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Resolvent estimates for 2-dimensional perturbations of plane Couette flow *

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Abstract

We present results concerning resolvent estimates for the linear operator associated with the system of differential equations governing 2 dimensional perturbations of plane Couette flow. We prove estimates on the L_2 norm of the resolvent of this operator showing this norm to be proportional to the Reynolds number R for a region of the unstable half plane. For the remaining region, we show that the problem can be reduced to estimating the solution of a homogeneous ordinary differential equation with non-homogeneous boundary conditions. Numerical approximations indicate that the norm of the resolvent is proportional to R in the whole region of interest.

1 Introduction

We consider the initial boundary-value problem

$$\begin{gathered} w_t + (w \cdot \nabla)W + (W \cdot \nabla)w + \nabla p = \frac{1}{R}\Delta w + H \\ \nabla \cdot w = 0 \\ w(x,0,t) = w(x,1,t) = (0,0) \\ w(x,y,t) = w(x+1,y,t) \\ w(x,y,0) = (0,0) \end{gathered} \tag{1.1}$$

where $w : \mathbb{R} \times [0,1] \times [0,\infty) \to \mathbb{R}^2$ is the unknown function $w(x,y,t) = (u(x,y,t), v(x,y,t))$. W is The vector field $W = (y,0)$ and the Reynolds number R is a positive parameter. The forcing $H(x,y,t) = (F(x,y,t), G(x,y,t))$ is a given C^∞ function satisfying satisfying $\int_0^\infty ||H(\cdot,\cdot,t)||^2 dt < \infty$ and $\nabla \cdot H = 0$ for all $(x,y,t) \in \mathbb{R} \times [0,1] \times [0,\infty)$. These equations are the linearization of the equations governing 2 dimensional perturbations $w(x,y,t)$ of $W = (y,0)$,

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known as Couette flow, which is a steady solution of

$$\begin{gathered}
U_t + (U \cdot \nabla)U + \nabla P = \frac{1}{R}\Delta U \\
\nabla \cdot U = 0 \\
U(x,0,t) = (0,0) \\
U(x,1,t) = (1,0) \\
U(x,y,t) = U(x+1,y,t).
\end{gathered} \tag{1.2}$$

The pressure term $p(x,y,t)$ in (1.1) is determined in terms of w by the linear elliptic equation

$$\begin{gathered}
\Delta p = -\nabla \cdot ((w \cdot \nabla)W) - \nabla \cdot ((W \cdot \nabla)w) \\
p_y(x,0,t) = \frac{1}{R} v_{yy}(x,0,t) \\
p_y(x,1,t) = \frac{1}{R} v_{yy}(x,1,t).
\end{gathered} \tag{1.3}$$

We note that p depends linearly on w, and is determined up to a constant. The estimates derived in this paper are independent of p. With p given by the above equation, the solution w of (1.1) remains divergence free for all $t \geq 0$. Therefore we drop the continuity equation and write the problem as

$$\begin{gathered}
w_t = \frac{1}{R}\Delta w - (w \cdot \nabla)W - (W \cdot \nabla)w - \nabla p + H =: \mathcal{L}_R w + H \\
w(x,y,0) = (0,0)
\end{gathered} \tag{1.4}$$

with w satisfying the boundary conditions described in (1.1). Note that $\mathcal{L}_R$ is a linear operator on $L_2(\Omega)$, $\Omega = [0,1] \times [0,1]$. The L_2 inner product and norm over Ω are denoted by

$$\langle w_1, w_2 \rangle = \int_\Omega \overline{w}_1 \cdot w_2 \, dx\, dy\,, \quad \|w\|^2 = \langle w, w \rangle.$$

As motivation, consider a general linear evolution equation

$$\begin{gathered}
w_t = \mathcal{L} w + H \\
w(x,y,0) = 0.
\end{gathered} \tag{1.5}$$

Formally, after Laplace transformation with respect to t, the transformed equation is $s\widetilde{w} = \mathcal{L}\widetilde{w} + \widetilde{H}$ for $\mathrm{Re}(s) \geq 0$, where $\widetilde{w}$ is the Laplace transform of w. The solution of the transformed equation is given by

$$\widetilde{w} = (s\mathcal{I} - \mathcal{L})^{-1}\widetilde{H}. \tag{1.6}$$

where $\mathcal{I}$ is the identity operator, and then we get the solution of (1.5) by applying the inverse Laplace transform to $\widetilde{w}$. Moreover, from (1.6),

$$\|\widetilde{w}(\cdot,s)\|^2 \leq \|(s\mathcal{I} - \mathcal{L})^{-1}\|^2 \|\widetilde{H}(\cdot,s)\|^2. \tag{1.7}$$

We recall the following definition.

Definition. Let $\mathcal{L}$ be a linear operator in a Banach space X and $\mathcal{I}$ the identity operator. The resolvent set $\rho(\mathcal{L}) \subseteq \mathbb{C}$ of $\mathcal{L}$ is the set of all complex numbers s such that the operator $(s\mathcal{I} - \mathcal{L})^{-1}$ exists, is bounded and has a dense domain in X. The linear operator $(s\mathcal{I} - \mathcal{L})^{-1}$, for $s \in \rho(\mathcal{L})$, is said to be the resolvent of $\mathcal{L}$. The set $\sigma(\mathcal{L}) = \mathbb{C} \setminus \rho(\mathcal{L})$ is the spectrum of $\mathcal{L}$.

