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Existence of global solutions to a quasilinear wave equation with general nonlinear damping *

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Abstract

In this paper we prove the existence of a global solution and study its decay for the solutions to a quasilinear wave equation with a general nonlinear dissipative term by constructing a stable set in $H^2 \cap H_0^1$.

1 Introduction

We consider the problem

$$\begin{gathered} u'' - \Phi(\|\nabla_x u\|_2^2)\Delta_x u + g(u') + f(u) = 0 \quad \text{in } \Omega \times [0, +\infty[, \\ u = 0 \quad \text{on } \Gamma \times [0, +\infty[, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on } \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma$, $\Phi(s)$ is a C^1- class function on $[0, +\infty[$ satisfying $\Phi(s) \geq m_0 > 0$ for $s \geq 0$ with m_0 constant.

For the problem (1.1), when $\Phi(s) \equiv 1$ and $g(x) = \delta x$ ($\delta > 0$), Ikehata and Suzuki [11] investigated the dynamics, they have shown that for sufficiently small initial data (u_0, u_1), the trajectory $(u(t), u'(t))$ tends to $(0,0)$ in $H_0^1(\Omega) \times L^2(\Omega)$ as $t \to +\infty$. When $g(x) = \delta|x|^{m-1}x$ ($m \geq 1$) and $f(y) = -\beta|y|^{p-1}y$ ($\beta > 0$, $p \geq 1$), Georgiev and Todorova [6] have shown that if the damping term dominates over the source, then a global solution exists for any initial data. Quite recently, Ikehata [8] proved that a global solution exists with no relation between p and m, and Todorova [27] proved that the energy decay rate is $E(t) \leq (1+t)^{-2/(m-1)}$ for $t \geq 0$, she used a general method on the energy decay introduced by Nakao [19]. Unfortunately this method does not seem to be applicable to the case of more general functions g.

Aassila [2] proved the existence of a global decaying H^2 solution when $g(x)$ has not necessarily a polynomial growth near zero and a source term of the form $\beta|y|^{p-1}y$, but with small parameter β. The decay rate of the global solution

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depends on the polynomial growth near zero of $g(x)$ as it was proved in [3, 27, 15].

When $\Phi(s)$ is not a constant function, $g(x) \equiv 0$ and $f(y) \equiv 0$ the equation is often called the wave equation of Kirchhoff type. This equation was introduced to study the nonlinear vibrations of an elastic strings by Kirchhoff [14], and the existence of global solutions was investigated by many authors [25, 13, 7]. In [9], the authors discussed the existence of a global decaying solution in the case $\Phi(s) = m_0 + s^{\frac{(\gamma+2)}{2}}$, $\gamma \geq 0$, $g(v) = |v|^r v$, $0 \leq r \leq 2/(n-2)$ ($0 \leq r \leq \infty$ if $n = 1, 2$), $f(u) = -|u|^\alpha u$, $0 < \alpha \leq 4/(n-2)$ ($0 < \alpha < \infty$ if $n = 1, 2$) by use of a stable set method due to Sattinger [26]. But, then, the method in [9] cannot be applied to the case $\alpha > 4/(n-2)$, which is caused by the construction of stable set in H_0^1. Quite recently, in [10] (see also [1]) Ikehata, Matsuyama and Nakao have constructed a stable set in $H_0^1 \cap H^2$ to obtain a global decaying solution to the initial boundary value problem for quasilinear visco-elastic wave equations.

Our purpose in this paper is to give a global solvability in the class $H_0^1 \cap H^2$ and energy decay estimates of the solutions to problem (1.1) for a general nonlinear damping g. We use some new techniques introduced in [2] to derive a decay rate of the solution. So we use the argument combining the method in [2] with the concept of stable set in $H_0^1 \cap H^2$. We also use some ideas from [17] introduced in the study of the decay rates of solutions to the wave equation $u_{tt} - \Delta u + g(u_t) = 0$ in $\Omega \times \mathbb{R}^+$.

We conclude this section by stating our plan and giving some notations. In section 2 we shall prepare some lemmas needed for our arguments. Section 3 is devoted to the proof of the global existence and decay estimates to the problem (1.1). Section 4 is devoted to the proof of the global existence and decay estimates to the problem (1.1) in the case $\alpha = 0$, i.e., $f(u) = -u$. In this case the smallness of $|\Omega|$ (the volume of Ω) will play an essential role in our argument. In the last section we shall treat the case $\Phi \equiv 1$, we prove only the global decaying H_0^1 solution, but we obtain more results than the case when $\Phi \not\equiv 1$. The condition that β (k_1 in our paper) is small is removed here, also we extend some results obtained by Ono [24] and Martinez [17].

Throughout this paper the functions considered are all real valued. We omit the space variable x of $u(t,x)$, $u_t(t,x)$ and simply denote $u(t,x)$, $u_t(t,x)$ by $u(t)$, $u'(t)$, respectively, when no confusion arises. Let l be a number with $2 \leq l \leq \infty$. We denote by $\| \,.\, \|_l$ the L^l norm over Ω. In particular, L^2 norm is denoted $\| \,.\, \|_2$. $(\,.\,)$ denotes the usual L^2 inner product. We use familiar function spaces H_0^1, H^2.

2 Preliminaries

Let us state the precise hypotheses on Φ, g and f.

(H1) Φ is a C^1-class function on $\mathbb{R}^+$ and satisfies

$$\Phi(s) \geq m_0 \quad \text{and} \quad |\Phi'(s)| \leq m_1 s^{\gamma/2} \quad \text{for } 0 \leq s < \infty \tag{2.1}$$

for some constants $m_0 > 0$, $m_1 \geq 0$, and $\gamma \geq 0$.

(H2) g is a C^1 odd increasing function and

$$c_2|x| \leq |g(x)| \leq c_3|x|^q \quad \text{if} \quad |x| \geq 1 \quad \text{with} \quad 1 \leq q \leq \frac{N+2}{(N-2)^+},$$

where c_1, c_2 and c_3 are positive constants.

(H3) $f(.)$ is a $C^1(\mathbb{R})$ satisfying

$$|f(u)| \leq k_2|u|^{\alpha+1} \quad \text{and} \quad |f'(u)| \leq k_2|u|^{\alpha} \quad \text{for all} u \in \mathbb{R} \tag{2.2}$$

with some constant $k_2 > 0$, and

$$0 < \alpha < \frac{2}{(N-4)^+}, \tag{2.3}$$

where $(N-4)^+ = \max\{N-4, 0\}$. A typical example of these functions is $f(u) = -|u|^\alpha u$.

We first state three well known lemmas, and then we prove two other lemmas that will be needed later.

