$

with $g(v, w) = \rho vw/(1 + v + \kappa v^2)$ and seven parameters $a, b, \alpha, \beta, \gamma, \rho$ and $\kappa$. Murray's work is based on a previous model experimentally posed by D. Thomas [14]. However, the fascinating aspects of Thomas' model, as shown by Murray, is certainly unexpected. In connecting to our work presented in this article, one would like to know if the elementary functions choice of the vector supply terms $F$ in the reaction-diffusion systems is adequate for the common purposes.

The author is very indebted to the anonymous referee for her/his valuable discussions.

## 2. Constructive proof of Theorem 1.1

This is divided into two main steps: the construction of the smooth function $A$ as well as the smooth map $f$ in the nonlinearity $F_K(\phi, v) = (A(\phi), A'(\phi)v + f(\phi))$ and the analysis of the resulted reaction-diffusion system (1.1b).

**2A. Tools** First we recall two construction results from [4]. Starting with a Banach space $E$, we define the $\omega-$*limit set* $\omega(\psi, E)$ (resp., $\alpha-$*limit set* $\alpha(\psi, E)$) of a continuous path $\psi : (a, b) \to E$ to be the set consisting of each point $p \in E$ for which there exists a sequence $(s_k)_{k\geq 1} \subset (a, b)$ such that $\|\psi(s_k) - p\| \to 0$ as $k \to \infty$ and $s_k \to b$ as $k \to \infty$ (resp., $s_k \to a$ as $k \to \infty$). Equivalently, we have the following more compact representations

$$\omega(\psi, E) = \bigcap_{k=2}^{\infty} \overline{\psi([b - (b - a)/k, b))}, \quad \alpha(\psi, E) = \bigcap_{k=2}^{\infty} \overline{\psi(a, a + (b - a)/k]}$$

Here the closure is taken in $E$. These two limit sets describe the behavior of the path $\psi$ near the endpoints $a$ and $b$. In general, we do not assume that $\psi(s)$ has a limit as $s$ approaching to endpoints; namely, a path $\psi$ which serves to satisfy our following requirements should be very irregular near endpoints.

It is not difficult to establish the following two useful properties of limit sets: Each $\omega-$limit (resp., $\alpha-$limit) set is closed and connected once it is not empty. Recall that a closed subset in a Banach space is said *connected* if this set can not be decomposed into the union of two disjoint closed subsets. Thus, connectedness is sometimes refered to as "indecomposibility".

**Lemma 2.1.** *Let $(E, \|\cdot\|)$ be a given Banach space.*
**(i)** *For each connected compact subset $K$ in $E$ there exists a path $\psi : (0,1] \to E$ of class $C^\infty$ such that $\alpha(\psi, E) = K$.*
**(ii)** *For every $C^\infty$ path $\varphi : (0,1] \to E$ there exists a $C^\infty$ function $A : \mathbb{R} \to \mathbb{R}$ satisfying the conditions (c1)-(c2) below:*
**(c1)** *$A$ is an odd function which is strictly negative and concave in the interval $(0,\infty)$, i.e., there holds $-A(-s) = A(s) < 0$ and $A''(s) < 0$ for all $s > 0$. Moreover, $A^{(m)}(0) = 0$ for all $m \in \{0\} \cup \mathbb{N}$.*
**(c2)** *The function $A$ smooths the given path $\varphi$ in such a way that*

$$\lim_{s\to 0} |A(s)^{(m)}| \cdot \|\varphi(s)^{(l)}\| = 0 \quad \forall m, l \in \{0\} \cup \mathbb{N}.$$

The topological assertion in (i) has just been proved in [4, Lemma 1] (a sketch proof can be also found in the Appendix of [5]). It is worthy noting that this assertion implies the following fact: A compact subset $K$ of a Banach space $E$ is connected if and only if it is the $\alpha-$limit set of some smooth path in $E$. The assertion in Lemma 2.1-(ii) is very similar to the ones in [4, Lemma 2]. However, the concavity and the convexity of the function $A$ here is new. For the sake of completeness, below we give a sketch of its proof.