Therefore, if the complex number s is such that $s \in \rho(\mathcal{L})$, the formal argument above to express the transformed function $\widetilde{w}$ in terms of the transformed forcing $\widetilde{H}$ by (1.6) is valid and we have the estimate (1.7). According to Romanov [4], the unstable half plane $\operatorname{Re} s \geq 0$ is contained in the resolvent set of the linear operator $\mathcal{L}_R$ defined in (1.4) , for all positive Reynolds numbers R, and the eigenvalue of $\mathcal{L}_R$ with largest real part is at a distance from the imaginary axis proportional to $1/R$. Our aim in this paper is to estimate the dependence on R of the L_2 norm of the resolvent $\sup_{Re\, s \geq 0} \|(s\mathcal{I} - \mathcal{L}_R)^{-1}\|$. For each R, we derive estimates for the region $|s| \geq 2\sqrt{2}(1 + \sqrt{R})$, showing the norm of the resolvent to be proportional to R. For the remaining region, we prove that we can reduce the problem to estimating the norm of the solution of a homogeneous ordinary differential equation with non-homogeneous boundary conditions. This is the main contribution of the paper, since in principle it simplifies the problem either if one wants to prove the estimates analytically or use numerical computations. We perform numerical computations indicating the norm of the resolvent to be proportional to R also for $|s| < 2\sqrt{2}(1 + \sqrt{R})$. There are results in Liefvendahl & Kreiss[3], Reddy & Henningson[5], Trefethen *et al.*[6], Kreiss *et al.*[2], also partly based in computations, indicating the L_2 norm of the resolvent to be proportional to R^2 in the case of 3 space dimensions. In the papers above, the computations are performed using different methods than the one used here.

2 Resolvent estimates

First we study the case when s bounded away from 0.

Theorem 1 *If $|s| \geq 2\sqrt{2}(1 + \sqrt{R})$, then $\|(s\mathcal{I} - \mathcal{L}_R)^{-1}\|^2 \leq \frac{8}{|s|^2}(1 + \sqrt{R})^2 \leq 1$.*

Proof: After applying the Laplace transformation in t, the first equation in (1.1) is transformed to

$$s\widetilde{w} = \frac{1}{R}\Delta\widetilde{w} - (\widetilde{w} \cdot \nabla)W - (W \cdot \nabla)\widetilde{w} - \nabla\tilde{p} + \widetilde{H} = \mathcal{L}_R\widetilde{w} + \widetilde{H} \tag{2.1}$$

with $\widetilde{w}$ satisfying the boundary conditions in the space variables described in (1.1). Taking the inner product of (2.1) with $\widetilde{w}$, we have

$$\langle \widetilde{w}, s\widetilde{w} \rangle = \langle \widetilde{w}, \frac{1}{R}\Delta\widetilde{w} \rangle - \langle \widetilde{w}, (\widetilde{w} \cdot \nabla)W \rangle - \langle \widetilde{w}, (W \cdot \nabla)\widetilde{w} \rangle - \langle \widetilde{w}, \nabla\widetilde{p} \rangle + \langle \widetilde{w}, \widetilde{H} \rangle. \tag{2.2}$$

Integrating by parts and using the divergence free and boundary conditions, one can prove that $\langle \widetilde{w}, \nabla \widetilde{p} \rangle = 0$ and $\langle \widetilde{w}, (W \cdot \nabla)\widetilde{w} \rangle$ is purely imaginary. Therefore,

$$s\|\widetilde{w}\|^2 + \frac{1}{R}\|D\widetilde{w}\|^2 = -\langle \widetilde{w}, (\widetilde{w} \cdot \nabla)W \rangle - \langle \widetilde{w}, (W \cdot \nabla)\widetilde{w} \rangle + \langle \widetilde{w}, \widetilde{H} \rangle =: P \tag{2.3}$$

where $D\widetilde{w}$ denotes the derivative of $\widetilde{w}$ with respect to the space variables, and $\|D\widetilde{w}\|$ its Frobenius norm. P satisfies

$$|P| \leq \|\widetilde{w}\|\|(\widetilde{w} \cdot \nabla)W\| + \|\widetilde{w}\|\|(W \cdot \nabla)\widetilde{w}\| + \|\widetilde{w}\|\|\widetilde{H}\| \tag{2.4}$$

which implies, since $\|DW\| = 1$,

$$|P| \leq \|\widetilde{w}\|^2 + \|\widetilde{w}\|\|D\widetilde{w}\| + \|\widetilde{w}\|\|\widetilde{H}\|. \tag{2.5}$$

Since $\langle \widetilde{w}, (W \cdot \nabla)\widetilde{w} \rangle$ is purely imaginary, taking the real part of (2.3) gives

$$\operatorname{Re} s\|\widetilde{w}\|^2 + \frac{1}{R}\|D\widetilde{w}\|^2 \leq |\langle \widetilde{w}, (\widetilde{w} \cdot \nabla)W \rangle| + |\langle \widetilde{w}, \widetilde{H} \rangle| \leq \|\widetilde{w}\|^2 + \|\widetilde{w}\|\|\widetilde{H}\| \tag{2.6}$$

and therefore

$$\operatorname{Re} s\|\widetilde{w}\|^2 \leq \|\widetilde{w}\|^2 + \|\widetilde{w}\|\|\widetilde{H}\|. \tag{2.7}$$

Note that (2.6) and (2.7) are valid for all s satisfying $\operatorname{Re} s \geq 0$. To complete the proof, we consider two separate cases:

Case 1: $\operatorname{Re} s \geq |\operatorname{Im} s|$**.** In this case, $\operatorname{Re} s \geq |s|/\sqrt{2}$, and we choose s such that $|s|/(2\sqrt{2}) \geq 1$. Using (2.7),

$$\frac{|s|}{2\sqrt{2}}\|\widetilde{w}\|^2 = \Big(\frac{|s|}{\sqrt{2}} - \frac{|s|}{2\sqrt{2}}\Big)\|\widetilde{w}\|^2 \leq (\operatorname{Re} s - 1)\|\widetilde{w}\|^2 \leq \|\widetilde{w}\|\|\widetilde{H}\|$$

which implies

$$|s|^2\|\widetilde{w}\|^2 \leq 8\|\widetilde{H}\|^2. \tag{2.8}$$

Case 2: $|\operatorname{Im} s| \geq \operatorname{Re} s \geq 0$**.** In this case, $|\operatorname{Im} s| \geq |s|/\sqrt{2}$. Taking the imaginary part of equation (2.3) gives

$$|\operatorname{Im} s|\|\widetilde{w}\|^2 \leq |P| \leq \|\widetilde{w}\|^2 + \|\widetilde{w}\|\|D\widetilde{w}\| + \|\widetilde{w}\|\|\widetilde{H}\|. \tag{2.9}$$

