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Generalized quasilinearization method for a second order three point boundary-value problem with nonlinear boundary conditions *

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Abstract

The generalized quasilinearization technique is applied to obtain a monotone sequence of iterates converging uniformly and quadratically to a solution of three point boundary value problem for second order differential equations with nonlinear boundary conditions. Also, we improve the convergence of the sequence of iterates by establishing a convergence of order k.

1 Introduction

The method of quasilinearization pioneered by Bellman and Kalaba [1] and generalized by Lakshmikantham [8, 9] has been applied to a variety of problems [2, 10, 11, 12, 13, 16].

Multipoint boundary value problems for second order differential equations have also been receiving considerable attention recently. Kiguradze and Lomtatidze [7] and Lomtatidze [14, 15] have studied closely related problems. Gupta et.al. [4, 5, 6] have studied problems related to three point boundary value problems. More recently, Paul Eloe and Yang Gao [3] discussed the method of quasilinearization for a three point boundary value problem. In this paper, we develop the method of generalized quasilinearization for a three point boundary value problem involving nonlinear boundary conditions and obtain a monotone sequence of approximate solutions converging uniformly and quadratically to a solution of the problem. Also, we have discussed the convergence of order k.

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Basic Results

Consider the three point boundary value problem with nonlinear boundary conditions

$$\begin{aligned} x'' &= f(t, x(t)), \quad t \in [0,1] = J \\ x(0) &= a, \quad x(1) = g(x(\frac{1}{2})), \end{aligned} \tag{1.1}$$

where $f \in C[J \times \mathbb{R}, \mathbb{R}]$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous. Let $G(t,s)$ denote the Green's function for the conjugate or Dirichlet boundary value problem and is given by

$$G(t,s) = \begin{cases} t(s-1), & 0 \le t < s \le 1 \\ s(t-1), & 0 \le s < t \le 1. \end{cases}$$

We note that $G(t,s) < 0$ on $(0,1) \times (0,1)$. If $x(t)$ is the solution of (1.1) and (1.2), then

$$x(t) = a(1-t) + g(x(\frac{1}{2}))t + \int_0^1 G(t,s) f(s, x(s)) ds. \tag{1.2}$$

Let $\alpha, \beta \in C^2[0,1]$. We say that α is a lower solution of the BVP (1.1), if

$$\begin{aligned} \alpha'' &\ge f(t, \alpha), \quad t \in [0,1] \\ \alpha(0) &\le a, \quad \alpha(1) \le g(\alpha(\frac{1}{2})), \end{aligned}$$

and β be an upper solution of the BVP (1.1), if

$$\begin{aligned} \beta'' &\le f(t, \beta), \quad t \in [0,1] \\ \beta(0) &\ge a, \quad \beta(1) \ge g(\beta(\frac{1}{2})). \end{aligned}$$

Now, we state the following theorems without proof [3].

Theorem 1.1 *Assume that f is continuous with $f_x > 0$ on $[0,1] \times \mathbb{R}$ and g is continuous with $0 \le g' < 1$ on $\mathbb{R}$. Let β and α be the upper and lower solutions of the BVP (1.1) respectively. Then $\alpha(t) \le \beta(t)$, $t \in [0,1]$.*

Theorem 1.2 (Method of upper and lower solutions) *Assume that f is continuous on $[0,1] \times \mathbb{R}$ and g is continuous on $\mathbb{R}$ satisfying $0 \le g' < 1$. Further, we assume that there exists an upper solution β and a lower solution α of the BVP (1.1) such that $\alpha(t) \le \beta(t)$, $t \in [0,1]$. Then there exists a solution x of the BVP (1.1) such that*

$$\alpha(t) \le x \le \beta(t), \quad t \in [0,1].$$

2 Main Result

Theorem 2.1 (Generalized quasilinearization method)