**Sketch proof of Lemma 2.1-(ii):** We begin with $k \in \{0\} \cup \mathbb{N}$ and put

$$\rho_k(s) := \max_{s\le r\le 1} \|\varphi^{(k)}(r)\| \quad (s \in (0,1]).$$

Each $\rho_k$ is nonincreasing and continuous. Note that for each $s \in (0,1]$ there are at most finitely many $k \in \{0\} \cup \mathbb{N}$ such that $k < 1/s$. Therefore, the following

$$\tilde\rho(s) := \sum_{\{k<1/s\}} \rho_k(s) \quad (0 < s \le 1)$$

well defines a function on $(0,1]$, which is again nonincreasing and continuous. Setting $\rho(s) := s/(1+\tilde\rho(s))$ $(0 < s \le 1)$, $\rho(0) := 0$ and $\rho(s) := \rho(1)$ for all $s > 1$, we obtain a nondecreasing continuous function $\rho$. Define

$$g(s) := \int_0^s \beta(s-r)\rho(r)\,dr \quad (s \ge 0)$$

where the function $\beta : [0,\infty) \to \mathbb{R}$ is given by $\beta(r) := r\exp(-1/r)$ for $r > 0$ and $\beta(0) := 0$. An elementary computation yields that $\beta^{(m)}(0) = 0$ for all $m \in \mathbb{N}$, $\beta''(r) = r^{-4}\beta(r)$ for all $r > 0$. This implies that the function $g$ is an increasing and convex function of class $C^\infty$ and

$$g^{(m)}(s) = \int_0^s \beta^{(m)}(s-r)\rho(r)\,dr \tag{2.1}$$

for all $m \ge 1$ and all $s \ge 0$. Finally, the odd function $A$ is defined by $A(s) := -g(s)$ for $s \le 0$ and $A(s) := -g(-s)$ for $s < 0$. The desired properties in (c1) follow directly from the related ones of the function $g$. To see the smoothing property in (c2), we fix $l, m \in \mathbb{N} \cup \{0\}$ and consider $s \in (0,1)$ such that $s < 1/(1+l)$. On one hand, we have by definition that $\tilde\rho(s) \ge \rho_l(s) \ge \|\varphi^{(l)}(s)\|$ and thus

$$\rho(s) = s/(1+\tilde\rho(s)) \le s/(1+\|\varphi^{(l)}(s)\|). \tag{2.2a}$$

On the other hand, we use (2.1) combining with the monotonicity of the function $\rho$ to find that

$$|A^{(m)}(s)| = |g^{(m)}(s)| \le c_m \cdot \rho(s) \tag{2.2b}$$

with $c_m := \int_0^1 |\beta^{(m)}(r)|\,dr$. Taking into account the estimates in (2.2a-b), we see that

$$|A^{(m)}(s)| \cdot \|\varphi^{(l)}(s)\| \le c_m \cdot s$$

for all $s \in (0,1)$ being such that $s < 1/(1+l)$; yielding the convergence $|A^{(m)}(s)| \cdot \|\varphi^{(l)}(s)\| \to 0$ as $s \to 0$.

**2B. Construction of the vector supply term** From now on $K \subset \mathbb{R}^n$ is a given connected compact subset of arbitrary complexity.

As the first step of the construction we will apply Lemma 2.1 under the choice $E = \mathbb{R}^n \subset C(\bar\Omega)^n$ to find the function $A$ and the map $f$. Here, as before, we have identified $\mathbb{R}^n$ with the subspace of constant functions in the product Banach space $C(\bar\Omega)^n$. Starting with the set $K$ we first apply Lemma 2.1-(i) to obtain a $C^\infty$ path $\tilde\psi : (0,1] \to \mathbb{R}^n$ such that $\alpha(\tilde\psi, \mathbb{R}^n) = K$. We then apply Lemma 2.1-(ii) under the choice of the path $\varphi = \tilde\psi$ to obtain an odd $C^\infty$ function $A : \mathbb{R} \to \mathbb{R}$ satisfying the conditions (c1)-(c2) there.