By (2.6),

$$\frac{1}{R}\|D\widetilde{w}\|^2 \leq \|\widetilde{w}\|^2 + \|\widetilde{w}\|\|\widetilde{H}\| \leq \Big(\|\widetilde{w}\| + \|\widetilde{H}\|\Big)^2$$

and then

$$\|D\widetilde{w}\| \leq \sqrt{R}\left(\|\widetilde{w}\| + \|\widetilde{H}\|\right). \tag{2.10}$$

Using this estimate in (2.9), we get

$$|\operatorname{Im} s|\|\widetilde{w}\|^2 \leq (1 + \sqrt{R})\|\widetilde{w}\|^2 + (1 + \sqrt{R})\|\widetilde{w}\|\|\widetilde{H}\|. \tag{2.11}$$

Choosing s such that $\frac{|s|}{2\sqrt{2}} \geq 1+\sqrt{R}$, we have

$$\frac{|s|}{2\sqrt{2}}\|\widetilde{w}\|^2 \leq \Big(|\operatorname{Im} s| - (1+\sqrt{R})\Big)\|\widetilde{w}\|^2 \leq (1+\sqrt{R})\|\widetilde{w}\|\|\widetilde{H}\|$$

and this implies

$$|s|^2\|\widetilde{w}\|^2 \leq 8(1+\sqrt{R})^2\|\widetilde{H}\|^2. \tag{2.12}$$

From the two cases, we conclude that

$$|s| \geq 2\sqrt{2}(1+\sqrt{R}) \Rightarrow |s|^2\|\widetilde{w}\|^2 \leq 8(1+\sqrt{R})^2\|\widetilde{H}\|^2 \tag{2.13}$$

which implies the desired estimate

$$\|(s\mathcal{I}-\mathcal{L}_R)^{-1}\|^2 \leq \frac{8}{|s|^2}(1+\sqrt{R})^2 \leq 1, \tag{2.14}$$

valid in the region $|s| \geq 2\sqrt{2}(1+\sqrt{R})$ of the unstable half plane $\operatorname{Re} s \geq 0$. $\square$

Now we study the case when $|s| < 2\sqrt{2}(1+\sqrt{R})$. For this case, write the problem (1.1) componentwise:

$$\begin{gathered}
u_t + yu_x + v + p_x = \frac{1}{R}\Delta u + F\\
v_t + yv_x + p_y = \frac{1}{R}\Delta v + G\\
u_x + v_y = 0\\
u(x,0,t) = u(x,1,t) = v(x,0,t) = v(x,1,t) = 0\\
u(x,y,t) = u(x+1,y,t)\\
v(x,y,t) = v(x+1,y,t)\\
u(x,y,0) = v(x,y,0) = 0
\end{gathered} \tag{2.15}$$

Introduce the stream function ψ by

$$\psi_x = v, \quad \psi_y = -u. \tag{2.16}$$

Apply the Laplace transform to problem (2.15) with respect to t, expand in a Fourier series in the periodic direction x. The equations for the transformed functions $\widehat{u}(k,y,s)$, $\widehat{v}(k,y,s)$, $\widehat{p}(k,y,s)$, $\widehat{F}(k,y,s)$, $\widehat{G}(k,y,s)$ are

$$\begin{gathered}
s\widehat{u} + iky\widehat{u} + \widehat{v} + ik\widehat{p} = -\frac{k^2}{R}\widehat{u} + \frac{1}{R}\widehat{u}_{yy} + \widehat{F}\\
s\widehat{v} + iky\widehat{v} + \widehat{p}_y = -\frac{k^2}{R}\widehat{v} + \frac{1}{R}\widehat{v}_{yy} + \widehat{G}\\
iku + \widehat{v}_y = 0.
\end{gathered} \tag{2.17}$$

Differentiating the first equation in (2.15) with respect to y, the second with respect to x and subtracting the second from the first, we eliminate the pressure

p, and using the divergence free condition, we see that the transformed stream function $\widehat{\psi}$ satisfies the following boundary value problem for an ordinary differential equation with three parameters s, R, k:

$$\begin{gathered}\frac{1}{R}\widehat{\psi}'''' - \Big(s + \frac{2k^2}{R} + iky\Big)\widehat{\psi}'' + \Big(sk^2 + \frac{k^4}{R} + ik^3y\Big)\widehat{\psi} = I \\ \widehat{\psi}(k,0,s) = \widehat{\psi}(k,1,s) = \widehat{\psi}'(k,0,s) = \widehat{\psi}'(k,1,s) = 0\end{gathered} \tag{2.18}$$

where $I := \widehat{F}_y - ik\widehat{G}$ and $'$ denotes the derivative with respect to y. We also use the notation $\mathcal{D} = \frac{\partial}{\partial y}$, and write the problem as

$$\begin{gathered}TT_0\widehat{\psi} = \Big(\frac{1}{R}\mathcal{D}^2 - (s + \frac{k^2}{R} + iky)\Big)(\mathcal{D}^2 - k^2)\widehat{\psi} = I \\ \widehat{\psi}(k,0,s) = \widehat{\psi}(k,1,s) = \widehat{\psi}_y(k,0,s) = \widehat{\psi}_y(k,1,s) = 0\end{gathered} \tag{2.19}$$

where $T = \frac{1}{R}\mathcal{D}^2 - \big(s + \frac{k^2}{R} + iky\big)$ and $T_0 = \mathcal{D}^2 - k^2$ are differential operators depending on the parameters R, k and s. To derive the resolvent estimates, we use the following Lemma.