Lemma 2.1 (Sobolev-Poincaré inequality) *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$, then there is a constant $c_* = c(\Omega, q)$ such that*

$$\|u\|_q \leq c_*\|\nabla u\|_2 \quad \text{for} \quad u \in H_0^1(\Omega).$$

Lemma 2.2 (Gagliardo-Nirenberg) *Let $1 \leq r < q \leq +\infty$ and $p \leq q$. Then, the inequality*

$$\|u\|_{W^{m,q}} \leq C\|u\|_{W^{m,p}}^{\theta}\|u\|_r^{1-\theta} \quad \text{for} \quad u \in W^{m,p} \cap L^r$$

holds with some $C > 0$ and

$$\theta = \big(\frac{k}{n} + \frac{1}{r} - \frac{1}{q}\big)\big(\frac{m}{n} + \frac{1}{r} - \frac{1}{p}\big)^{-1}$$

provided that $0 < \theta \leq 1$ (we assume $0 < \theta < 1$ if $q = +\infty$).

Lemma 2.3 ([15]) *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $p \geq 1$ and $A > 0$ such that*

$$\int_S^{+\infty} E^{\frac{p+1}{2}}(t)\,dt \leq AE(S), \quad 0 \leq S < +\infty.$$

Then

$$E(t) \leq cE(0)(1+t)^{\frac{-2}{p-1}} \quad \forall t \geq 0, \quad \text{if} \quad p > 1\,,$$
$$E(t) \leq cE(0)e^{-\omega t} \quad \forall t \geq 0, \quad \text{if} \quad p = 1,$$

where c and ω are positive constants independent of the initial energy $E(0)$.

Lemma 2.4 ([17]) *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^2 function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as } t \to +\infty.$$

Assume that there exist $p \geq 1$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{\frac{p+1}{2}}(t)\phi'(t)\,dt \leq AE(S). \quad 0 \leq S < +\infty,$$

Then

$$E(t) \leq cE(0)(1+\phi(t))^{-2/(p-1)} \quad \forall t \geq 0, \quad \text{if} \quad p > 1,$$
$$E(t) \leq cE(0)e^{-\omega\phi(t)} \quad \forall t \geq 0, \quad \text{if} \quad p = 1,$$

where c and ω are positive constants independent of the initial energy $E(0)$.

Proof Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we remark that ϕ^{-1} has a sense by the hypotheses assumed on ϕ). f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\begin{aligned}
\int_{\phi(S)}^{\phi(T)} f(x)^{\frac{p+1}{2}}\,dx &= \int_{\phi(S)}^{\phi(T)} E(\phi^{-1}(x))^{(p+1)/2}\,dx \\
&= \int_S^T E(t)^{\frac{p+1}{2}}\phi'(t)\,dt \\
&\leq AE(S) = Af(\phi(S)) \quad 0 \leq S < T < +\infty.
\end{aligned}$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\int_s^{+\infty} f(x)^{\frac{p+1}{2}}\,dx \leq Af(s) \quad 0 \leq s < +\infty.$$

Thanks to Lemma 2.3, we deduce the desired results. □

Remark 2.5 The use of a 'weight function' $\phi(t)$ to establish the decay rate of solutions to hyperbolic PDE was successfully done by Aassila [3], Martinez [17], and Mochizuki and Motai [18].

Lemma 2.6 ([17]) *There exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}$ increasing and such that ϕ is concave and $\phi(t) \to +\infty$ as $t \to +\infty$, $\phi'(t) \to 0$ as $t \to +\infty$, and*

$$\int_1^{+\infty} \phi'(t)\left(g^{-1}(\phi'(t))\right)^2\,dt < +\infty.$$

Proof. If such a function exists, we can assume that $\phi(1) = 1$. Setting $s := \phi(t)$ we obtain

$$\begin{aligned}\int_1^{+\infty} \phi'(t)\left(g^{-1}(\phi'(t))\right) dt &= \int_1^{+\infty} \left(g^{-1}(\phi'(\phi^{-1}(s)))\right)^2 ds \\ &= \int_1^{+\infty} g^{-1}\Big(\frac{1}{(\phi^{-1})'(s)}\Big)^2 ds.\end{aligned}$$

Let us define

$$\psi(t) := 1 + \int_1^t \frac{1}{g(1/s)}\, ds, \quad t \geq 1.$$

Note that ψ is increasing, of class C^2, and

$$\psi'(t) = \frac{1}{g(1/t)} \to +\infty \quad \text{as} \quad t \to +\infty.$$

Hence $\psi(t) \to +\infty$ as $t \to +\infty$ and

$$\int_1^{+\infty} \Big(g^{-1}\big(\frac{1}{\psi'(s)}\big)\Big)^2 ds = \int_1^{+\infty} \frac{1}{s^2}\, ds < +\infty.$$

Furthermore ψ' is non-decreasing, and hence ψ is convex. Let us verify that ψ^{-1} is concave: from $\psi(\psi^{-1}(s)) = s$ we have

$$(\psi^{-1})''(s) = -\frac{\psi''(\psi^{-1}(s))\left((\psi^{-1})'(s)\right)^2}{\psi'\left(\psi^{-1}(s)\right)} = -\frac{\psi''\left(\psi^{-1}(s)\right)}{\left(\psi'(\psi^{-1}(s))\right)^3} \leq 0.$$

In conclusion, if we set $\phi(t) := \psi^{-1}(t)$ for all $t \geq 1$, we see that ϕ verify all the hypotheses of lemma 2.6. $\square$

First, we shall construct a stable set in $H_0^1 \cap H^2$. For this, we define the following functionals:

$$\begin{gathered} J(u) \equiv \frac{1}{2}\int_0^{\|\nabla_x u\|_2^2} \Phi(s)\, ds + \int_\Omega \int_0^u f(\eta)\, d\eta\, dx \quad \text{for } u \in H_0^1, \\ \tilde{J}(u) \equiv \Phi(\|\nabla_x u\|_2^2)\|\nabla_x u\|_2^2 + \int_\Omega f(u)u\, dx \quad \text{for } u \in H_0^1 \\ E(u,v) \equiv \frac{1}{2}\|v\|_2^2 + J(u) \quad \text{for } (u,v) \in H_0^1 \times L^2. \end{gathered}$$

Lemma 2.7 *Let $0 < \alpha < 4/(N-4)^+$. Then, for any $K > 0$, there exists a number $\varepsilon_0 \equiv \varepsilon_0(K) > 0$ such that if $\|\Delta_x u\| \leq K$ and $\|\nabla_x u\| \leq \varepsilon_0$, we have*

$$J(u) \geq \frac{m_0}{4}\|\nabla_x u\|_2^2 \quad \textit{and} \quad \tilde{J}(u) \geq \frac{m_0}{2}\|\nabla_x u\|_2^2. \tag{2.4}$$