To construct the smooth map $f$, we need extend the path $\tilde\psi$ to a smooth path $\psi$ defined on $(0,\infty)$. To this end, we choose a $C^\infty$ function $\gamma$ on $\mathbb{R}$ such that $\gamma(s) = 1$ for all $s < 1/4$ and $\gamma(s) = 0$ for all $s > 1/2$. The method of constructing such functions is well-known, see e.g. [3, p.42]. We set $\psi(s) := \gamma(s)\tilde\psi(s)$ for $s \in (0,1]$ and $\psi(s) = \mathbf{0}$ for all $s > 1$. Then $\psi : (0,\infty) \to \mathbb{R}^n$ is a smooth path such that

$$\alpha(\psi, \mathbb{R}^n) = \alpha(\tilde\psi, \mathbb{R}^n) = K. \tag{2.3a}$$

We define the map $f : \mathbb{R} \to \mathbb{R}^n$ by

$$f(\phi) := \begin{cases} A(\phi)\psi'(\phi) - A'(\phi)\psi(\phi) & \text{for } \phi > 0, \\ 0 & \text{for } \phi = 0, \\ A(-\phi)\psi'(-\phi) - A'(-\phi)\psi(-\phi) & \text{for } \phi < 0. \end{cases} \tag{2.3b}$$

Clearly, $f$ vanishes outside the compact interval $[-1/4, 1/4]$. Moreover, $f$ is an even map, i.e., $f(-\phi) = f(\phi)$ for all $\phi \in \mathbb{R}$. With $A, f$ constructed as above, we take the smooth map

$$F_K(u) := (A(\phi), A'(\phi)v + f(\phi)) \quad \text{for} \quad u = (\phi, v) \in \mathbb{R} \times \mathbb{R}^n \tag{2.4a}$$

as the vector supply term, and thus have a reaction-diffusion system

$$\begin{gathered} u_t = D\Delta u + F_K(u) \quad x \in \Omega, t > 0 \\ \frac{du}{dn} = 0 \quad \text{on} \quad \partial\Omega \end{gathered} \tag{2.4b}$$

accompanying with the zero flux boundary condition. As observed in the Introduction, with $D = \operatorname{diag}(d_0, \tilde D)$ the reaction-diffusion system (2.4a-b) decomposes into the reaction-diffusion equation for the *activator* consisting of the first component $\phi$ :

$$\begin{gathered} \phi_t = d_0\Delta\phi + A(\phi), \quad x \in \Omega, t > 0, \\ \frac{d\phi}{dn} = 0 \quad \text{on} \quad \partial\Omega \end{gathered} \tag{2.5a}$$

and the linear inhomogeneous reaction-diffusion system for the *inhibitor* consisting of the rest $n$ components $v$ :

$$\begin{gathered} v_t = \tilde D\Delta v + A'(\phi)v + f(\phi), \quad x \in \Omega, t > 0, \\ \frac{dv}{dn} = 0 \quad \text{on} \quad \partial\Omega. \end{gathered} \tag{2.5b}$$

Since the function $A$ is odd so that $A'$ is even and the map $f$ is even, it follows from the equivalent form (2.5) that $u = (\phi, v)$ solves (2.4) if and only if $\tilde{u} := (-\phi, v)$ solves (2.4).

**2C. Analysis of the activator system (2.5a)** We study first its kinetic reduction $\xi' = A(\xi)$. Since the smooth function $A$ is negative in the interval $(0, \infty)$ (cf. condition (c1)), the following initial value problem

$$\xi'(t) = A(\xi(t)), t \geq 0; \quad \xi(0) = s \in \mathbb{R}. \tag{2.6$a$}$$

admits a unique smooth solution $\xi(t, s)$ which exists for all $t \geq 0$. As a consequence of the oddness of $A$ and the uniqueness of the solutions, there holds $\xi(t, -s) = -\xi(t, s)$ for all $(t, s)$ and $\xi(t, s) > 0$ if $s > 0$. Moreover, the negativity $A(s) < 0$ $(s > 0)$ implies that

$$\lim_{t\to\infty} \xi(t, s) = 0 = \lim_{t\to\infty} \xi(t, -s) \quad \text{for all} \quad s \geq 0. \tag{2.6$b$}$$

