Electronic Journal of Differential Equations, Vol. 2002(2002), No. 86, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

AN EMBEDDING NORM AND THE LINDQVIST TRIGONOMETRIC FUNCTIONS

CHRISTER BENNEWITZ & YOSHIMI SAITŌ

Abstract. We shall calculate the operator norm $\|T\|_p$ of the Hardy operator $Tf = \int_0^x f$, where $1 \le p \le \infty$. This operator is related to the Sobolev embedding operator from $W^{1,p}(0,1)/\mathbb{C}$ into $W^p(0,1)/\mathbb{C}$. For $1 < p < \infty$, the extremal, whose norm gives the operator norm $\|T\|_p$, is expressed in terms of the function $\sin_p$ which is a generalization of the usual sine function and was introduced by Lindqvist [6].

1. Introduction

Evans, Harris and Saitō [3] give the following result: Let $W^{1,p}(0,1)$ be the complex first order Sobolev space given by

$$W^{1,p}(0,1) = \big\{ f \mid \int_0^1 (|f|^p + |f'|^p) < \infty \big\},$$

where $1 < p < \infty$. Then

$$\sup \frac{\|f\|_{p,s}}{\|f'\|_p} = \|T\|_p/2, \tag{1.1}$$

where the supremum is over all non-zero functions in $W^{1,p}(0,1)$, $\|\cdot\|_p$ is the norm of $L^p(0,1)$, $\|\cdot\|_{p,s}$ is given by

$$\|f\|_{p,s} = \inf_{C\in\mathbb{C}} \|f - C\|_p,$$

and $\|T\|_p$ is the operator norm of the Hardy operator

$$T : L^p(0,1) \ni f \mapsto Tf(x) = \int_0^x f \in L^p(0,1).$$

The left-hand side of (1.1) is the norm of the embedding from the factor space $W^{1,p}(0,1)/\mathbb{C}$ into the factor space $L^p(0,1)/\mathbb{C}$. Since the Poincaré inequality holds on $(0,1)$, the norm $\|f'\|_p$, $f \in W^{1,p}(0,1)$, is one of the equivalent norms of the space $W^{1,p}(0,1)/\mathbb{C}$. For more details on the embedding operator we refer to Evans and Harris [4], [5], in addition to [3].

In this note we calculate the norm $\|T\|_p = p^{1/q}q^{1/p}\sin(\pi/p)/\pi$, $1 < p < \infty$, where $1/p + 1/q = 1$, and $\|T\|_1 = \|T\|_\infty = 1$. Our main arguments depend on classical and elementary calculus of variations. We will also calculate all extremals,

2000 *Mathematics Subject Classification.* 46E35, 33D05.
Key words and phrases. Sobolev embedding operator, Volterra operator.

Submitted September 12, 2002. Published October 9, 2002.

i.e., functions f for which $\|Tf\|_p = \|T\|_p\|f\|_p$. In particular, if $1 < p < \infty$ these are expressed in terms of '$\sin_p$' and '$\cos_p$' functions which has been introduced in Lindqvist [6]. These are natural generalizations of the usual trigonometric functions when the Euclidean norm in $\mathbb{R}^2$ is replaced by a p-norm.

After finishing our work, we found two recent works, Edmunds and Lang [2] and Drabek and Manásevich [1], which, among others, produced quite a similar results in real L^p spaces on intervals using the results on the p-Laplacian.

In Section 2 we shall give a short account on $\sin_p$ and $\cos_p$ functions. In Section 3 we shall first deal with the cases that $p = 1$ and $p = \infty$. Then we shall study the case that $1 < p < \infty$. After showing that we have only to find the nonnegative extremals (Lemmas 3 and 4), we shall show that the extremal is expressed in terms of the $\sin_p$ function and the norm $\|T\|_p$ is computed using some properties of the $\sin_p$ and $\cos_p$ functions (Theorem 3.1). We shall discuss the operator

$$T : L^p(0,1) \ni f \mapsto Tf(x) = \int_0^x f \in L^q(0,1).$$

in Section 4.