Proof: We see from the Gagliardo-Nirenberg inequality that

$$\|u\|_{\alpha+2}^{\alpha+2} \le C\|u\|_{\frac{2N}{(N-2)}}^{(\alpha+2)(1-\theta)}\|\Delta_x u\|_2^{(\alpha+2)\theta} \le C\|\nabla_x u\|_2^{(\alpha+2)(1-\theta)}\|\Delta_x u\|_2^{(\alpha+2)\theta} \quad (2.5)$$

with

$$\theta = \Big(\frac{N-2}{2N} - \frac{1}{\alpha+2}\Big)^+\Big(\frac{2}{N} + \frac{N-2}{2N} - \frac{1}{2}\Big)^{-1} = \frac{((N-2)\alpha-4)^+}{2(\alpha+2)} \ (\le 1). \quad (2.6)$$

Here, we note that

$$(\alpha+2)(1-\theta)-2 = \begin{cases} \alpha & \text{if } 0<\alpha\le\frac{4}{N-2} \\ & (0<\alpha<\infty \text{ for } N=1,2), \\ \frac{(4-N)\alpha+4}{2} & \text{if } \frac{4}{N-2}<\alpha<\frac{4}{N-4} \\ & (\frac{4}{N-2}<\alpha<\infty \text{ for } N=3,4). \end{cases} \quad (2.7)$$

Hence, if $\|\Delta_x u\|_2 \le K$, we have

$$\begin{aligned} J(u) &\ge \frac{m_0}{2}\|\nabla_x u\|_2^2 - \frac{k_2}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2} \\ &\ge \frac{m_0}{2}\|\nabla_x u\|_2^2 - C\|\nabla_x u\|_2^{(\alpha+2)(1-\theta)}\|\Delta_x u\|_2^{(\alpha+2)\theta} \\ &\ge \big\{\frac{m_0}{2} - CK^{(\alpha+2)\theta}\|\nabla_x u\|_2^{(\alpha+2)(1-\theta)-2}\big\}\|\nabla_x u\|_2^2. \end{aligned} \quad (2.8)$$

Using (2.7), we define $\varepsilon_0 \equiv \varepsilon_0(K)$ by

$$CK^{(\alpha+2)\theta}\varepsilon_0^{(\alpha+2)(1-\theta)-2} = \frac{m_0}{4}.$$

Thus, we obtain

$$J(u) \ge \frac{m_0}{4}\|\nabla_x u\|_2^2 \quad (2.9)$$

if $\|\nabla_x u\|_2 \le \varepsilon_0$. It is clear that (2.9) is valid for $\tilde{J}(u)$. $\square$

Let us define a stable in $H_0^1 \cap H^2$ as follows: For some $K>0$,

$$\begin{aligned} \mathcal{W}_K \equiv \Big\{&(u,v) \in (H_0^1\cap H^2)\times H_0^1 : \|\Delta_x u\|_2 < K, \\ &\|\nabla_x v\|_2 < K \text{ and } \sqrt{4m_0^{-1}E(u,v)} < \varepsilon_0\Big\} \end{aligned}$$

Remark 2.8 If $f(u)u \ge 0$, we do not need $\varepsilon_0(K)$, and $\mathcal{W}_K$ is replaced by

$$\tilde{\mathcal{W}}_K \equiv \{(u,v) \in (H_0^1\cap H^2)\times H_0^1 : \|\Delta_x u\|_2 < K, \|\nabla_x v\|_2 < K\}$$

3 Global Existence and Asymptotic Behavior

A simple computation shows that

$$E'(t) = -\int_\Omega u' g(u')\,dx \le 0,$$

hence the energy is non-increasing and in particular $E(t) \le E(0)$ for all $t \ge 0$.

Lemma 3.1 *Let $u(t)$ be a strong solution satisfying $(u(t), u'(t)) \in \mathcal{W}_K$ on $[0, T[$ for some $K > 0$. Then we have*

$$E(t) \le cE(0)\Big(G^{-1}(\frac{1}{t})\Big)^2 \quad \text{on } [0, T[,$$

where c is a positive constant independent of the initial energy $E(0)$ and $G(x) = xg(x)$. Furthermore, if $x \mapsto g(x)/x$ is non-decreasing on $[0, \eta]$ for some $\eta > 0$, then

$$E(t) \le cE(0)\Big(g^{-1}(\frac{1}{t})\Big)^2 \quad \text{on } [0, T[,$$

where c is a positive constant independent of the initial energy $E(0)$.

Proof of lemma 3.1 For the rest of this article, we denote by c various positive constants which may be different at different occurences. We multiply the first equation of (1.1) by $E\phi' u$, where ϕ is a function satisfying all the hypotheses of lemma 2.6, we obtain

$$\begin{aligned}
0 =& \int_S^T E\phi' \int_\Omega u(u'' - \Phi(\|\nabla_x u\|_2^2)\Delta u + g(u') + f(u))\,dx\,dt \\
=& \Big[E\phi' \int_\Omega uu'\,dx\Big]_S^T - \int_S^T (E'\phi' + E\phi'') \int_\Omega uu'\,dx\,dt - 2\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt \\
&+ \int_S^T E\phi' \int_\Omega \big(u'^2 + \Phi(\|\nabla_x u\|_2^2)|\nabla u|^2 + f(u)u\big)\,dx\,dt \\
&+ \int_S^T E\phi' \int_\Omega ug(u')\,dx\,dt\,.
\end{aligned}$$

Under the assumption $(u(t), u'(t)) \in \mathcal{W}_K$, the functionals $J(u(t))$ and $\tilde{J}(u(t))$ are both equivalent to $\|\nabla_x u(t)\|_2^2$, by lemma 2.7. So we deduce that

$$\begin{aligned}
\int_S^T E^2\phi'\,dt \le& -\Big[E\phi' \int_\Omega uu'\,dx\Big]_S^T + \int_S^T (E'\phi' + E\phi'') \int_\Omega uu'\,dx\,dt \\
&+ 2\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt - \int_S^T E\phi' \int_\Omega ug(u')\,dx\,dt \\
\le& -\Big[E\phi' \int_\Omega uu'\,dx\Big]_S^T + \int_S^T (E'\phi' + E\phi'') \int_\Omega uu'\,dx\,dt
\end{aligned}$$

$$+ 2\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt + c(\varepsilon)\int_S^T E\phi' \int_{|u'|\le 1} g(u')^2\,dx\,dt$$
$$+ \varepsilon\int_S^T E\phi' \int_{|u'|\le 1} u^2\,dx\,dt - \int_S^T E\phi' \int_{|u'|>1} ug(u')\,dx\,dt$$