Let $\sigma, \tau$, $\sigma > \tau > 0$, be fixed. Set $\xi_1(t) := \xi(t, \sigma)$, $\xi_2(t) := \xi(t, \tau)$ and $\eta(t) := \xi_1(t) - \xi_2(t)$ for $t \geq 0$. Since $\sigma > \tau$, an elementary comparison result yields that $\xi_1(t) > \xi_2(t)$ and thus $\eta(t) > 0$ for all $t \geq 0$. Using the concavity of the function $A(s)$ for $s > 0$, we have

$$A(\xi_1(t)) - A(\xi_2(t)) \leq A'(\xi_2(t))(\xi_1(t) - \xi_2(t)).$$

Therefore,

$$\eta'(t) = A(\xi_1(t)) - A(\xi_2(t)) \leq A'(\xi_2(t))\eta(t).$$

Integrating this inequality yields

$$\eta(t) \leq \eta(0) \cdot \exp(\int_0^t A'(\xi_2(s))\, ds).$$

To compute the integral, we differentiate the equation $\xi_2'(t) = A(\xi_2(t))$ and obtain

$$A'(\xi_2(t)) = \xi_2''(t)/\xi_2'(t) = \frac{d}{dt} \ln(-\xi_2'(t)).$$

It follows that

$$\int_0^t A'(\xi_2(s))\, ds = \ln(\xi_2'(t)/\xi_2'(0)) = \ln(A(\xi_2(t))/A(\tau))$$

and thus $\eta(t) \leq \eta(0)A(\xi_2(t))/A(\tau)$. In terms of the solutions $\xi(\cdot, \sigma)$ and $\xi(\cdot, \tau)$, this implies that

$$0 < \xi(t, \sigma) - \xi(t, \tau) \leq \sigma \cdot \frac{A(\xi(t, \tau))}{A(\tau)} \quad (t \geq 0); \quad \sigma > \tau > 0. \tag{2.6$c$}$$

Next we turn to the study of solutions for the activator system (2.5a). Consider an initial value $\phi_0 \in C(\bar{\Omega})$ and set

$$\tau_0 := \min_{x\in\bar{\Omega}} \phi_0(x), \quad \sigma_0 := \max_{x\in\bar{\Omega}} \phi_0(x). \tag{2.7$a$}$$

By the classical existence and uniqueness theorem (cf. e.g. [12]) there exists a $T > 0$ such that the equation (2.5a) has a unique solution $\phi$ in the time interval $[0, T)$, $\phi(0) = \phi_0$, with the property that $\phi$ is continuous in $\bar{\Omega} \times [0, T)$, $\phi_t, \Delta\phi$ as well as all partial derivatives $\frac{\partial \phi}{\partial x_i}$ are continuous in $\bar{\Omega} \times (0, T)$. Moreover, if $T < \infty$, then there should hold $\lim_{t\to T} \|\phi(t)\|_\infty = \infty$. We claim that $T = \infty$, i.e., the solution $\phi$

to (2.5a) with initial value $\phi(0) = \phi_0 \in C(\bar\Omega)$ exists for all times $t \geq 0$. To see this, we note that both $\xi(\cdot, \sigma_0)$ and $\xi(\cdot, \tau_0)$ are also solutions of (2.5a) such that

$$\xi(0, \tau_0) = \tau_0 \leq \phi_0(x) \leq \sigma_0 = \xi(0, \sigma_0) \quad \forall x \in \bar\Omega.$$

Hence, the classical comparison theorem (see e.g. [12, Theorem 10.1, p.94]) yields that

$$\xi(t, \tau_0) \leq \phi(x, t) \leq \xi(t, \sigma_0) \quad \forall x \in \bar\Omega, t \in [0, T). \tag{2.7b}$$