(A_1) *f, f_x are continuous on $[0,1]\times\mathbb{R}$ and f_{xx} exists on $[0,1]\times\mathbb{R}$. Further, $f_x > 0$ and $f_{xx}+\phi_{xx}\le 0$, where ϕ, ϕ_x are continuous on $[0,1]\times\mathbb{R}$ and $\phi_{xx}\le 0$.*

(A_2) *g, g' are continuous on $\mathbb{R}$ and g'' exists and $0\le g' < 1$, $g''(x)\ge 0$, $x\in\mathbb{R}$.*

(A_3) *α and β are lower and upper solutions of the BVP (1.1) respectively.*

Then there exists a monotone sequence $\{w_n\}$ of solutions converging quadratically to the unique solution x of the BVP (1.1).

Proof. Define $F:[0,1]\times\mathbb{R}\to\mathbb{R}$ as

$$F(t,x) = f(t,x)+\phi(t,x).$$

Then, in view of (A_1), we note that F, F_x are continuous on $[0,1]\times\mathbb{R}$, and F_{xx} exists such that

$$F_{xx}(t,x)\le 0. \tag{2.1}$$

Using the mean value theorem and the assumptions (A_1) and (A_2), we obtain

$$f(t,x)\le F(t,y)+F_x(t,y)(x-y)-\phi(t,x), \tag{2.2}$$
$$g(x)\ge g(y)+g'(y)(x-y), \tag{2.3}$$

where $x,y\in\mathbb{R}$ such that $x\ge y$ and $t\in[0,1]$. Here, we remark that (2.2) and (2.3) are also valid independent of the requirement $x\ge y$. Define the functions $\overset{*}{F}(t,x,y)$ and $h(x,y)$ as

$$\overset{*}{F}(t,x,y) = F(t,y)+F_x(t,y)(x-y)-\phi(t,x),$$
$$h(x,y) = g(y)+g'(y)(x-y).$$

We observe that

$$f(t,x) = \min_y F^*(t,x,y). \tag{2.4}$$

Further

$$\begin{aligned}\overset{*}{F}_x(t,x,y) &= F_x(t,y)-\phi_x(t,x)\ge F_x(t,x)-\phi_x(t,x)\\ &= f_x(t,x) > 0,\end{aligned} \tag{2.5}$$

implies that $\overset{*}{F}(t,x,y)$ is increasing in x for each fixed $(t,y)\in[0,1]\times\mathbb{R}$. Similarly

$$g(x) = \max_y h(x,y), \tag{2.6}$$

$$0\le h'(x,y) < 1. \tag{2.7}$$

Now, set $\alpha = w_0$, and consider the three point BVP

$$\begin{aligned} x'' &= \overset{*}{F}(t, x(t), w_0(t)), \quad t \in [0,1] = J \\ x(0) &= a, \quad x(1) = h(x(\tfrac{1}{2}), w_0(\tfrac{1}{2})). \end{aligned} \tag{2.8}$$

Using (A_3) together with (2.4) and (2.6), we have

$$\begin{gathered} w_0'' \geq f(t, w_0) = \overset{*}{F}(t, w_0, w_0), \quad t \in [0,1] \\ w_0(0) \leq a, \quad w_0(1) \leq g(w_0(\tfrac{1}{2})) = h(w_0(\tfrac{1}{2}), w_0(\tfrac{1}{2})), \end{gathered}$$

and

$$\begin{gathered} \beta'' \leq f(t, \beta) \leq \overset{*}{F}(t, \beta, w_0), \quad t \in [0,1] \\ \beta(0) \geq a, \quad \beta(1) \geq g(\beta(\tfrac{1}{2})) \geq h(\beta(\tfrac{1}{2}), w_0(\tfrac{1}{2})), \end{gathered}$$

which imply that w_0 and β are lower and upper solutions of the BVP (2.8) respectively. In view of (2.5) (2.7) and the fact that w_0 and β are lower and upper solutions of the BVP (2.8) respectively, it follows by Theorems 1.1 and 1.2 that there exists a unique solution w_1 of the BVP (2.8) such that

$$w_0(t) \leq w_1(t) \leq \beta(t), \quad t \in [0,1].$$

Now, consider the BVP

$$\begin{aligned} x'' &= \overset{*}{F}(t, x(t), w_1(t)), \quad t \in [0,1] = J \\ x(0) &= a, \quad x(1) = h(x(\tfrac{1}{2}), w_1(\tfrac{1}{2})). \end{aligned} \tag{2.9}$$