Lemma 2 *If for all $k \in \mathbb{Z}$ and for all $s \in \mathbb{C}$ such that $|s| < 2\sqrt{2}(1 + \sqrt{R})$ the solution $\widehat{\psi}(k,y,s)$ of (2.18) satisfies*

$$\begin{gathered}k^2\|\widehat{\psi}(k,\cdot,s)\|^2 \le CR^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2) \\ \|\widehat{\psi}'(k,\cdot,s)\|^2 \le CR^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2)\end{gathered} \tag{2.20}$$

where C is a constant independent of s, R, k, $\widehat{F}$, $\widehat{G}$, then

$$\|(s\mathcal{I} - \mathcal{L}_R)^{-1}\|^2 \le 2CR^2$$

for $|s| < 2\sqrt{2}(1 + \sqrt{R})$.

Proof: If for all $k \in \mathbb{Z}$ and for all $s \in \mathbb{C}$ such that $|s| < 2\sqrt{2}(1 + \sqrt{R})$

$$\begin{gathered}k^2\|\widehat{\psi}(k,\cdot,s)\|^2 \le CR^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2) \\ \|\widehat{\psi}'(k,\cdot,s)\|^2 \le CR^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2)\end{gathered} \tag{2.21}$$

then by the definition (2.16) of the stream function,

$$\begin{gathered}\|\widehat{v}(k,\cdot,s)\|^2 = k^2\|\widehat{\psi}(k,\cdot,s)\|^2 \le CR^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2) \\ \|\widehat{u}(k,\cdot,s)\|^2 = \|\widehat{\psi}'(k,\cdot,s)\|^2 \le CR^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2)\,.\end{gathered} \tag{2.22}$$

Since

$$\|\widetilde{u}(\cdot,\cdot,s)\|^2 = \sum_{k=-\infty}^{\infty} \|\widehat{u}(k,\cdot,s)\|^2$$
$$\|\widetilde{v}(\cdot,\cdot,s)\|^2 = \sum_{k=-\infty}^{\infty} \|\widehat{v}(k,\cdot,s)\|^2$$
$$\|\widetilde{F}(\cdot,\cdot,s)\|^2 + \|\widetilde{G}(\cdot,\cdot,s)\|^2 = \sum_{k=-\infty}^{\infty} (\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2),$$

(2.22) implies

$$\begin{aligned} \|\widetilde{u}(\cdot,\cdot,s)\|^2 &\le CR^2(\|\widetilde{F}(\cdot,\cdot,s)\|^2 + \|\widetilde{G}(\cdot,\cdot,s)\|^2) \\ \|\widetilde{v}(\cdot,\cdot,s)\|^2 &\le CR^2(\|\widetilde{F}(\cdot,\cdot,s)\|^2 + \|\widetilde{G}(\cdot,\cdot,s)\|^2) \end{aligned} \tag{2.23}$$

and this implies $\|(s\mathcal{I} - \mathcal{L}_R)^{-1}\|^2 \le 2CR^2$ for $|s| < 2\sqrt{2}(1+\sqrt{R})$. □

This Lemma shows that to estimate the norm of the resolvent, it is enough to prove estimates of the form (2.21) for the solution $\widehat{\psi}$ of (2.18). To prove those estimates for $\widehat{\psi}$, take the inner product of the differential equation in (2.18) with $\widehat{\psi}$ and obtain

$$\frac{1}{R}\langle\widehat{\psi},\widehat{\psi}''''\rangle - \langle\widehat{\psi},(s+\frac{2k^2}{R}+iky)\widehat{\psi}''\rangle + \langle\widehat{\psi},(sk^2+\frac{k^4}{R}+ik^3y)\widehat{\psi}\rangle = \langle\widehat{\psi},I\rangle. \tag{2.24}$$

Through integration by parts,

$$\langle\widehat{\psi},\widehat{\psi}''''\rangle = \|\widehat{\psi}''\|^2 \quad\text{and}\quad -\langle\widehat{\psi},\widehat{\psi}''\rangle = \|\widehat{\psi}'\|^2. \tag{2.25}$$

Therefore, equation (2.24) becomes

$$\frac{1}{R}\|\widehat{\psi}''\|^2 + \big(\frac{2k^2}{R}+s\big)\|\widehat{\psi}'\|^2 + \big(sk^2+\frac{k^4}{R}\big)\|\widehat{\psi}\|^2 - ik\langle\widehat{\psi},y\widehat{\psi}''\rangle + ik^3\langle\widehat{\psi},y\widehat{\psi}\rangle = \langle\widehat{\psi},I\rangle.$$

Again through integration by parts, we have $\langle\widehat{\psi},y\widehat{\psi}''\rangle = -\langle\widehat{\psi},\widehat{\psi}'\rangle - \langle y\widehat{\psi}',\widehat{\psi}'\rangle$, and since ψ is a real function, $\langle\widehat{\psi},\widehat{\psi}'\rangle$ is purely imaginary, $\langle y\widehat{\psi}',\widehat{\psi}'\rangle$ and $\langle\widehat{\psi},y\widehat{\psi}\rangle$ are real numbers. Therefore, taking the real part of the previous equation we get

$$\frac{1}{R}\|\widehat{\psi}''\|^2 + \big(\frac{2k^2}{R}+\operatorname{Re}s\big)\|\widehat{\psi}'\|^2 + \big((\operatorname{Re}s)k^2+\frac{k^4}{R}\big)\|\widehat{\psi}\|^2 = \operatorname{Re}\langle\widehat{\psi},I\rangle - \operatorname{Re}(ik\langle\widehat{\psi},\widehat{\psi}'\rangle)$$

and since $\operatorname{Re}s \ge 0$, this equation implies

$$\frac{1}{R}\|\widehat{\psi}''\|^2 + \frac{2k^2}{R}\|\widehat{\psi}'\|^2 + \frac{k^4}{R}\|\widehat{\psi}\|^2 - |k||\langle\widehat{\psi},\widehat{\psi}'\rangle| \le |\langle\widehat{\psi},I\rangle|. \tag{2.26}$$

We separate the analysis into three cases for $k \in \mathbb{Z}$: $|k| > \sqrt{R/\sqrt{2}}$, $0 < |k| \le \sqrt{R/\sqrt{2}}$, and the special case $k = 0$.