for all $\varepsilon > 0$. Choosing ε small enough, we deduce that

$$\begin{aligned}
&\int_S^T E^2\phi'\,dt \\
&\le -\Big[E\phi' \int_\Omega uu'\,dx\Big]_S^T + \int_S^T (E'\phi' + E\phi'')\int_\Omega uu'\,dx\,dt + c\int_S^T E\phi'\int_\Omega u'^2\,dx\,dt \\
&\le cE(S) - \int_S^T E\phi' \int_{|u'|>1} ug(u')\,dx\,dt + c\int_S^T E\phi'\int_\Omega u'^2\,dx\,dt.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\int_S^T E\phi' \int_{|u'|>1} ug(u')\,dx\,dt \\
&\le \int_S^T E\phi'\Big(\int_\Omega |u|^q\,dx\Big)^{1/(q+1)}\Big(\int_{|u'|>1} |g(u')|^{\frac{(q+1)}{q}}\,dx\Big)^{q/(q+1)} \\
&\le c\int_S^T E^{3/2}\phi'\Big(\int_{|u'|>1} u'g(u')\,dx\Big)^{q/(q+1)} \le \int_S^T \phi' E^{3/2}(-E')^{\frac{q}{(q+1)}} \\
&\le c\int_S^T \phi'(E^{\frac{3}{2}-\frac{q}{q+1}})\Big((-E')^{\frac{q}{(q+1)}}E^{\frac{q}{q+1}}\Big) \\
&\le c(\varepsilon')\int_S^T \phi'(-E'E)\,dt + \varepsilon'\int_S^T \phi' E^{(q+1)(\frac{3}{2}-\frac{q}{(q+1)})}\,dt \\
&\le c(\varepsilon')E(S)^2 + \varepsilon' E(0)^{(q-1)/2}\int_S^T \phi' E^2\,dt
\end{aligned}$$

for every $\varepsilon' > 0$. Choosing ε' small enough, we obtain

$$\int_S^T E^2\phi'\,dt \le cE(S) + c\int_S^T E\phi'\int_\Omega u'^2\,dx\,dt$$

We want to majorize the last term of the above inequality, we have

$$\begin{aligned}
\int_S^T E\phi'\int_\Omega u'^2\,dx\,dt = &\int_S^T E\phi'\int_{\Omega_1} u'^2\,dx\,dt + \int_S^T E\phi'\int_{\Omega_2} u'^2\,dx\,dt \\
&+ \int_S^T E\phi'\int_{\Omega_3} u'^2\,dx\,dt,
\end{aligned}$$

where, for $t \ge 1$,

$$\Omega_1 := \{x \in \Omega : |u'| \le h(t)\}, \quad \Omega_2 := \{x \in \Omega : h(t) < |u'| \le h(1)\},$$

$$\Omega_3 := \{x \in \Omega : |u'| > h(1)\},$$

and $h(t) := g^{-1}(\phi'(t))$, which is a positive non-increasing function and satisfies $h(t) \to 0$ as $t \to +\infty$. Because

$$\begin{aligned}\int_S^T E\phi' \int_{\Omega_1} u'^2 \,dx\,dt \le & c\int_S^T E(t)\phi'(t)\Big(\int_{\Omega_1} h(t)^2\,ds\Big)\,dt \\ \le & cE(S)\int_S^T \phi'(t)(g^{-1}(\phi'(t)))^2\,dt \le cE(S),\end{aligned}$$

we have the following: Since g is non-decreasing, for $x \in \Omega_2$ we have $\phi'(t) = g(h(t)) \le |g(u')|$, and hence

$$\begin{aligned}\int_S^T E\phi' \int_{\Omega_2} u'^2\,dx\,dt \le & \int_S^T E\int_{\Omega_2} |g(u')|u'^2\,dx\,dt \\ \le & h(1)\int_S^T E\int_{\Omega_2} u'g(u')\,dx\,dt \le \frac{h(1)}{2}E(S)^2\,;\end{aligned}$$

and since $g(x) \ge cx$ for $x \ge h(1)$, we have

$$\begin{aligned}\int_S^T E\phi' \int_{\Omega_3} u'^2\,dx\,dt \le & c\int_S^T E\phi'\int_{\Omega} u'g(u')\,dx\,dt \\ \le & c\int_S^T E(-E')\,dx\,dt \le cE(S)^2.\end{aligned}$$

Then we deduce that

$$\int_S^T E^2\phi'\,dt \le cE(S),$$

and thanks to Lemma 2.6, we obtain

$$E(t) \le \frac{c\;E(0)}{\phi(t)}, \qquad \forall t \ge 1.$$

Let s_0 be such that $g(1/s_0) \le 1$, since g is non-decreasing we have

$$\psi(s) \le 1 + (s-1)\frac{1}{g(1/s)} \le s\frac{1}{g(1/s)} = \frac{1}{G(1/s)} \quad \forall s \ge s_0,$$

hence $s \le \phi\big(1/G(1/s)\big)$ and

$$\frac{1}{\phi(t)} \le \frac{1}{s} \quad \text{with} \quad t := \frac{1}{G(1/s)}.$$

Thus

$$\frac{1}{\phi(t)} \le G^{-1}(1/t).$$

Now define $H(x) := g(x)/x$, H is non-decreasing, $H(0) = 0$, then we use the function $h(t) := H^{-1}(\phi'(t))$. On Ω_2 it holds that

$$\phi'(t)(u')^2 \le |H(u')|(u')^2 = u'g(u').$$

The same calculations as above with

$$\phi^{-1}(t) = 1 + \int_1^t \frac{1}{H(1/s)}\, ds$$

yield $E(t) \le c\, E(0)\big(g^{-1}(1/t)\big)^2$. □

Lemma 3.2 *Let $u(t)$ be a strong solution satisfying $(u(t), u'(t)) \in \mathcal{W}_K$ on $[0, T[$ for some $K > 0$. Assume that*

$$\int_0^{+\infty} \big(g^{-1}(1/t)\big)^{\min\{\gamma+1, \alpha(1-\theta_0)\}}\, dt < +\infty.$$

Then we have

$$\|\nabla u'(t)\|_2^2 + \|\Delta u(t)\|_2^2 \le Q_1^2(I_0, I_1, K),$$

with $\lim_{I_0\to 0} Q_1^2(I_0, I_1, K) = I_1^2$ and where we set

$$I_0^2 = E(0) = \frac{1}{2}\|u_1\|_2^2 + J(u_0), \quad I_1^2 = \|\nabla u_1\|_2^2 + \Phi(\|\nabla_x u_0\|_2^2)\|\Delta u_0\|_2^2$$

Proof Multiplying the first equation of (1.1) by $-\Delta u'(t)$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big[\|\nabla u'(t)\|_2^2 + \Phi(\|\nabla_x u\|_2^2)\|\Delta u(t)\|_2^2\Big] + \Big(\nabla g(u'(t)), \nabla u'(t)\Big)\\ = -\int_\Omega f'(u)\nabla u.\nabla u'(t)\, dx\Big) + \Phi'(\|\nabla_x u\|_2^2)(\nabla u'(t), \nabla u(t))\|\Delta_x u\|_2^2.\end{aligned}$$

We set

$$E_1(t) \equiv \|\nabla_x u'\|_2^2 + \Phi(\|\nabla_x u\|_2^2)\|\Delta_x u\|_2^2$$