Again, using (A_3), (2.4) and (2.6), we find that w_1 and β are lower and upper solutions of (2.9) respectively, that is,

$$\begin{gathered} w_1'' = \overset{*}{F}(t, w_1, w_0) \geq \overset{*}{F}(t, w_1, w_1), \quad t \in [0,1] \\ w_1(0) = a, \quad w_1(1) = h(w_1(\tfrac{1}{2}), w_0(\tfrac{1}{2})) \leq h(w_1(\tfrac{1}{2}), w_1(\tfrac{1}{2})), \end{gathered}$$

and

$$\begin{gathered} \beta'' \leq f(t, \beta) \leq \overset{*}{F}(t, \beta, w_1), \quad t \in [0,1] \\ \beta(0) \geq a, \quad \beta(1) \geq g(\beta(\tfrac{1}{2})) \geq h(\beta(\tfrac{1}{2}), w_1(\tfrac{1}{2})). \end{gathered}$$

Hence, by Theorems 1.1 and 1.2, there exists a unique solution w_2 of (2.9) such that

$$w_1(t) \leq w_2(t) \leq \beta(t), \quad t \in [0,1].$$

Continuing this process successively, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(t) \le w_1(t) \le \cdots \le w_n(t) \le \beta(t), \quad t \in [0,1],$$

where each element w_n of the sequence is a solution of the BVP

$$\begin{gathered} x'' = \overset{*}{F}(t, x(t), w_{n-1}(t)), \quad t \in [0,1] = J \\ x(0) = a, \quad x(1) = h(x(\frac{1}{2}), w_{n-1}(\frac{1}{2})), \end{gathered}$$

and

$$w_n(t) = a(1-t) + h(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2}))t + \int_0^1 G(t,s) \overset{*}{F}(s, w_n, w_{n-1})ds. \tag{2.10}$$

Employing the fact that $[0,1]$ is compact and the monotone convergence is pointwise, it follows that the convergence of the sequence is uniform. If $x(t)$ is the limit point of the sequence, then passing onto the limit $n \to \infty$, (2.10) gives

$$\begin{aligned} x(t) &= a(1-t) + h(x(\frac{1}{2}), x(\frac{1}{2}))t + \int_0^1 G(t,s) \overset{*}{F}(s, x(s), x(s))ds \\ &= a(1-t) + g(x(\frac{1}{2}))t + \int_0^1 G(t,s) f(s, x(s))ds. \end{aligned}$$

Thus, $x(t)$ is the solution of the BVP (1.1). Now, we show that the convergence of the sequence is quadratic. For that, set

$$e_n(t) = x(t) - w_n(t), \quad t \in [0,1].$$

Observe that

$$\begin{gathered} e_n(t) \ge 0, \quad e_n(0) = 0, \\ e_n(1) = g(x(\frac{1}{2})) - h(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2})). \end{gathered}$$

Using the mean value theorem repeatedly, (A_1) and the nonincreasing property of F_x, we have