Case 1: $|k| > \sqrt{R/\sqrt{2}}$ For this case, we prove the following theorem.

Theorem 3 *If* $|k| > \sqrt{R/\sqrt{2}}$, *then*

$$\begin{gathered}k^2\|\widehat{\psi}(k,\cdot,s)\|^2 \le 16R^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2)\\ \|\widehat{\psi}'(k,\cdot,s)\|^2 \le 16R^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2).\end{gathered}$$

Proof: Since $|\langle\widehat{\psi},\widehat{\psi}'\rangle| \le \|\widehat{\psi}\|\|\widehat{\psi}'\| \le \frac{R}{4|k|}\|\widehat{\psi}\|^2 + \frac{|k|}{R}\|\widehat{\psi}'\|^2$, inequality (2.26) implies

$$\frac{1}{R}\|\widehat{\psi}''\|^2 + \big(\frac{2k^2}{R} - \frac{k^2}{R}\big)\|\widehat{\psi}'\|^2 + \big(\frac{k^4}{R} - \frac{R}{4}\big)\|\widehat{\psi}\|^2 \le |\langle\widehat{\psi}, I\rangle| \tag{2.27}$$

and since $|k| > \sqrt{R/\sqrt{2}}$, we have $\frac{k^4}{R} - \frac{R}{4} > \frac{k^4}{2R}$ and (2.27) implies

$$\frac{1}{R}\|\widehat{\psi}''\|^2 + \frac{k^2}{R}\|\widehat{\psi}'\|^2 + \frac{k^4}{2R}\|\widehat{\psi}\|^2 \le |\langle\widehat{\psi}, I\rangle|. \tag{2.28}$$

Since equation (2.18) is linear and the forcing is $I = \widehat{F}_y - ik\widehat{G}$, it suffices, to get the desired estimates for the solution $\widehat{\psi}$, to prove estimates for $\widehat{\psi}_1$ and $\widehat{\psi}_2$, solutions of the boundary-value problems

$$\begin{gathered}\frac{1}{R}\widehat{\psi}_1'''' - \big(s + \frac{2k^2}{R} + iky\big)\widehat{\psi}_1'' + \big(sk^2 + \frac{k^4}{R} + ik^3y\big)\widehat{\psi}_1 = \widehat{F}_y\\ \widehat{\psi}_1(k,0,s) = \widehat{\psi}_1(k,1,s) = \widehat{\psi}_1'(k,0,s) = \widehat{\psi}_1'(k,1,s) = 0\end{gathered} \tag{2.29}$$

and

$$\begin{gathered}\frac{1}{R}\widehat{\psi}_2'''' - \big(s + \frac{2k^2}{R} + iky\big)\widehat{\psi}_2'' + \big(sk^2 + \frac{k^4}{R} + ik^3y\big)\widehat{\psi}_2 = ik\widehat{G}\\ \widehat{\psi}_2(k,0,s) = \widehat{\psi}_2(k,1,s) = \widehat{\psi}_2'(k,0,s) = \widehat{\psi}_2'(k,1,s) = 0,\end{gathered} \tag{2.30}$$

since $\widehat{\psi}$ is given by $\widehat{\psi} = \widehat{\psi}_1 - \widehat{\psi}_2$.

Estimates for $\widehat{\psi}_1$: Through integration by parts,

$$|\langle\widehat{\psi}_1, \widehat{F}_y\rangle| = |\langle\widehat{\psi}_1', \widehat{F}\rangle|.$$

Using the Cauchy-Schwarz inequality, (2.28) with forcing $\widehat{F}_y$ yields

$$\frac{1}{R}\|\widehat{\psi}_1''\|^2 + \frac{k^2}{R}\|\widehat{\psi}_1'\|^2 + \frac{k^4}{2R}\|\widehat{\psi}_1\|^2 \le \|\widehat{\psi}_1'\|\|\widehat{F}\|\,. \tag{2.31}$$

Since all the terms on the left hand side of this equation are positive, we have

$$\frac{k^2}{R}\|\widehat{\psi}_1'\|^2 \le \|\widehat{\psi}_1'\|\|\widehat{F}\| \Rightarrow k^4\|\widehat{\psi}_1'\|^2 \le R^2\|\widehat{F}\|^2\,, \tag{2.32}$$

$$\frac{k^4}{2R}\|\widehat{\psi}_1\|^2 \le \|\widehat{\psi}_1'\|\|\widehat{F}\| \Rightarrow k^6\|\widehat{\psi}_1\|^2 \le 2R^2\|\widehat{F}\|^2\,, \tag{2.33}$$

$$\frac{1}{R}\|\widehat{\psi}_1''\|^2 \le \|\widehat{\psi}_1'\|\|\widehat{F}\| \Rightarrow k^2\|\widehat{\psi}_1''\|^2 \le R^2\|\widehat{F}\|^2. \tag{2.34}$$

Estimates for $\widehat{\psi}_2$: By (2.28) with forcing $ik\widehat{G}$ and the Cauchy-Schwarz inequality, we have

$$\frac{1}{R}\|\widehat{\psi}_2''\|^2 + \frac{k^2}{R}\|\widehat{\psi}_2'\|^2 + \frac{k^4}{2R}\|\widehat{\psi}_2\|^2 \le |k|\|\widehat{\psi}_2\|\|\widehat{G}\|\,. \tag{2.35}$$