This implies in particular the uniform boundedness of the solution $\phi(t)$ in the time interval $[0, T)$ and thus we have $T = \infty$. Moreover, since both solutions $\xi(t, \tau_0)$ and $\xi(t, \sigma_0)$ converge to 0 as $t \to \infty$, we have that $\|\phi(t)\|_\infty \to 0$ as $t \to \infty$. To find one more result, we assume $\phi_0 > 0$. It follows that $\tau_0 > 0$. Hence, we are in the position to apply (2.6c) under the choice of the positive constants $\tau, \sigma$ such that $\tau \leq \tau_0$ and $\sigma \geq \sigma_0 = \|\phi_0\|_\infty$, and obtain that

$$\begin{gathered} \|\phi(t) - \xi(t, \tau)\|_\infty \leq \sigma \cdot \frac{A(\xi(t, \tau))}{A(\tau)} \quad (t \geq 0); \\ \forall \tau, \sigma : 0 < \tau \leq \tau_0 \leq \sigma_0 \leq \sigma. \end{gathered} \tag{2.7c}$$

**2D. Final step of the proof of Theorem 1.1** We prove first the assertion in (i). For this purpose, we consider an initial value $u_0 = (\phi_0, v_0) \in C(\bar\Omega) \times C(\bar\Omega)^n$. By the result in 2C, the equation (2.5a) has a unique global solution $\phi$, $\phi(0) = \phi_0$, with the property that $\phi$ is continuous in $\bar\Omega \times [0, \infty)$, $\phi_t, \Delta\phi$ as well as all partial derivatives $\frac{\partial \phi}{\partial x_i}$ are continuous in $\bar\Omega \times (0, \infty)$. Moreover, there holds the estimate in (2.7b) for the general case and the estimate in (2.7c) for the particular case $\phi_0 > 0$. Substituting this global solution $\phi$ into (2.5b), we have a linear system. It follows from the basic linear theory (see e.g., [11], [2]) that the corresponding linear system (2.5b) has a unique global solution $v$, $v(0) = v_0$, with the property that $v$ is continuous in $\bar\Omega \times [0, \infty)$, $v_t, \Delta v$ as well as all partial derivatives $\frac{\partial v}{\partial x_i}$ are continuous in $\bar\Omega \times (0, \infty)$. Thus, we have established the global existence and uniqueness assertion in (i).

To prove the convergence assertion in (ii), we note that if $u(t) = (\phi(t), v(t)) \in C(\bar\Omega)^{1+n}$ is a global solution of (2.5a-b) then $\tilde u(t) := (-\phi(t), v(t))$ is also a solution of (2.5a-b). Hence, it suffices to prove the assertion in (ii) for the case $\phi_0 > 0$. To this end, we fix two positive constants $\sigma, \tau$, $\tau < \sigma$. We consider a global solution $u = (\phi, v)$ with initial value $u_0 = (\phi_0, v_0)$ such that $\phi_0 \in C(\bar\Omega)$ satisfies

$$\tau \leq \phi_0(x) \leq \sigma \quad \forall x \in \bar\Omega. \tag{2.8a}$$

Then, we have, using (2.7b-c), that

$$\begin{gathered} \|\phi(t) - \zeta(t)\|_\infty \leq c_1 \cdot (-A(\zeta(t))) \\ \zeta(t) \leq \phi(x, t) \quad (\forall x \in \bar\Omega) \end{gathered} \quad (t \geq 0) \tag{2.8b}$$

with the positive constant $c_1 := \sigma/(-A(\tau)) > 0$ and positive function $\zeta(t) := \xi(t, \tau)$ $(t \geq 0)$. Since $A$ is strictly concave in the interval $(0, \infty)$, the function $A'$ is decreasing there and thus

$$A'(\phi(x, t)) \leq A'(\zeta(t)) \quad \forall x \in \bar\Omega, t \geq 0. \tag{2.8c}$$