$$\begin{aligned} e''_{n+1}(t) &= x''(t) - w''_{n+1}(t) \\ &= f(t,x) - [F(t,w_n) + F_x(t,w_n)(w_{n+1} - w_n) - \phi(t, w_{n+1})] \\ &= F_x(t,c_1)(x - w_n) - F_x(t,w_n)(x - w_n) + F_x(t,w_n)(x - w_{n+1}) \\ &\quad -\phi_x(t,c_2)(x - w_{n+1}) \\ &= (F_{xx}(t,c_3)(c_1 - w_n)(x - w_n) + (F_x(t,w_n) - \phi_x(t,c_2))(x - w_{n+1}) \\ &\ge F_{xx}(t,c_3)(x - w_n)^2 + (F_x(t,c_2) - \phi_x(t,c_2))(x - w_{n+1}) \\ &= F_{xx}(t,c_3)(e_n)^2 + f_x(t,c_2)e_{n+1} \\ &\ge F_{xx}(t,c_2)(e_n)^2 \ge -M \parallel e_n \parallel^2, \end{aligned}$$

where M is a bound on $F_{xx}(t,x)$ for $t \in [0,1]$, $w_n < c_3 < c_1 < x(t)$, $w_{n+1} < c_2 < x(t)$, and $||\cdot||$ denotes the supremum norm on $C[0,1]$. Thus, we have

$$\begin{aligned}
e_{n+1}(t) &= [g(x(\frac{1}{2})) - h(w_{n+1}(\frac{1}{2}), w_n(\frac{1}{2}))]t + \int_0^1 G(t,s)e''_{n+1}(s)ds \\
&\leq [g(x(\frac{1}{2})) - g(w_n(\frac{1}{2})) - g'(w_n(\frac{1}{2}))(w_{n+1}(\frac{1}{2}) - w_n(\frac{1}{2}))]t \\
&\quad + \int_0^1 G(t,s)M\|e_n\|^2 ds \\
&\leq [g'(c_o)(x(\frac{1}{2}) - w_n(\frac{1}{2})) - g'(w_n(\frac{1}{2}))(w_{n+1}(\frac{1}{2}) - w_n(\frac{1}{2}))]t \\
&\quad + M\|e_n\|^2 \int_0^1 |G(t,s)|\, ds \\
&= [g''(c_1)(c_o - w_n(\frac{1}{2}))(x(\frac{1}{2}) - w_n(\frac{1}{2})) + g'(w_n(\frac{1}{2}))e_{n+1}]t \\
&\quad + M_1\|e_n\|^2 \\
&\leq [g''(c_1)e_n^2(t) + g'(w_n(\frac{1}{2}))e_{n+1}]t + M_1 \parallel e_n \parallel^2,
\end{aligned}$$

where $w_n(\frac{1}{2}) < c_1 < c_o < x(\frac{1}{2})$. Taking the maximum over the interval $[0,1]$, we get

$$\|e_{n+1}\| \leq M_2\|e_n\|^2 + \lambda\|e_{n+1}\| + M_1\|e_n\|^2.$$

Solving algebraically, we get

$$\|e_{n+1}\| \leq \frac{M_3}{1-\lambda}\|e_n\|^2,$$

where, $|g'| \leq \lambda < 1$, M_1 provides a bound on $M\int_0^1 \mid G(t,s) \mid ds$, M_2 provides a bound for $|g''|$ on $[w_n(\frac{1}{2}), x(\frac{1}{2})]$ and $M_3 = M_1 + M_2$. This establishes the quadratic convergence.

3 Rapid Convergence

Theorem 3.1 *Assume that*

(B_1) $\frac{\partial^i}{\partial x^i}f(t,x)$ $(i = 0,1,2,\dots k)$ *are continuous on* $[0,1]\times\mathbb{R}$ *satisfying*

$$\frac{\partial^i}{\partial x^i}f(t,x) \geq 0, \quad (i = 0,1,2,\dots k-1)$$
$$\frac{\partial^k}{\partial x^k}(f(t,x) + \phi(t,x)) \leq 0,$$

where $\frac{\partial^i}{\partial x^i}\phi(t,x)$ $(i = 0,1,2,\dots k)$ *are continuous and* $\frac{\partial^k}{\partial x^k}\phi(t,x) < 0$ *for some function* $\phi(t,x)$.