2. $\sin_p$, $\cos_p$ AND $\tan_p$ FUNCTIONS

Suppose $1 < p < \infty$ and consider the function $x \mapsto \int_0^x \frac{dt}{(1-t^p)^{1/p}}$, $0 \le x \le 1$. This is strictly increasing, and for $p = 2$ we obtain $\arcsin x$. By analogy we denote the inverse for general p by $\sin_p x$. It is defined on the interval $[0, \pi_p/2]$ where $\pi_p = 2\int_0^1 \frac{dt}{(1-t^p)^{1/p}}$, it is strictly increasing on this interval, $\sin_p 0 = 0$ and $\sin_p(\pi_p/2) = 1$. We may extend the definition to the interval $[0, \pi_p]$ by setting $\sin_p x = \sin_p(\pi_p - x)$ for $\pi_p/2 \le x \le \pi_p$, and to $[-\pi_p, \pi_p]$ by extending it as an odd function. Finally, we may define it on all of $\mathbb{R}$ by extending it as a periodic function of period $2\pi_p$.

Next we define $\cos_p x = \frac{d}{dx}\sin_p x$, which is an even, $2\pi_p$-periodic function, odd around $\pi_p/2$. In $[0, \pi_p/2]$, setting $y = \sin_p x$, we have

$$\cos_p x = \tfrac{d}{dx}\sin_p x = (1 - y^p)^{1/p} = (1 - (\sin_p x)^p)^{1/p},$$

so that $\cos_p$ is strictly decreasing on this interval, $\cos_p 0 = 1$ and $\cos_p(\pi_p/2) = 0$. Furthermore

$$|\cos_p x|^p + |\sin_p x|^p = 1,$$

first in $[0, \pi_p/2]$, but then by symmetry and periodicity in all of $\mathbb{R}$. One may also introduce an analogue of the tangent function as $\tan_p x = \sin_p x / \cos_p x$. We then obtain $\frac{d}{dx}\cos_p x = -|\tan_p x|^{p-2}\sin_p x$. We also get $\frac{d}{dx}\tan_p x = 1/|\cos_p x|^p = 1 + |\tan_p x|^p$, so that the inverse of the restriction of $\tan_p$ to the interval $(-\pi_p/2, \pi_p/2)$ has derivative $1/(1 + |y|^p)$, $y \in \mathbb{R}$. The constant π_p is easily calculated. In fact, by a change of variable $t = s^{1/p}$ we obtain

$$\pi_p = \tfrac{2}{p}\int_0^1 (1-s)^{-1/p}s^{1/p-1}\,ds = \tfrac{2}{p}B(1-1/p, 1/p) = 2\frac{\pi/p}{\sin(\pi/p)}, \tag{2.1}$$

where B is the classical beta function. If q is the conjugate exponent to p, so that $1/p + 1/q = 1$, this means in particular that $p\pi_p = q\pi_q$.

The cases $p = 1$ and $p = \infty$ are somewhat degenerate, especially for $p = 1$. For $p = \infty$ one gets $\pi_\infty = 2$, and $\sin_p$ becomes a triangular wave, with $\sin_p x = x$ on $[-1, 1]$, and $\cos_p$ the corresponding square wave. For $p = 1$ one gets $\pi_1 = \infty$, and $\sin_p x$ is the odd extension of $1 - e^{-x}$, $x \ge 0$, whereas $\cos_p x = e^{-|x|}$.

3. An operator norm

Consider the linear operator $Tf(x) = \int_0^x f$ on $L^p(0,1)$, $1 \le p \le \infty$. We shall determine the norm $\|T\|_p$ and all *extremals*, *i.e.*, all functions $f \in L^p(0,1)$ for which $\|Tf\|_p = \|T\|_p \|f\|_p$. In particular, we shall show that if q is the conjugate exponent to p, then $\|T\|_p = \|T\|_q$. This may be proved by duality if $1 < p < \infty$, but will follow from our explicit calculations below.

Theorem 3.1. *For $1 < p < \infty$, $\|T\|_p = 2p^{1/q}q^{1/p}/(p\pi_p) = p^{1/q}q^{1/p}\sin(\pi/p)/\pi$, where q is the dual exponent to p. The corresponding extremals are all multiples of $\cos_p(\pi_p x/2)$, where π_p is given by (2.1). Furthermore, $\|T\|_1 = \|T\|_\infty = 1$, and the extremals for $p = \infty$ are all constants, whereas no extremals exist in $L^1(0,1)$ for the case $p = 1$. If one extends T to the space of finite Borel measures on $[0,1]$, however, the extremals are all multiples of the Dirac measure at 0.*

In particular $\|T\|_p = \|T\|_q$ since $p\pi_p = q\pi_q$. The statement of the theorem indicates that it will be advantageous, for $p = 1$, to extend T to operate on finite Borel measures on $[0,1]$, normed by total variation, thus considering $L^1(0,1)$ as the subset of absolutely continuous Borel measures on $[0,1]$. We will do this in the sequel without further comment. The proof of the theorem is almost immediate in the cases $p = 1$ and $p = \infty$.