Using the assumptions on Φ, g et f, we have

$$\begin{aligned}\frac{d}{dt}E_1(t) \le& C\|\nabla_x u\|_2^{\gamma+1}\|\nabla_x u'\|_2\|\Delta_x u\|_2^2 + 2k_2\int_\Omega |u|^\alpha |\nabla_x u||\nabla_x u'|\, dx\\ \le& C\Big\{E(t)^{(\gamma+1)/2}K^3 + \Big(\int_\Omega |u|^{2\alpha}|\nabla_x u|^2\, dx\Big)^{1/2}\Big(\int_\Omega |\nabla_x u'|\, dx\Big)^{1/2}\Big\}\end{aligned} \tag{3.1}$$

Here, we see from the Gagliardo-Nirenberg inequality that

$$
\begin{aligned}
\Big(\int_\Omega |u|^{2\alpha}|\nabla_x u|^2\,dx\Big)^{1/2} &\le \|u(t)\|_{N\alpha}^{\alpha}\|\nabla_x u(t)\|_{\frac{2N}{(N-2)}} \\
&\le C\|u(t)\|_{\frac{2N}{(N-2)}}^{\alpha(1-\theta_0)}\|\Delta_x u(t)\|_2^{\alpha\theta_0}\|\Delta_x u(t)\|_2 \\
&\le C\|\nabla_x u(t)\|_2^{\alpha(1-\theta_0)}\|\Delta_x u(t)\|_2^{\alpha\theta_0+1} \\
&\le CE(t)^{\alpha(1-\theta_0)}K^{\alpha\theta_0+1}
\end{aligned}
\tag{3.2}
$$

with

$$
\theta_0 = \Big(\frac{N-2}{2} - \frac{1}{\alpha}\Big)^+ = \frac{((N-2)\alpha-2)^+}{2\alpha} \quad (\le 1).
$$

Hence, it follows from (3.1) and (3.2) that

$$
\frac{d}{dt}E_1(t) \le C\Big\{E(t)^{\frac{(\gamma+1)}{2}}K^3 + E(t)^{\frac{\alpha(1-\theta_0)}{2}}K^{\alpha\theta_0+2}\Big\}. \tag{3.3}
$$

we conclude that

$$
\begin{aligned}
&\|\Delta_x u(t)\|_2^2 + \|\nabla_x u'(t)\|_2^2 \\
&\le \frac{1}{\min\{1,m_0\}}\Big\{I_1^2 + CK^3\int_0^\infty E(t)^{(\gamma+1)/2}\,dt + CK^{\alpha\theta_0+2}\int_0^\infty E(t)^{\alpha(1-\theta_0)/2}\,dt\Big\}
\end{aligned}
$$

Example Let $g(x)$ be the inverse function of

$$
M(0)=0 \quad\text{and}\quad M(x) = \frac{x^\sigma}{(\log(-\log x))^\beta} \quad \text{for } 0<x<x_0, \quad (\beta,\sigma>0).
$$

The function g exists and satisfies the hypothesis (H2), when $0<\sigma<1$ (see Appendix). So

$$
g^{-1}(1/t) = \frac{1}{t^\sigma(\log(\log t))^\beta}
$$

the conditions in the Lemma 3.2 give

$$
\int_{t_0}^\infty \frac{1}{t^{\sigma(\gamma+1)}(\log(\log t))^{\beta(\gamma+1)}}\,dt < \infty, \tag{3.4}
$$

$$
\int_{t_0}^\infty \frac{1}{t^{\sigma\alpha(1-\theta_0)}(\log(\log t))^{\beta\alpha(1-\theta_0)}}\,dt < \infty, \tag{3.5}
$$

which are similar to Bertrand integrals. So, when $\gamma=0$, the first integral (3.4) is not finite, we obtain the following cases: if $\sigma(\gamma+1)>1$, the integral is finite, if $\sigma(\gamma+1)=1$, and $\beta(\gamma+1)>1$, also the integral is finite. The second integral (3.5), is fine under the following conditions:

$$
\sigma^{-1} < \alpha \le \frac{2}{(N-2)^+} \quad \text{for } N=1,2,3
$$

or

$$\alpha > \frac{2(1-\sigma)}{\sigma} \quad \text{for } N = 3$$

or

$$\alpha = \sigma^{-1} \quad \text{and} \quad \beta^{-1} < \alpha \leq \frac{2}{(N-2)^+} \quad \text{for } N = 1, 2, 3$$

or

$$\alpha = \frac{2(1-\sigma)}{\sigma} \quad \text{and} \quad \alpha > \frac{2(1-\beta)}{\beta} \quad \text{for } N = 3.$$

Hence, we must restrict ourselves to $1 \leq N \leq 3$.

Remark 3.3 When $\Phi \equiv 1$, $g(x) = |x|^{p-1}x$, $p \geq 1$, and $f(y) = -|y|^{q-1}y$ with $q \geq 1$, we obtain

$$\begin{aligned} E(t) &\leq cE(0)e^{-\omega t} \quad \forall t \geq 0,\ c > 0,\ \omega > 0, \quad \text{if } p = 1 \\ E(t) &\leq \frac{cE(0)}{(1+t)^{2/(p-1)}} \quad \forall t \geq 0,\ c > 0 \quad \text{if } p > 1. \end{aligned}$$

Also

$$Q_1^2(I_0, I_1, K) = I_1^2 + cK^2 I_0^{q-1}, \quad Q_2^2(I_0, I_1, K) = I_1^2 + cK^{(q-1)\theta+2} I_0^{(q-1)(1-\theta)}.$$

When $g(x) = |x|^{p-1}x$, $p \geq 1$, $f(y) \equiv 0$, and $p < \gamma + 2$, we obtain the same above results (see [1]).

Theorem 3.4 *Under the hypotheses of lemma 3.1 and 3.2 there exists an open set $S_1 \subset (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, which includes $(0,0)$ such that if $(u_0, u_1) \in S_1$, the problem (1.1) has a unique global solution u satisfying*

$$u \in L^\infty([0,\infty[; H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{1,\infty}([0,\infty[; H_0^1(\Omega)) \cap W^{2,\infty}([0,\infty[; L^2(\Omega)),$$

furthermore we have the decay estimate

$$E(t) \leq c\ E(0) \left(g^{-1}(1/t)\right)^2 \quad \forall t > 0. \tag{3.6}$$

Proof of theorem 3.4

Let $K > 0$. Put

$$S_K \equiv \{(u_0, u_1) \in \mathcal{W}_K \,|\, Q_1(I_0, I_1, K) < K\}, \quad S_1 \equiv \bigcup_{K>0} S_K\,.$$

Note that if E_0, E_1 are sufficiently small, then S_K is not empty.