(B_2) *$\alpha, \beta \in C^2[J, \mathbb{R}]$ are lower and upper solutions of the BVP (1.1).*

(B_3) *$\frac{d^i}{dx^i} g(x)$ ($i = 0, 1, 2, \dots k$) are continuous on $\mathbb{R}$ satisfying*

$$0 \le \frac{d^i}{dx^i} g(x) < \frac{M}{(\beta - \alpha)^{i-1}},$$

with $0 < M < \frac{1}{3}$ and $\frac{d^k}{dx^k} g(x) \ge 0$.

Then there exists a monotone sequence of solutions $\{w_n\}$ that converge to the unique solution, x, of the BVP (1.1) with the order of convergence $k \ge 2$.

Proof. Define $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ as

$$F(t, x) = f(t, x) + \phi(t, x).$$

Using (B_1), (B_3) and the generalized mean value theorem, we obtain

$$f(t, x) \le \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} F(t, y) \frac{(x - y)^i}{i!} - \phi(t, x),$$

$$g(x) \ge \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x - y)^i}{i!}.$$

Define

$$\overset{**}{F}(t, x, y) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} F(t, y) \frac{(x - y)^i}{i!} - \phi(t, x), \tag{3.1}$$

and

$$\overset{*}{h}(x, y) = \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x - y)^i}{i!}. \tag{3.2}$$

Observe that $\overset{**}{F}(t, x, y)$ and $\overset{*}{h}(x, y)$ are continuous and further

$$f(t, x) = \min_y \overset{**}{F}(t, x, y), \tag{3.3}$$

$$g(x) = \max_y \overset{*}{h}(x, y). \tag{3.4}$$

Using generalized mean value theorem, (3.1) can be written as

$$\overset{**}{F}(t, x, y) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, y) \frac{(x - y)^i}{i!} - \frac{\partial^k}{\partial x^k} \phi(t, \xi) \frac{(x - y)^k}{k!}. \tag{3.5}$$

Differentiating (3.5) and using (B_1), we get

$$\overset{**}{F}_x(t, x, y) > \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, y) \frac{(x - y)^{i-1}}{(i-1)!} \ge 0, \tag{3.6}$$

which implies that $\overset{**}{F}(t,x,y)$ is increasing in x for each $(t,y) \in [0,1] \times \mathbb{R}$. Similarly, differentiation of (3.2), in view of (B_3), yields

$$\overset{*}{h}{}'(x,y) = \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^{i-1}}{(i-1)!}.$$

Clearly $\overset{*}{h}{}'(x,y) \geq 0$ and

$$\begin{aligned}
\overset{*}{h}{}'(x,y) &= \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^{i-1}}{(i-1)!} \leq \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(\beta-\alpha)^{i-1}}{(i-1)!} \\
&\leq \sum_{i=0}^{k-1} \frac{M}{(i-1)!} < M(1 + \sum_{i=0}^{k-2} \frac{1}{2^{k-1}}) = M(3 - \frac{1}{2^{k-3}}) \\
&< 3M < 1.
\end{aligned}$$

Now, set $\alpha = w_0$, and consider the linear BVP

$$\begin{gathered}
x'' = \overset{**}{F}(t, x(t), w_0(t)), \quad t \in [0,1] = J \\
x(0) = a, \quad x(1) = \overset{*}{h}(x(\tfrac{1}{2}), w_0(\tfrac{1}{2})).
\end{gathered} \tag{3.7}$$

Using (B_2), (3.3) and (3.4), we find that

$$\begin{gathered}
w_0'' \geq f(t, w_0) = \overset{**}{F}(t, w_0, w_0), \quad t \in [0,1] \\
w_0(0) \leq a, \quad w_0(1) \leq g(w_0(\tfrac{1}{2})) = \overset{*}{h}(w_0(\tfrac{1}{2}), w_0(\tfrac{1}{2})),
\end{gathered}$$