Then, as in the estimates for $\widehat{\psi}_1$ above, we obtain

$$\begin{aligned} k^6\|\widehat{\psi}_2\|^2 &\le 4R^2\|\widehat{G}\|^2 \\ k^4\|\widehat{\psi}_2'\|^2 &\le 2R^2\|\widehat{G}\|^2 \\ k^2\|\widehat{\psi}_2''\|^2 &\le 2R^2\|\widehat{G}\|^2. \end{aligned} \tag{2.36}$$

Therefore, $\widehat{\psi}$, the solution of problem (2.18), satisfies

$$k^2\|\widehat{\psi}''\|^2 + k^4\|\widehat{\psi}'\|^2 + k^6\|\widehat{\psi}\|^2 \le 16R^2(\|\widehat{F}\|^2 + \|\widehat{G}\|^2) = 16R^2\|\widehat{H}\|^2.$$

Case 2: $0 < |k| \le \sqrt{R/\sqrt{2}}$ For this case, we prove that we can reduce the problem to the study of the solutions of a homogenous 4th order ordinary differential equation with non-homogenous boundary conditions. We compute the norms of those functions numerically. The computations indicate that the norm of the solutions of these simplified problems do not grow as R grows. This implies the norm of the resolvent of the operator $\mathcal{L}_R$ to be proportional to R. We begin by noting that we can restrict ourselves to the case when $s = i\xi$ is purely imaginary. This is a consequence of the following theorem for holomorphic mappings in Banach spaces (Chae [1]).

Theorem 4 (Maximum Modulus Theorem) *Let U be a connected, open subset of $\mathbb{C}$ and $f : U \to E$ a holomorphic mapping, where E is a Banach space. If $\|f(z)\|$ has a maximum at a point in U, then $\|f(z)\|$ is constant on U.*

Applying the theorem to the holomorphic function $f(s) = (s\mathcal{I} - \mathcal{L}_R)^{-1}$ defined on $\operatorname{Re} s > 0$, taking values in the Banach space of bounded linear operators on $L_2(\Omega)$ and noting that (2.14) implies $\lim_{|s|\to\infty} \|f(s)\| = 0$, we conclude

$$\sup_{Res\ge 0} \|(s\mathcal{I} - \mathcal{L}_R)^{-1}\| = \sup_{\xi\in\mathbb{R}} \|(i\xi\mathcal{I} - \mathcal{L}_R)^{-1}\|. \tag{2.37}$$

We note that since we just need to consider $s = i\xi$, $\xi \in \mathbb{R}$, and in this case $|s| = |\xi| = |\operatorname{Im} s|$, the proof of Theorem 1 is valid for $|s| = |\operatorname{Im} s| > 2(1 + \sqrt{R})$. This restricts the parameter range for the numerical calculations. We now reduce our problem to the study of solutions of a homogenous ordinary differential equation. To this end, first consider the second order system

$$Th = \big(\frac{1}{R}\mathcal{D}^2 - (s + \frac{k^2}{R} + iky)\big)h = I = \widehat{F}_y - ik\widehat{G}, \quad h(0) = h(1) = 0$$
$$T_0 g = (\mathcal{D}^2 - k^2)g = h, \quad g(0) = g(1) = 0$$

for $s = i\xi$, $\xi \in \mathbb{R}$. Taking the inner product with h in the first equation and with g in the second, and since the boundary conditions for both equations imply that the boundary terms after integration by parts vanish, we get

$$\begin{gathered} \|g''\|^2 + k^2\|g'\|^2 + k^4\|g\|^2 \le C_1\|h\|^2 \\ \|h'\|^2 + k^2\|h\|^2 \le C_2R^2(\|\widehat{F}\|^2 + \|\widehat{G}\|^2) \end{gathered} \tag{2.38}$$

where C_1, C_2 are constants independent of R, s, k, $\widehat{F}$, $\widehat{G}$. Combining those two inequalities, it follows that

$$k^2\|g''\|^2 + k^4\|g'\|^2 + k^6\|g\|^2 \le CR^2(\|\widehat{F}\|^2 + \|\widehat{G}\|^2) \tag{2.39}$$

where C is a constant independent of R, s, k, $\widehat{F}$, $\widehat{G}$. Note that g satisfies

$$\begin{gathered} TT_0g = \Big(\frac{1}{R}\mathcal{D}^2 - \big(s + \frac{k^2}{R} + iky\big)\Big)(\mathcal{D}^2 - k^2)g = \widehat{F}_y - ik\widehat{G} \\ g(0) = g(1) = 0. \end{gathered} \tag{2.40}$$

Let $g'(0) = \alpha$ and $g'(1) = \beta$. The numbers α and β can be estimated using a 1-dimensional Sobolev type inequality. Indeed, $|g'|_\infty^2 \le \|g'\|^2 + \|g'\|\|g''\|$. Using the estimates (2.39) and that $k \in \mathbb{Z}$ satisfies $1 \le |k|$, we conclude that

$$|g'|_\infty^2 \le CR^2(\|\widehat{F}\|^2 + \|\widehat{G}\|^2)\,. \tag{2.41}$$

Therefore,

$$\begin{gathered} |\alpha|^2 = |g'(0)|^2 \le CR^2(\|\widehat{F}\|^2 + \|\widehat{G}\|^2) \\ |\beta|^2 = |g'(1)|^2 \le CR^2(\|\widehat{F}\|^2 + \|\widehat{G}\|^2), \end{gathered} \tag{2.42}$$

where C represents a constant independent of s, R, k, $\widehat{F}$, $\widehat{G}$. Now, let $\delta(k, y, s)$ be the solution of the problem

$$\begin{gathered} TT_0\delta = \Big(\frac{1}{R}\mathcal{D}^2 - \big(s + \frac{k^2}{R} + iky\big)\Big)(\mathcal{D}^2 - k^2)\delta = 0 \\ \delta(0) = \delta(1) = 0 \\ \delta'(0) = \alpha \\ \delta'(1) = \beta. \end{gathered} \tag{2.43}$$