If $(u_0, u_1) \in S_K$ for some $K > 0$, then an assumed strong solution $u(t)$ exist globally and satisfies $(u(t), u'(t)) \in \mathcal{W}_K$ for all $t \geq 0$. Let $\{w_j\}_{j=1}^\infty$ be the basis of H_0^1 consisted by the eigenfunction of $-\Delta$ with Dirichlet condition. We define the approximation solution u_m (m=1, 2, ...) in the form

$$u_m = \sum_{j=1}^{m} g_{jm} w_j$$

where $g_{jm}(t)$ are determined by

$$\begin{aligned}(u_m''(t), w_j) + \Phi(\|\nabla_x u_m(t)\|_2^2)(\nabla_x u_m(t), \nabla_x w_m)\\ +(g(u_m'(t)), w_j) + (f(u_m(t)), w_j) = 0\end{aligned} \tag{3.7}$$

for $j \in \{1, 2, \dots, m\}$ with the initial data where $u_m(0)$ and $u_m'(0)$ are determined in such a way that

$$u_m(0) = u_{0m} = \sum_{j=1}^{m} (u_0, w_j) w_j \to u_0 \text{ strongly in } H_0^1 \cap H^2 \text{ as } m \to \infty,$$

$$u_m'(0) = u_{1m} = \sum_{j=1}^{m} (u_1, w_j) w_j \to u_1 \text{ strongly in } H_0^1 \text{ as } m \to \infty.$$

By the theory of ordinary differential equations, (3.7) has a unique solution $u_m(t)$. Suppose that $(u_0, u_1) \in S_K$ for $K > 0$. Then, $(u_m(0), u_m'(0)) \in S_K$ for large m. It is clear that all the estimates obtained above are valid for $u_m(t)$ and, in particular, $u_m(t)$ exists on $[0, \infty[$. Thus, we conclude that $(u_m(t), u_m'(t)) \in \mathcal{W}_K$ for all $t \geq 0$ and all the estimates are valid for $u_m(t)$ for all $t \geq 0$.

Thus, $u_m(t)$ converges along a subsequence to $u(t)$ in the following way:

$$\begin{aligned} u_m(.) &\to u(.) \text{ weakly * in } L^\infty_{\rm loc}([0, \infty); H_0^1 \cap H^2),\\ u_m'(.) &\to u_t(.) \text{ weakly * in } L^\infty_{\rm loc}([0, \infty); H_0^1),\\ u_m(.) &\to u_{tt}(.) \text{ weakly * in } L^\infty_{\rm loc}([0, \infty); L^2),\end{aligned}$$

and hence,

$$\begin{aligned}\Phi(\|\nabla_x u_m(.)\|_2^2)\nabla_x u_m(.) &\to \Phi(\|\nabla_x u(.)\|_2^2)\nabla_x u(.) \text{ weakly * in } L^\infty_{\rm loc}([0, \infty); H_0^1),\\ g(u_m(.)) &\to g(u(.)) \text{ weakly * in } L^\infty_{\rm loc}([0, \infty); H_0^1),\end{aligned}$$

Therefore, the limit function $u(t)$ is a desired solution belonging to

$$L^\infty([0, \infty[; H_0^1 \cap H^2) \cap W^{1,\infty}([0, \infty[; H_0^1) \cap W^{2,\infty}([0, \infty[; L^2)$$

The uniqueness can be proved by use of the monotonicity of g, $n\alpha < 2n/(n-4)$ and $\sup_{0 \leq t \leq T}(\|u(t)\|_{H^2} + \|u'(t)\|_{H_0^1}) \leq C(T) < \infty$ (see [2]). □

4 The case $\alpha = 0$

In this section we shall discuss the existence of a global solution to the problem (1.1) with $f(u) \equiv -u$. More precisely, we impose an assumption on $f(u)$ instead of (H3) as follows:

(H.3)' $f(.)$ satisfies $f(u) = -k_3 u$ for $u \in \mathbb{R}$ with $k_3 C(\Omega) < m_0$, $k_3 > 0$, where $C(\Omega)$ is a quantity such that

$$C(\Omega) = \sup_{u \in H_0^1 \setminus \{0\}} \frac{\|u\|_2}{\|\nabla_x u\|_2} \tag{4.1}$$

Remark 4.1 The condition $k_3 C(\Omega) < m_0$ implies that $|\Omega|$ is small in some sense. On the other hand, if $f(u) = u$, we need not take $C(\Omega)$ into consideration.

Our result reads as follows.

Theorem 4.2 *Under the hypotheses of Lemma 3.1 (we replace* (H.3) *by* (H.3)'*) and 3.2 , there exists an open unbounded set S_2 in $(H^2 \cap H_0^1) \times H_0^1$, which includes $(0,0)$, such that if $(u_0, u_1) \in S_2$, the problem* (1.1) *has a unique solution u in the sense of theorem 3.4 which satisfies the decay estimate (3.6).*

Proof of theorem 4.2

This proof is also given in parallel way to the proof of theorem 3.4 so se just sketch the outline.

First, let $k_3 C(\Omega) < m_0$. Then, by (4.1,)

$$J(u) = \frac{1}{2} \int_0^{\|\nabla_x u\|_2^2} \Phi(s)\, ds - \frac{k_3}{2} \|u\|_2^2 \geq \frac{1}{2} (m_0 - k_3 C(\Omega)) \|\nabla_x u\|_2^2. \tag{4.2}$$

We may assume $\tilde{J}(u)$ also satisfies (4.2). If $u(t)$ is a strong solution satisfying $\|\nabla_x u(t)\|_2 < K$ and $\|\nabla_x u'(t)\|_2 < K$ on $[0, T[$ for some $K > 0$, then as in lemma 3.1, we derive the decay estimate

$$E(t) \leq c \left(g^{-1}(1/t) \right)^2 . \tag{4.3}$$

Multiplying the equation by $-\Delta_x u'$, we see

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_1(t) &\leq |\Phi'(\|\nabla_x u(t)\|_2^2)| (\nabla_x u(t), \nabla_x u'(t)) \|\Delta_x u(t)\|_2^2 + \frac{k_3}{2} \frac{d}{dt} \|\nabla_x u(t)\|_2^2 \\ &\leq C K^3 E(t)^{(\gamma+1)/2} + \frac{k_3}{2} \frac{d}{dt} \|\nabla_x u(t)\|_2^2 \end{aligned} \tag{4.4}$$

where we set

$$E_1(t) = \Phi(\|\nabla_x u(t)\|_2^2) \|\Delta_x u(t)\|_2^2 + \|\nabla_x u'(t)\|_2^2.$$

we integrate (4.4) to obtain

$$\begin{aligned} &\|\Delta_x u(t)\|_2^2 + \|\nabla_x u(t)\|_2^2 \\ &\leq \frac{1}{\min\{1, m_0\}} \Big\{ I_1^2 + C K^3 \int_0^\infty E(t)^{\frac{(\gamma+1)}{2}} \, dt + k_3 \|\nabla_x u(t)\|_2^2 - k_3 \|\nabla_x u_0\|_2^2 \Big\} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\min\{1, m_0\}} \Big\{ I_1^2 + C I_0^2 + C\ I_0^{\gamma+1}\ K^3 \int_0^\infty \big(g^{-1}(1/t)\big)^{(\gamma+1)}\ dt \Big\} \\
&\equiv Q_2^2(I_0, I_1, K) \quad \text{on } [0, T[.
\end{aligned}
$$

Defining

$$
S_K \equiv \{(u_0, u_1) \in H_0^1 \cap H^2 : Q_2(I_0, I_1, K) < K\}, \quad S_2 \equiv \bigcup_{K>0} S_K
$$

we conclude that if $(u_0, u_1) \in S_2$, the corresponding solution to the problem (1.1) exists globally and satisfies the estimate

$$
E(t) \leq c\,\big(g^{-1}(1/t)\big)^2 \quad \text{and} \quad \|\Delta_x u(t)\|_2^2 + \|\nabla_x u'(t)\|_2^2 < K^2,
$$

for all $t > 0$. The proof of theorem 4.2 is complete.