We want to show that there exist positive constants $c_2, c_3$, depending only on the given constants $\sigma, \tau$, such that

$$\begin{array}{c}\|w(t)\|_\infty \le c_2 \cdot \|w(0)\|_\infty \cdot (-A(\zeta(t))) + c_3 \cdot \zeta(t/2) \\ \text{with} \quad w(t) := v(t) - \psi(\xi(t))\end{array} \quad (\forall t \ge 0). \tag{2.9}$$

Once (2.9) was proved, we derive the asymptotic stability of the solution $u$ and the equality $\omega(u, C) = \tilde{K}$ as follows. On the one hand, we use (2.9) combining with the fact that $\zeta(t) \to 0$ and $A(\zeta(t)) \to 0$ as $t \to \infty$ (see (2.6b)) as well as the equality $\alpha(\psi, \mathbb{R}^n) = K$ (see (2.3a)) to find that

$$\omega(v, C) = \alpha(\psi, C) = \alpha(\psi, \mathbb{R}^n) = K,$$

giving the desired equality $\omega(u, C) = \{0\} \times K = \tilde{K}$. On the other hand, we let $\hat{u} = (\hat{v}, \hat{\phi})$ be another global solution of (2.5a-b) with an initial value $\hat{u}_0 = (\hat{v}_0, \hat{\phi}_0)$ such that

$$\tau \le \hat{\phi}_0(x) \le \sigma \quad \forall x \in \bar{\Omega}.$$

Then we can apply (2.9) to the new solution $\hat{u}$ and obtain that

$$\|\hat{w}(t)\|_\infty \le c_2 \cdot \|\hat{w}(0)\|_\infty \cdot (-A(\zeta(t))) + c_3 \cdot \zeta(t/2) \quad \forall t \ge 0$$

with $\hat{w}(t) := \hat{v}(t) - \psi(\xi(t))$ $(t \ge 0)$. It yields that

$$\|v(t) - \hat{v}(t)\|_\infty \le c_2(\|w(0)\|_\infty + \|\hat{w}(0)\|_\infty) \cdot (-A(\zeta(t))) + 2c_3 \cdot \zeta(t/2)$$

for all $t \ge 0$. Obviously, this implies the asymptotic stability of the solution $u$. Now we prove (2.9). To this end, we note that

$$\begin{aligned} w_t &= v_t - \zeta'\psi'(\zeta) = v_t - A(\zeta)\psi'(\zeta) \\ &= \tilde{D}\Delta(w + \psi(\zeta)) + A'(\phi)(w + \psi(\zeta)) + f(\phi) - A(\zeta)\psi'(\zeta) \\ &= \tilde{D}\Delta w + A'(\phi)w + [A'(\phi) - A'(\zeta)]\psi(\zeta) + [f(\phi) - f(\zeta)] \\ &\quad + [f(\zeta) + A'(\zeta)\psi(\zeta) - A(\zeta)\psi'(\zeta)]. \end{aligned}$$

By the definition of the map $f$ (see (2.3b)), the term in the last bracket vanishes. Therefore,

$$w_t = \tilde{D}\Delta w + A'(\phi)w + g(t) \tag{2.10$a$}$$

with

$$g(t) := [A'(\phi(t)) - A'(\zeta(t))]\psi(\zeta(t)) + [f(\phi(t)) - f(\zeta(t))] \quad (t \ge 0). \tag{2.10$b$}$$