Lemma 3.2. *For $p = 1$ extremals exist, they are precisely the non-zero multiples of the Dirac measure at 0, and $\|T\|_1 = 1$.*

Proof. If μ is a finite Borel measure on $[0,1]$ and we define $T\mu(x) = \int_{[0,x]} d\mu$, we obtain

$$\begin{aligned}\|T\mu\|_1 = \int_0^1 \Big| \int_{[0,x]} d\mu \Big|\, dx &\le \int_0^1 \int_{[0,x]} |d\mu|\, dx \\ &= \int_{[0,1]} (1-t)|d\mu(t)| \le \int_{[0,1]} |d\mu| = \|\mu\|_1, \end{aligned} \tag{3.1}$$

where the middle equality follows from Fubini's theorem. Thus $\|T\|_1 \le 1$. However, we obtain equality throughout in (3.1) precisely if $\int_{[0,1]} t|d\mu(t)| = 0$, *i.e.*, the support of μ consists of the point 0. Thus, if μ is a multiple of the Dirac measure at 0. □

The proof in the case $p = \infty$ is even simpler.

Lemma 3.3. *For $p = \infty$ extremals exist, they are precisely the functions which are a.e. constant, and $\|T\|_\infty = 1$.*

Proof. If $f \in L^\infty(0,1)$ we obtain

$$\|Tf\|_\infty = \sup_{0 \le x \le 1} \Big| \int_0^x f \Big| \le \int_0^1 |f| \le \|f\|_\infty,$$

and we have equality throughout if and only if f is a.e. a multiple of $\|f\|_\infty$ (see also the proof of Lemma 3.4). □

The case when $1 < p < \infty$ is less trivial, and the first step in the proof is to show that we may restrict ourselves to positive functions when looking for extremals.

Lemma 3.4. *Any extremal for $1 \le p \le \infty$ is a multiple of a non-negative extremal. Therefore, the norm $\|T\|_p$ when T is defined on the complex L^p space is the same as the one when T is defined on the real L^p space.*

Proof. We already know this if $p = 1$ or $p = \infty$, so we now assume that $1 < p < \infty$. Suppose f is an extremal and put $g = |f|$. Since $\|f\|_p = \|g\|_p$ and $\big|\int_0^x f\big| \le \int_0^x g$ it follows that if f is an extremal, then so is g. Furthermore, it follows that $\|Tf\|_p = \|Tg\|_p$. Now $\big|\int_0^x f\big| \le \int_0^x g$ for all $x \in [0,1]$, so in order that $\|Tf\|_p = \|Tg\|_p$ we must have equality throughout $[0,1]$. In particular, we must have

$$|\int_0^1 f| = \int_0^1 g,$$

i.e., we must have equality in the triangle inequality. This proves the lemma, since it requires that f and g are a.e. proportional. We remind the reader why this is so.

If we choose $\theta \in \mathbb{R}$ so that $e^{i\theta}\int_0^1 f$ is positive, and we have equality in the triangle inequality $\big|\int_0^1 f\big| \le \int_0^1 |f|$, then we have $\int_0^1 \operatorname{Re}(e^{i\theta} f) = \int_0^1 |f|$ while at the same time $\operatorname{Re}(e^{i\theta} f) \le |e^{i\theta} f| = |f|$. Thus $|f| = e^{i\theta} f$ a.e. □

The next step in the proof of Theorem 3.1 is to show the existence of extremals.