Then, $\widehat{\psi}(k, y, s) = g(k, y, s) - \delta(k, y, s)$ is the solution of (2.19). Indeed,

$$TT_0\widehat{\psi} = TT_0(g - \delta) = TT_0g - TT_0\delta = \widehat{F}_y - ik\widehat{G}$$

and

$$\begin{gathered} \widehat{\psi}(k, 0, s) = g(k, 0, s) - \delta(k, 0, s) = 0 \\ \widehat{\psi}(k, 1, s) = g(k, 1, s) - \delta(k, 1, s) = 0 \\ \widehat{\psi}'(k, 0, s) = g'(k, 0, s) - \delta'(k, 0, s) = \alpha - \alpha = 0 \\ \widehat{\psi}'(k, 1, s) = g'(k, 1, s) - \delta'(k, 1, s) = \beta - \beta = 0. \end{gathered}$$

Since we already have suitable estimates for g, our problem is reduced to deriving estimates for the solution of

$$\begin{aligned} TT_0\delta = \Big(\frac{1}{R}\mathcal{D}^2 - \big(s + \frac{k^2}{R} + iky\big)\Big)(\mathcal{D}^2 - k^2)\delta = 0 \\ \delta(0) = \delta(1) = 0 \\ \delta'(0) = \alpha \\ \delta'(1) = \beta \end{aligned} \tag{2.44}$$

where $|\alpha|^2 \le CR^2(\|\widehat{F}\|^2 + \|\widehat{G}\|^2)$ and $|\beta|^2 \le CR^2(\|\widehat{F}\|^2 + \|\widehat{G}\|^2)$. Therefore, it remains to estimate $k^2\|\delta(k,\cdot,s)\|^2$ and $\|\delta'(k,\cdot,s)\|^2$ for the parameter range $(k,\xi) \in \mathbb{Z}\times\mathbb{R}$, $1 \le |k| \le \sqrt{R/\sqrt{2}}$, $|\xi| < 2(1+\sqrt{R})$, where $s = i\xi$. To study this problem numerically, we simplify it as follows: let δ_1 be the solution of

$$\begin{aligned} TT_0\delta_1 = 0 \\ \delta_1(0) = \delta_1(1) = 0 \\ \delta_1'(0) = 1 \\ \delta_1'(1) = 0 \end{aligned} \tag{2.45}$$

and δ_2 be the solution of

$$\begin{aligned} TT_0\delta_2 = 0 \\ \delta_2(0) = \delta_2(1) = 0 \\ \delta_2'(0) = 0 \\ \delta_2'(1) = 1. \end{aligned} \tag{2.46}$$

Then $\delta = \alpha\delta_1 + \beta\delta_2$ is the solution of (2.44). Therefore,

$$\begin{aligned} k^2\|\delta(k,\cdot,s)\|^2 \le 2|\alpha|^2k^2\|\delta_1(k,\cdot,s)\|^2 + 2|\beta|^2k^2\|\delta_2(k,\cdot,s)\|^2 \\ \|\delta'(k,\cdot,s)\|^2 \le 2|\alpha|^2\|\delta_1'(k,\cdot,s)\|^2 + 2|\beta|^2\|\delta_2'(k,\cdot,s)\|^2. \end{aligned}$$

Thus we can restrict ourselves to the dependence on R of $k^2\|\delta_j(k,\cdot,s)\|^2$ and $\|\delta_j'(k,\cdot,s)\|^2$, for $j = 1,2$. Moreover, since $\|\psi(k,\cdot,s)\|^2 = \|\psi(-k,\cdot,-s)\|^2$, where ψ is the solution of (2.18), we can restrict ourselves to $0 \le \xi < 2(1+\sqrt{R})$.

We did numerical computations using the MATLAB 6.0 built-in boundary value problem solver BVP4C, for the parameter range $1 \le |k| \le \sqrt{R/\sqrt{2}}$, $0 \le \xi \le 2(1+\sqrt{R})$ and values of R from 1 up to 10000. For ξ, we used a mesh with variable number of points for each R. BVP4C makes use of a collocation method, and we performed computations with different absolute and relative tolerances, using continuation in the Reynolds number for the initial guess of the solution. The results were similar for all cases. Therefore, even though the problem is stiff for some of the parameter values, the results shown in figures (1), (2), (3), (4) in the following pages should be reliable. For different values of the Reynolds number R, we plot the maximum of $k^2\|\delta_j(k,\cdot,s)\|^2$ and $\|\delta_j'(k,\cdot,s)\|^2$ for $1 \le k \le \sqrt{R/\sqrt{2}}$, $0 \le \xi \le 2(1+\sqrt{R})$. The results indicate that

$$k^2\|\delta_j(k,\cdot,s)\|^2 \le 1 \quad \text{and} \quad \|\delta_j'(k,\cdot,s)\|^2 \le 1 \quad j = 1,2. \tag{2.47}$$

Therefore,

$$\begin{aligned} k^2\|\delta(k,\cdot,s)\|^2 \le & 2|\alpha|^2k^2\|\delta_1(k,\cdot,s)\|^2 + 2|\beta|^2k^2\|\delta_2(k,\cdot,s)\|^2 \\ \le & 4CR^2(\|\widehat{F}\|^2 + \|\widehat{G}\|^2), \end{aligned}$$

that is,

$$k^2\|\delta(k,\cdot,s)\|^2 \le \widetilde{C}R^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2) \tag{2.48}$$

and similarly

$$\|\delta'(k,\cdot,s)\|^2 \le \widetilde{C}R^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2) \tag{2.49}$$

for $1 \le |k| \le \sqrt{R/\sqrt{2}}$, $|\xi| \le 2(1+\sqrt{R})$, where $\widetilde{C}$ is a constant independent of R, s, k, $\widehat{F}$, $\widehat{G}$. Those inequalities imply

$$\begin{aligned} k^2\|\widehat{\psi}(k,\cdot,s)\|^2 \le CR^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2) \\ \|\widehat{\psi}'(k,\cdot,s)\|^2 \le CR^2(\|\widehat{F}(k,\cdot,s)\|^2 + \|\widehat{G}(k,\cdot,s)\|^2), \end{aligned}$$

where C is a constant independent of R, s, k, $\widehat{F}$, $\widehat{G}$. Those are the desired estimates for $\widehat{\psi}$.