To estimate the term $g(t)$, we note that the spatially uniform solution $\zeta$ is bounded in time, and (2.8a) implies that the solution $\phi$ is uniformly bounded in space-time. Moreover, the smooth map $f$ has a compact support, and the smooth path $\psi$ is bounded by the compactness of the given set $K$. Hence, there exists a positive constant $c_4$ (essentially depending only on the given constants $\sigma, \tau$ as well as on the map $\psi$) such that

$$\|g(t)\|_\infty \le c_4 \cdot \|\phi(t) - \zeta(t)\|_\infty$$

for all $t \ge 0$. This implies by (2.8b) that

$$\|g(t)\|_\infty \le c_5 \cdot (-A(\zeta(t))) \quad (t \ge 0) \tag{2.10$c$}$$

with $c_5 := c_4\sigma/(-A(\tau)) > 0$. To treat the solution $w$ of the inhomogeneous linear evolution equation (2.10a), we recall from the basic linear theory (see e.g., [11], [2]) the following representation

$$w(t) = U(t,0)w(0) + \int_0^t U(t,s)g(s)\,ds \quad (t \geq 0), \tag{2.11}$$

where $\{U(t,s) : t \geq s \geq 0\}$ is the evolution family associated with the nonautonomous linear system

$$\begin{gathered} W_t = \tilde{D}\Delta W + A'(\phi)W, \quad x \in \Omega, t > 0, \\ \frac{dW}{dn} = 0 \quad \text{on} \quad \partial\Omega \end{gathered} \tag{2.12$a$}$$

settled in the state space $C(\bar{\Omega})^n$. Since solutions to (2.12a) with positive initial data keep positive, each operator $U(t,s)$ is positive, i.e.,

$$U(t,s)h \geq 0 \quad \text{whenever} \quad h \geq 0. \tag{2.12$b$}$$

We compare the linear system (2.12a) with the following nonautonomous linear system

$$\begin{gathered} \tilde{W}_t = \tilde{D}\Delta\tilde{W} + A'(\zeta(t))\tilde{W}, \quad x \in \Omega, t > 0, \\ \frac{d\tilde{W}}{dn} = 0 \quad \text{on} \quad \partial\Omega \end{gathered} \tag{2.13$a$}$$

settled in the same state space $C(\bar{\Omega})^n$. It is easily verified, using the equality $\zeta' = A(\zeta)$, that the evolution family $\{V(t,s) : t \geq s \geq 0\}$ associated with (2.13a) is given by

$$V(t,s) = \frac{A(\zeta(t))}{A(\zeta(s))}T(t-s) \quad (t \geq s \geq 0), \tag{2.13$b$}$$

where $\{T(t) : t \geq 0\}$ is the semigroup generated by the operator $\tilde{D}\Delta$ for which each operator $T(t)$ is a positive contraction on the state space $C(\bar{\Omega})^n$. We claim that

$$U(t,s) \leq V(t,s) \quad \forall t \geq s \geq 0. \tag{2.14$a$}$$

To see this, we fix some $s \geq 0$ and consider a solution $W$ of (2.12a) in the interval $[s,\infty)$ with a positive initial value $W(s) = h \geq 0$. It follows from the definition of evolution families that this solution $W$ is given by $W(t) = U(t,s)h$ for all $t \geq s$. We have $W(t) \geq 0$ for all $t \geq s$. Hence, by (2.8c) we find that

$$W_t = \tilde{D}\Delta W + A'(\phi)W \leq \tilde{D}\Delta W + A'(\zeta)W.$$

Applying the classical comparison theorem (see e.g., [12, Theorem 10.1, p.94]) to each component of $W$, we conclude that $W(t) \leq \tilde{W}(t)$ ($\forall t \geq s$), where $\tilde{W}$ is the positive solution of (2.13a) with the same initial value $\tilde{W}(s) = h$ and thus is given by $\tilde{W}(t) = V(t,s)h$ for all $t \geq s$, again by the definition of evolution families. This establishes (2.14a). As consequence of (2.14a), we find particularly that

$$\|U(t,s)\|_\infty \leq \|V(t,s)\|_\infty \leq \frac{A(\zeta(t))}{A(\zeta(s))} \quad (t \geq s \geq 0). \tag{2.14$b$}$$