Lemma 3.5. *If $1 < p < \infty$ there exists a function $f \in L^p(0,1)$ with non-zero norm such that $\|Tf\|_p = \|T\|_p \|f\|_p$.*

Proof. Let $f_1, f_2, \dots$ be a sequence of unit vectors in $L^p(0,1)$ such that

$$\|Tf_n\|_p \to \|T\|_p \quad \text{as } n \to \infty.$$

We may assume that $f_n \rightharpoonup f \in L^p(0,1)$ weakly in $L^p(0,1)$. For each fixed $x \in [0,1]$ the characteristic function of $[0,x]$ is in $L^q(0,1)$, $q = p/(p-1)$, so we obtain

$$F_n(x) = \int_0^x f_n \to \int_0^x f = F(x) \quad \text{as } n \to \infty.$$

Since $|F_n(x)| \le \int_0^1 |f_n| \le \|f_n\|_p = 1$ by Hölder's inequality, we obtain by dominated convergence that $\|T\|_p = \lim \|F_n\|_p = \|F\|_p$, and since clearly $\|T\|_p > 0$ we can not have $F = 0$, so that $\|f\|_p > 0$. Since $\|T\|_p = \|F\|_p \le \|T\|_p \|f\|_p$ it actually follows that $\|f\|_p = 1$. The lemma is proved. □

We are now ready to prove Theorem 3.1. We will do this by applying standard methods of the calculus of variations.

Proof of Theorem 3.1. We need only consider the case $1 < p < \infty$. Suppose $f \ge 0$ is an extremal with non-zero norm and put $F(x) = Tf(x) = \int_0^x f \ge 0$. Then setting $\lambda = \|F\|_p^p / \|F'\|_p^p$ we have $\|T\|_p = \lambda^{1/p}$. If $\varphi \in C^1(0,1)$ with $\varphi(0) = 0$, then $\|F + \varepsilon\varphi\|_p^p / \|F' + \varepsilon\varphi'\|_p^p$ has a maximum for $\varepsilon = 0$. Differentiating we get, for $\varepsilon = 0$, that

$$\int_0^1 F^{p-1}\varphi - \lambda \int_0^1 (F')^{p-1}\varphi' = 0,$$

so integrating by parts we obtain

$$\int_0^1 (\lambda (F')^{p-1} - \int_x^1 F^{p-1})\varphi' = 0 \tag{3.2}$$

for all $\varphi \in C^1(0,1)$ with $\varphi(0) = 0$, so that, by du Bois Reymond's lemma, $\lambda (F')^{p-1} - \int_x^1 F^{p-1}$ is constant. It follows that $(F')^{p-1}$ is continuously differentiable, and differentiating we obtain

$$F^{p-1} + \lambda((F')^{p-1})' = 0. \tag{3.3}$$

Thus, integrating by parts in (3.2) we obtain $(F'(1))^{p-1}\varphi(1) = 0$, so that $F'(1) = f(1) = 0$. Multiplying (3.3) by F' and integrating we obtain

$$F^p + \lambda(p-1)(F')^p = C, \tag{3.4}$$

where $C > 0$ is a constant and we should note that

$$(p-1)p^{-1}\frac{d}{dx}\Big[((F')^{p-1})^{p/(p-1)}\Big] = ((F')^{p-1})'F'.$$

Here, after multiplying F by an appropriate constant, we may assume that $C = 1$ in (3.4). Although (3.4) is then satisfied in any interval where $F = 1$, this will not satisfy (3.2). In any point where $F \neq 1$ we then obtain

$$\frac{F'}{(1-F^p)^{1/p}} = a,$$

where $a = (\lambda(p-1))^{-1/p}$. Integrating again we obtain $F(x) = \sin_p(ax)$, since $F(0) = 0$. Thus $f(x) = F'(x) = a\cos_p(ax)$. From $f(1) = 0$ it follows that a is a zero of $\cos_p$, and since we have $f \geq 0$, a is the first positive zero of $\cos_p$. Thus $a = \pi_p/2$. We also have $\lambda = a^{-p}/(p-1)$ so that $\|T\|_p = (p-1)^{-1/p}/a = 2(p-1)^{-1/p}/\pi_p$. Now, if q is the conjugate exponent to p we have $p(p-1)^{-1/p} = p^{1/q}q^{1/p}$ and we are done, in view of the formula (2.1). $\square$