and

$$\begin{gathered}
\beta'' \leq f(t, \beta) \leq \overset{**}{F}(t, \beta, w_0), \quad t \in [0,1] \\
\beta(0) \geq a, \quad \beta(1) \geq g(\beta(\tfrac{1}{2})) \geq \overset{*}{h}(\beta(\tfrac{1}{2}), w_0(\tfrac{1}{2})),
\end{gathered}$$

imply that w_0 and β are lower and upper solutions of the BVP (3.7) respectively. It follows by Theorems 1.1 and 1.2 that there exists a unique solution w_1 of the BVP (3.7) such that

$$w_0(t) \leq w_1(t) \leq \beta(t), \quad t \in [0,1].$$

Continuing this process successively, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(t) \leq w_1(t) \leq w_2(t) \leq \cdots \leq w_n(t) \leq \beta(t), \quad t \in [0,1],$$

where each element w_n of the sequence is a solution of the BVP

$$\begin{gathered}
x'' = \overset{**}{F}(t, x(t), w_{n-1}(t)), \quad t \in [0,1] = J \\
x(0) = a, \quad x(1) = \overset{*}{h}(x(\tfrac{1}{2}), w_{n-1}(\tfrac{1}{2})),
\end{gathered}$$

and is given by

$$w_n(t) = a(1-t) + \overset{*}{h}\,(w_n(\tfrac{1}{2}), w_{n-1}(\tfrac{1}{2}))t + \int_0^1 G(t,s)\overset{**}{F}(s, w_n, w_{n-1})ds. \quad (3.8)$$

Again, using the standard arguments employed in the last section, it follows that

$$\begin{aligned} x(t) &= a(1-t) + \overset{*}{h}\,(x(\tfrac{1}{2}), x(\tfrac{1}{2}))t + \int_0^1 G(t,s)\overset{**}{F}(s, x(s), x(s))ds \\ &= a(1-t) + g(x(\tfrac{1}{2}))t + \int_0^1 G(t,s) f(s, x(s))ds. \end{aligned}$$

Hence $x(t)$ is the solution of the BVP (1.1). Now, we show that the convergence of the sequence of iterates is of order $k \geq 2$. For that, we set

$$e_n(t) = x(t) - w_n(t), \quad a_n(t) = w_{n+1}(t) - w_n(t), \quad t \in [0,1].$$

Note that $e_n(t) \geq 0$, $a_n(t) \geq 0$, $e_n(t) - a_n(t) = e_{n+1}(t)$, and

$$e_n(0) = 0, \quad e_n(1) = g(x(\tfrac{1}{2})) - h(w_n(\tfrac{1}{2}), w_{n-1}(\tfrac{1}{2})).$$

Also, $e_n(t) \geq a_n(t)$ and hence by induction $e_n^k(t) \geq a_n^k(t)$. Using the generalized mean value theorem, we have

$$\begin{aligned} & e''_{n+1}(t) \\ &= x'' - w''_{n+1} \\ &= \Big[\sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n)\frac{(x-w_n)^i}{i!} + \frac{\partial^k}{\partial x^k} f(t,\xi)\frac{(x-w_n)^k}{k!}\Big] \\ &\quad -\Big[\sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n)\frac{(w_{n+1}-w_n)^i}{i!} - \frac{\partial^k}{\partial x^k} \phi(t,\xi)\frac{(w_{n+1}-w_n)^k}{k!}\Big] \\ &= \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n)\frac{(e_n^i - a_n^i)}{i!} + \frac{\partial^k}{\partial x^k} f(t,\xi)\frac{(e_n)^k}{k!} + \frac{\partial^k}{\partial x^k} \phi(t,\xi)\frac{(a_n)^k}{k!} \\ &\geq \Big(\sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n)\frac{1}{i!}\sum_{i=0}^{k-1} e_n^j a_n^{i-1-j}\Big)e_{n+1} + \Big(\frac{\partial^k}{\partial x^k} f(t,\xi) + \frac{\partial^k}{\partial x^k}\phi(t,\xi)\Big)\frac{(e_n)^k}{k!} \\ &\geq \frac{\partial^k}{\partial x^k} F(t,\xi)\frac{(e_n)^k}{k!} \geq -M\|e_n\|^k, \end{aligned} \quad (3.9)$$