5 The case $\Phi \equiv 1$

Usually, we study global existence for Kirchhoff equation (i.e. when $\Phi \not\equiv 1$) in the class $H^2 \cap H_0^1$ (also when $f \equiv g \equiv 0$). Thus the condition in Lemma 3.2 excludes some functions g which verify (H2), for example $g(x) = e^{-1/x}$ or $g(x) = e^{-e^{1/x}}$ or the example above. We consider the case $\Phi \equiv 1$ (or a constant function) and we prove a global H_0^1 solution that decays. Here we do not need the condition of Lemma 3.2 and we will take only $\alpha \leq 4/(n-2)^+$ because we work only in $H_0^1(\Omega)$.

Now, we consider the initial boundary-value problem

$$
\begin{gathered}
u'' - \Delta_x u + g(u') + f(u) = 0 \quad \text{in } \Omega \times [0, +\infty[, \\
u = 0 \quad \text{on } \Gamma \times [0, +\infty[, \\
u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) \quad \text{on } \Omega,
\end{gathered}
\tag{5.1}
$$

First, we shall construct a stable set in H_0^1. For this, we need define the following functionals:

$$
\begin{gathered}
J(u) \equiv \frac{1}{2}\|\nabla_x u\|_2^2 + \int_\Omega \int_0^u f(\eta)\, d\eta\, dx \quad \text{for } u \in H_0^1, \\
\tilde{J}(u) \equiv \|\nabla_x u\|_2^2 + \int_\Omega f(u) u\, dx \quad \text{for } u \in H_0^1, \\
E(u, v) \equiv \frac{1}{2}\|v\|_2^2 + J(u) \quad \text{for } (u, v) \in H_0^1 \times L^2.
\end{gathered}
$$

Then we can define the stable set

$$
\mathcal{W} = \{u \in H_0^1(\Omega) : \|\nabla_x u\|_2^2 - k_1 \|u\|_{\alpha+2}^{\alpha+2} > 0\} \cup \{0\}
$$

Lemma 5.1 **(i)** *If $\alpha < 4/[n-2]^+$, then $\mathcal{W}$ is an open neighborhood of 0 in $H_0^1(\Omega)$.*
(ii) *If $u \in \mathcal{W}$, then*

$$\|\nabla_x u\|_2^2 \leq d_* J(u) \quad \textit{with} \quad d_* = \frac{2(\alpha+2)}{\alpha}. \tag{5.2}$$

Proof. (**i**) From the Sobolev-Poincaré inequality (see lemma 2.1) we have

$$k_1 \|u\|_{\alpha+2}^{\alpha+2} \leq A k_1 \|\nabla_x u\|_2^\alpha \|\nabla_x u\|_2^2 \tag{5.3}$$

where $A = c_*^{\alpha+2}$. Let

$$U(0) \equiv \big\{ u \in H_0^1(\Omega) : \|\nabla_x u\|_2^\alpha < \frac{1}{Ak_1} \big\}.$$

Then, for any $u \in U(0) \backslash \{0\}$, we deduce from (5.3) that

$$k_1 \|u\|_{\alpha+2}^{\alpha+2} < \|\nabla_x u\|_2^2,$$

that is, $K(u) > 0$. This implies $U(0) \subset \mathcal{W}$.
(ii) By the definition of $K(u)$ and $J(u)$ we have the inequality

$$J(u) \geq \frac{1}{2}\|\nabla_x u\|_2^2 - \frac{k_1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2} \geq \frac{\alpha}{2(\alpha+2)}\|\nabla_x u\|_2^2$$

□

Lemma 5.2 *Let $u(t)$ be a strong solution of* (5.1)*. Suppose that*

$$u(t) \in \overline{\mathcal{W}} \quad \textit{and } \tilde{J}(u(t)) \geq \frac{1}{2}\|\nabla_x u(t)\|_2^2 \tag{5.4}$$

for $0 \leq t < T$. Then we have

$$E(t) \leq cE(0) \left(G^{-1}(1/t)\right)^2 \quad \textit{on } [0, T[,$$

where c is a positive constant independent of the initial energy $E(0)$ and $G(x) = xg(x)$. Furthermore, if $x \mapsto g(x)/x$ is non-decreasing on $[0, \eta]$ for some $\eta > 0$, then we have

$$E(t) \leq cE(0) \left(g^{-1}(1/t)\right)^2 \quad \textit{on } [0, T[,$$

where c is a positive constant independent of the initial energy $E(0)$.

Examples

1) If $g(x) = e^{-1/x^p}$ for $0 < x < 1$, $p > 0$, then $E(t) \leq c/(\ln t)^{2/p}$.

2) If $g(x) = e^{-e^{1/x}}$ for $0 < x < 1$, then $E(t) \leq c/(\ln(\ln t))^2$.

Proof of lemma 3.1 The functionals $J(u(t))$ and $\tilde{J}(u(t))$ are both equivalent to $\|\nabla_x u(t)\|_2^2$, indeed we have

$$\int_\Omega f(u)u\,dx \le k_1\|u\|_{\alpha+2}^{\alpha+2} \le \|\nabla_x u(t)\|_2^2$$

So, we have

$$\frac{1}{2}\|\nabla_x u\|_2^2 \le K(u(t)) \le \frac{3}{2}\|\nabla_x u\|_2^2.$$

Also, we have

$$|J(u(t))| \le \frac{1}{2}\|\nabla_x u(t)\|_2^2 + \frac{1}{\alpha+2}\|\nabla_x u\|_2^2 \le \frac{\alpha+4}{2(\alpha+2)}\|\nabla_x u(t)\|_2^2 .$$

Therefore,

$$K(u(t)) \ge \frac{1}{2}\|\nabla_x u\|_2^2 \ge \frac{\alpha+2}{\alpha+4}J(u). \tag{5.5}$$

Now, we can derive the decay estimate (3.6) by similar argument as lemma 3.1.