4. Generalizations

One may of course also consider the operator

$$T : L^p(0,1) \ni f \mapsto Tf(x) = \int_0^x f \in L^q(0,1),$$

where now p and q are unrelated exponents in $[1,\infty]$. Even in this case it is possible to calculate the norm $\|T\|_{p,q}$. We introduce the constant $\pi_{p,q} = 2\int_0^1 \frac{dt}{(1-t^q)^{1/p}}$, which is finite unless $p = 1$, $q < \infty$, and then the function $\sin_{\mathrm{p,q}}$, first on the interval $[0, \pi_{p,q}/2)$ as the inverse of the strictly increasing function

$$[0,1) \ni x \mapsto \int_0^x \frac{dt}{(1-t^q)^{1/p}},$$

and then suitably extended to an odd function, which is $2\pi_{p,q}$-periodic and even around $\pi_{p,q}/2$ if $\pi_{p,q}$ is finite. We next define $\cos_{\mathrm{p,q}} x = \frac{d}{dx}\sin_{\mathrm{p,q}} x$ and may the easily deduce that

$$|\cos_{\mathrm{p,q}} x|^p + |\sin_{\mathrm{p,q}} x|^q = 1,$$

except if $p = \infty$, $q < \infty$. Drabek and Manásevich [1] also introduced $\sin_{\mathrm{p,q}}$ and $\cos_{\mathrm{p,q}}$ functions. They are quite similar to ours but not the same.

Proposition 4.1. *For $p, q \in (1,\infty)$ we have $\pi_{p,q} = \frac{2}{q}B(1/p', 1/q)$, where p' is the dual exponent of p and B is the classical beta function. Furthermore,*

- $\pi_{p,\infty} = 2$, $1 \leq p \leq \infty$,
- $\pi_{\infty,q} = 2$, $1 \leq q \leq \infty$,
- $\pi_{p,1} = 2p'$, $1 \leq p \leq \infty$,
- $\pi_{1,q} = \infty$, $1 \leq q < \infty$.

We may now carry out the analysis of the operator T in much the same way as in Section 3, with the following conclusion.

Theorem 4.2. *For $p, q \in (1, \infty)$ we have*
$$\|T\|_{p,q} = (p' + q)^{1-\frac{1}{p'}-\frac{1}{q}} (p')^{1/q} q^{1/p'} / B(1/p', 1/q),$$
where p' is the dual exponent of p and B is the classical beta function. Extremals are all non-zero multiples of $\cos_{\mathrm{p,q}}(\pi_{p,q}x/2)$. Furthermore,

- $\|T\|_{p,\infty} = 1$, $1 \le p \le \infty$. *Extremals are all constants $\neq 0$. In the case $p = 1$ any non-zero multiple of a non-zero positive measure is an extremal.*
- $\|T\|_{\infty,q} = (1+q)^{-1/q}$, $1 \le q < \infty$. *Extremals are all constants $\neq 0$.*
- $\|T\|_{p,1} = (1+p')^{-1/p'}$, $1 < p \le \infty$. *Extremals are all non-zero multiples of $(1-x)^{1/(p-1)}$.*
- $\|T\|_{1,q} = 1$, $1 \le q < \infty$. *Extremals are all non-zero multiples of the Dirac measure at the origin.*

It will be seen that as a function of $(1/p, 1/q)$ the norm $\|T\|_{p,q}$ is continuous in the unit square $0 \le 1/p \le 1$, $0 \le 1/q \le 1$. Furthermore, it is clear that $\|T\|_{p,q} = \|T\|_{q',p'}$ for all $p, q \in [1, \infty]$, a fact that could presumably also be proved by duality.

References

[1] P. Drabek and R. Manásevich, *On the solutionto some p-Laplacian nonhmogeneous eigenvalue problems in closed form*, Preprint.

[2] D. E. Edmunds and Jan Lang, *Behaviour of the approximate number of a Sobolev embedding in the one-dimensional case*, Preprint.

[3] W. D. Evans, D. J. Harris and Y. Saitō, *On the approximation numbers of Sobolev embeddings on irregular domains and trees*, Preprint.

[4] W. D. Evans and D. J. Harris, *Sobolev embeddings for generalized ridged domains*, Proc. London Math. Soc., 54 (1987), 141–175.

[5] W. D. Evans and D. J. Harris, *Fractals, trees and the Neumann Laplacian*, Math. Ann., 206 (1993), 493–527.

[6] P. Lindqvist, *Some remarkable sine and cosine functions*, Ricerche di Matematica, XLIV, fasc. 2, (1995), 269-290.

Christer Bennewitz
Department of Mathematics, Lund University,
Box 118, 221 00 Sweden.
E-mail address: `christer.bennewitz@math.lu.se`

Yoshimi Saitō
Dept. of Mathematics, University of Alabama at Birmingham,
Birmingham, AL 35294, USA.
E-mail address: `saito@math.uab.edu`