where M is a bound on $\frac{1}{k!}\frac{\partial^k}{\partial x^k}F(t,\xi)$ for $t \in [0,1]$. Thus, in view of (3.9), we have

$$
\begin{aligned}
&e_{n+1}(t)\\
&= \big(g(x(\frac{1}{2})) - h(w_{n+1}(\frac{1}{2}), w_n(\frac{1}{2}))\big)t + \int_0^1 G(t,s)e''_{n+1}(t)ds\\
&\le \big(g(x(\frac{1}{2})) - h(w_{n+1}(\frac{1}{2}), w_n(\frac{1}{2}))\big)t + M\|e_n\|^k \int_0^1 |G(t,s)|\,ds\\
&= \Big[\sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(w_n(\frac{1}{2}))\frac{(x(\frac{1}{2}) - w_n(\frac{1}{2}))^i}{i!} + \frac{d^k}{dx^k} g(\xi(\frac{1}{2}))\frac{(x(\frac{1}{2}) - w_n(\frac{1}{2}))^k}{k!}\\
&\quad -\sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(w_n(\frac{1}{2}))\frac{(w_{n+1}(\frac{1}{2}) - w_n(\frac{1}{2}))^i}{i!}\Big]t + M_1\|e_n\|^k\\
&= \Big[\sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(w_n(\frac{1}{2}))\frac{(e_n^i(\frac{1}{2}) - a_n(\frac{1}{2}))^i}{i!} + \frac{d^k}{dx^k} g(\xi(\frac{1}{2}))\frac{(e_n(\frac{1}{2}))^k}{k!}\Big]t + M_1\|e_n\|^k\\
&= \Big[\sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(w_n(\frac{1}{2}))\frac{1}{i!}\sum_{i=0}^{k-1} e_n^j(\frac{1}{2})a_n^{i-1-j}(\frac{1}{2}))e_{n+1}(\frac{1}{2})\\
&\quad + \frac{d^k}{dx^k} g(\xi(\frac{1}{2}))\frac{(e_n(\frac{1}{2}))^k}{k!}\Big]t + M_1\|e_n\|^k \qquad (3.10)\\
&\le \Big[\sum_{i=1}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}}\frac{1}{i!}\sum_{j=0}^{i-1} e_n^{i-1-j}(\frac{1}{2})a_n^j(\frac{1}{2})\Big]e_{n+1}(\frac{1}{2}) + M_2\|e_n\|^k + M_1\|e_n\|^k.
\end{aligned}
$$

Letting

$$P_n(t) = \sum_{i=1}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}}\frac{1}{i!}\sum_{j=0}^{i-1} e_n^{i-1-j}(\frac{1}{2})a_n^j(\frac{1}{2}),$$

we observe that

$$\lim_{n\to\infty} P_n(t) = \lim_{n\to\infty}\sum_{i=1}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}}\frac{1}{i!}\sum_{j=0}^{i-1} e_n^{i-1-j}(\frac{1}{2})a_n^j(\frac{1}{2}) = M < \frac{1}{3}.$$

Therefore, we can choose $\lambda < 1/3$ and $n_0 \in N$ such that for $n \ge n_0$, we have $P_n(t) < \lambda$ and consequently (3.10) becomes

$$\|e_{n+1}\| < \lambda\|e_{n+1}\| + M_3\|e_n\|^k.$$

Solving algebraically, we obtain

$$\|e_{n+1}\| \le \frac{M_3}{1-\lambda}\|e_n\|^k,$$

where $M_3 = M_1 + M_2$, M_1 provides bound for $M\int_0^1 |G(t,s)|ds$, and M_2 provides bound for

$$\frac{d^k}{dx^k} g(\xi(\frac{1}{2}))\frac{1}{k!}.$$

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