Theorem 5.3 *Suppose that $\alpha \le 4/(n-2)$ ($\alpha < \infty$ if $n \le 2$), and suppose that initial data $\{u_0, u_1\}$ belongs to $\mathcal{W}$, and its initial energy $E(0)$ is sufficiently small such that*

$$C_4 E(0)^{\alpha/2} < 1, \tag{5.6}$$

where $C_4 = 2k_1 c_*^{\alpha+2} d_*^{\alpha/2}$. Then, Problem (5.1) *has a unique global solution $u \in \mathcal{W}$ satisfying*

$$u \in L^\infty([0,\infty[; H_0^1(\Omega)) \cap W^{1,\infty}([0,\infty[; L^2(\Omega));$$

furthermore, we have the decay estimate

$$E(t) \le c\ E(0)\left(g^{-1}(1/t)\right)^2 \quad \forall t > 0 . \tag{5.7}$$

Proof of Theorem 3.4

Since $u_0 \in \mathcal{W}$ and $\mathcal{W}$ is an open set, putting

$$T_1 = \sup\{t \in [0,+\infty) : u(s) \in \mathcal{W} \text{ for } 0 \le s \le t\},$$

we see that $T_1 > 0$ and $u(t) \in \mathcal{W}$ for $0 \le t < T_1$. If $T_1 < T_{\max} < \infty$, where $T_{\max}$ is the lifespan of the solution, then $u(T_1) \in \partial\mathcal{W}$; that is

$$K(u(T_1)) = 0 \text{ and } u(T_1) \ne 0. \tag{5.8}$$

We see from lemma 2.2 and lemma 5.1 that

$$k_1\|u(t)\|_{\alpha+2}^{\alpha+2} \le \frac{1}{2}B(t)\|\nabla_x u(t)\|_2^2 \tag{5.9}$$

for $0 \leq t \leq T_1$, where we set

$$B(t) = C_4 E(0)^{\alpha/2} \tag{5.10}$$

with $C_4 = 2k_1 c_*^{\alpha+2} d_*^{\alpha/2}$. Next, we put

$$T_2 \equiv \sup\{t \in [0, +\infty) : B(s) < 1 \text{ for } 0 \leq s < t\},$$

and then we see that $T_2 > 0$ and $T_2 = T_1$ because $B(t) < 1$ by (5.6). Then

$$K(u(t)) \geq \|\nabla_x u(t)\|_2^2 - \frac{1}{2} B(t) \|\nabla_x u(t)\|_2^2 \geq \frac{1}{2} \|\nabla_x u(t)\|_2^2 \tag{5.11}$$

for $0 \leq t \leq T_1$. Moreover, (5.8) and (5.11) imply

$$K(u(T_1)) \geq \frac{1}{2} \|\nabla_x u(T_1)\|_2^2 > 0$$

which is a contradiction, and hence, it might be $T_1 = T_{\max}$. Therefore, (5.7) hold true for $0 \leq T \leq T_{\max}$, and such estimate give the desired a priori estimate; that is, the local solution u can be extended globally (i.e., $T_{\max} = \infty$). The proof of theorem 5.3 is now complete. □

Remarks: **a**) By a similar argument as the proof of Theorem 4.2, we can extend Theorem 5.3 to the case $\alpha = 0$.
b) It seems to be interesting to study a global decaying H^2 solution for Kirchhoff equation with nonlinear source and boundary damping terms or with nonlinear boundary damping and source terms, also in the case of polynomial damping term i.e. the following problems

$$\begin{gathered}
u'' - \Phi(\|\nabla_x u\|_2^2) \Delta_x u + f(u) = 0 \quad \text{in } \Omega \times [0, +\infty[, \\
u = 0 \quad \text{on } \Gamma_0 \times [0, +\infty[, \\
\frac{\partial u}{\partial \nu} = -Q(x) g(u') \quad \text{on } \Gamma_1 \times [0, +\infty[, \\
u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) \quad \text{on } \Omega,
\end{gathered}$$

and

$$\begin{gathered}
u'' - \Phi(\|\nabla_x u\|_2^2) \Delta_x u = 0 \quad \text{in } \Omega \times [0, +\infty[, \\
u = 0 \quad \text{on } \Gamma_0 \times [0, +\infty[, \\
\frac{\partial u}{\partial \nu} = -Q(x) g(u') + f(u) \quad \text{on } \Gamma_1 \times [0, +\infty[, \\
u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) \quad \text{on } \Omega,
\end{gathered}$$

We plan to address these questions in a future investigation.

Appendix

Let $g(x)$ be the inverse of the function $M(x)$ defined by

$$M(0) = 0, \quad M(x) = \frac{x^\sigma}{(\log(-\log x))^\beta} \quad \text{for } 0 < x < x_0, \quad (\sigma, \beta > 0).$$

For $x = 1/t (0 < x < x_0)$ we have

$$g^{-1}(1/t) = \frac{1}{t^\sigma (\log(\log t))^\beta} \quad (t \geq t_0).$$

Now, we prove that the function $g(x)$ exists and verifies the hypothesis (H2). Indeed,

$$(M(x))' = \frac{x^\sigma \Big[\sigma(\log(-\log x)) - \frac{\beta}{\log x}\Big]}{(\log(-\log x))^{\beta+1}}, \quad (\sigma, \beta > 0).$$

When x is near 0 ($0 < x < x_0$), it is clear that $(M(x))' \geq 0$, so $M(x)$ is an increasing continuous function. Thus the function g exists. We have also

$$\frac{x}{M(x)} = \frac{(\log(-\log x))^\beta}{x^{\sigma-1}} \to 0$$

as $x \to 0$ if $0 < \sigma < 1$, so $M(x) \to 0$ (as $x \to 0$) not faster than x (near 0). We deduce that $g(x) \to 0$ as $x \to 0$ faster than x i.e. $|g(x)| \leq c|x|$. We obtain hypothesis (H2). Now, $M(x)/x$ is a decreasing function; indeed,

$$\Big(\frac{M(x)}{x}\Big)' = \frac{x^{\sigma-2}\Big[(\sigma-1)(\log(-\log x)) - \frac{\beta}{\log x}\Big]}{(\log(-\log x))^{\beta+1}}.$$

For $x = e^{-n}$, and n big, we see that $(M(x)/x)' \leq 0$. g is a bijective and decreasing function, so for each x and y near 0, such that $x \leq y$, we have $M(x)/x \geq M(y)/y$, also there exist unique x' and y' such that $M(x) = x'$ and $M(y) = y'$ (because M is a bijective function), also $M(x)$ is an increasing function, thus, we have

$$x \leq y \Longleftrightarrow M(x) = x' \leq M(y) = y'$$

Therefore,

$$\begin{aligned} x' \leq y' &\Longleftrightarrow \frac{x'}{g(x')} \geq \frac{y'}{g(y')} \\ &\Longleftrightarrow \frac{g(x')}{x'} \leq \frac{g(y')}{y'} \quad \text{for } 0 < x < x_0. \end{aligned}$$

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