We are now in the position to finish the proof of (2.9). Using the representation (2.11) of $w$, we have

$$\begin{aligned}\|w(t)\|_\infty &\le \|U(t,0)\|_\infty \cdot \|w(0)\|_\infty + \int_0^t \|U(t,s)\|_\infty \cdot \|g(s)\|_\infty \, ds \\ &\le \frac{A(\zeta(t))}{A(\tau)} \cdot \|w(0)\|_\infty + \int_0^t \frac{A(\zeta(t))}{A(\zeta(s))} \cdot \|g(s)\|_\infty \, ds \quad \text{(by (2.14b))} \\ &\le \frac{A(\zeta(t))}{A(\tau)} \cdot \|w(0)\|_\infty + \int_0^t \frac{A(\zeta(t))}{A(\zeta(s))} \cdot c_5 \cdot (-A(\zeta(s)) \, ds \quad \text{(by (2.10c))}.\end{aligned}$$

Consequently,

$$\|w(t)\|_\infty \le \frac{A(\zeta(t))}{A(\tau)} \cdot \|w(0)\|_\infty + c_5 \cdot (-tA(\zeta(t)) \quad \forall t \ge 0. \tag{2.15$a$}$$

To estimate the term $(-tA(\zeta(t))$, we note that the relation $\zeta' = A(\zeta) < 0$ combining with the monotonicity of the function $A$ implies that the function $t \mapsto A(\zeta(t))$ is increasing. Hence,

$$\frac{t}{2} A(\zeta(t)) \ge \int_{t/2}^t A(\zeta(s)) \, ds = \int_{t/2}^t \zeta'(s) \, ds = \zeta(t) - \zeta(t/2) \ge -\zeta(t/2)$$

and thus

$$(-tA(\zeta(t)) \le 2\zeta(t/2) \quad \forall t \ge 0. \tag{2.15$b$}$$

Taking (2.15a-b) together, we find finally that

$$\|w(t)\|_\infty \le \frac{A(\zeta(t))}{A(\tau)} \cdot \|w(0)\|_\infty + 2c_5 \cdot \zeta(t/2) \quad \text{for all} \quad t \ge 0. \tag{2.16}$$

Therefore, the estimate (2.9) holds true under the choice of the positive constants $c_2 := 1/(-A(\tau))$ and $c_3 := 2c_5$. □

## References


[1] M. C. Cross and P. C. Hohenberg, Pattern formation outside of equilibrium, *Rev. Modern Phys.* **65** (1993), 851-1112.
[2] K.-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, 2000.
[3] M. W. Hirsch, Differential Topology (4th Ed.), Springer-Verlag, 1991.
[4] S.-Z. Huang, Constructing attractors with arbitrary complexity, Preprint 2000.
[5] S.-Z. Huang, Constructing Navier-Stokes equations with attractors of arbitrary complexity, *Physics Letters A* **282** (2001), 1-8.
[6] B. B. Mandelbrot, The Fractal Geometry of Nature, Freeman, New York, 1983.
[7] J. D. Murray, Nonlinear Differential Equation Models in Biology, Oxford, 1977.
[8] J. D. Murray, On pattern formation mechanisms for lepidopteran wing patterns and mammalian coat markings, *Phil. Trans. Roy. Soc. London* **B295** (1981), 473-496.
[9] J. D. J. Murray, A pre-pattern formation mechanism for animal coat markings, *J. Theor. Biol.* **88** (1981), 161-199.
[10] J. D. Murray, Mathematical Biology, Biomathematics Texts **19**, 2nd Ed., Springer-Verlag, 1993.
[11] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, 1983.
[12] J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, 1983.
[13] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, 1988.

[14] D. Thomas, Artificial enzyme membrane, transport, memory, and oscillatory phenomena. *In*: D. Thomas and J.-P. Kernevez (eds.) Analysis and Control of Immobilized Enzyme Systems. Springer-Verlag, 1975, pp. 115-150.

[15] A.M. Turing, The chemical basis of morphogenesis, *Phil. Trans. Roy. Soc. London* **B237** (1952), 37-72.

Sen-Zhong Huang
Fachbereich Mathematik, Universität Rostock,
Universitätsplatz 1, D-18055 Rostock, F. R. Germany
*E-mail address*: `sen-zhong.huang@mathematik.uni-rostock.de`