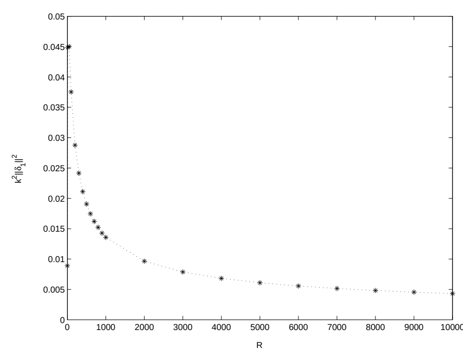

Figure 1: $\max_{k,\xi} k^2\|\delta_1(k,\cdot,i\xi)\|^2$ for $1 \le k \le \sqrt{R/\sqrt{2}}$, $-2(1+\sqrt{R}) \le \xi \le 2(1+\sqrt{R})$.

Case 3: $k = 0$ For this case, the differential equation for $\widehat{\psi}$ is reduced to

$$\begin{gathered} \frac{1}{R}\widehat{\psi}'''' - i\xi\widehat{\psi}'' = \widehat{F}_y \\ \widehat{\psi}(0,0,s) = \widehat{\psi}(0,1,s) = \widehat{\psi}'(0,0,s) = \widehat{\psi}'(0,1,s) = 0. \end{gathered} \tag{2.50}$$

By an energy technique, after integration by parts, we get

$$\frac{1}{R}\|\widehat{\psi}''\|^2 + i\xi\|\widehat{\psi}'\|^2 = \langle\widehat{\psi}', \widehat{F}\rangle. \tag{2.51}$$

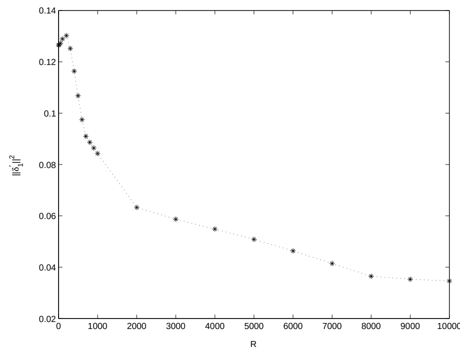

Figure 2: $\max_{k,\xi} \|\delta_1'(k,\cdot,i\xi)\|^2$ for $1 \le k \le \sqrt{R/\sqrt{2}}$, $-2(1+\sqrt{R}) \le \xi \le 2(1+\sqrt{R})$.

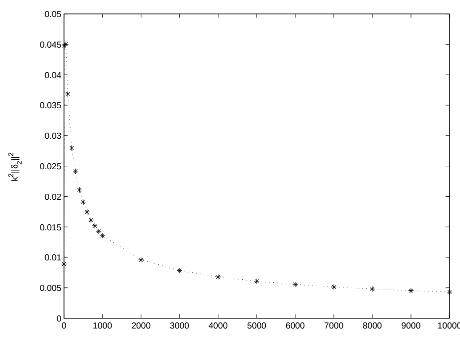

Figure 3: $\max_{k,\xi} k^2\|\delta_2(k,\cdot,i\xi)\|^2$ for $1 \le k \le \sqrt{R/\sqrt{2}}$, $-2(1+\sqrt{R}) \le \xi \le 2(1+\sqrt{R})$.

Taking the real part of this equation and using Poincaré 's inequality, we conclude that $\|\widehat{\psi}'\|^2 \le \frac{R^2}{\pi^4}\|\widehat{F}\|^2$. Applying the Poincaré's inequality again, we get

$$\|\widehat{\psi}(0,\cdot,s)\|^2 \le \frac{R^2}{\pi^6}\|\widehat{F}(0,\cdot,s)\|^2. \tag{2.52}$$

which is the desired estimate. $\square$

Using Lemma 2, we conclude from the three cases above that

$$\|(s\mathcal{I} - \mathcal{L}_R)^{-1}\|^2 \le CR^2 \tag{2.53}$$

for $|s| < 2(1+\sqrt{R})$, where C is a constant independent of s, R, $\widehat{F}$, $\widehat{G}$.

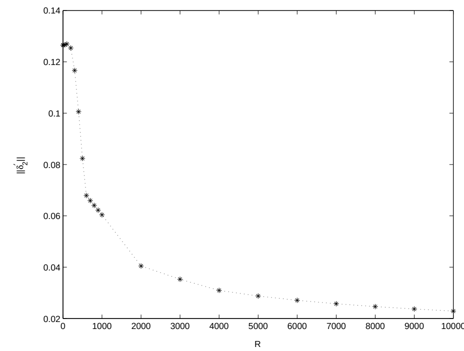

Figure 4: $\max_{k,\xi} \|\delta_2'(k,\cdot,i\xi)\|^2$ for $1 \le k \le \sqrt{R/\sqrt{2}}$, $-2(1+\sqrt{R}) \le \xi \le 2(1+\sqrt{R})$.

Conclusions

The estimates derived for s bounded away from 0 and computed numerically for the remaining region of the unstable half plane indicate the L_2 norm of the resolvent of the operator $\mathcal{L}_R$ to be proportional to R. Deriving the estimates analytically for the whole unstable half plane is still an open problem as far as we know. The method proposed here can be helpful to solve it. It may also be possible to adapt this method for the 3 dimensional problem, and with this approach it may be possible to give a satisfactory solution to the problem, or at least to perform more convincing computations. We hope to address these questions in the future.

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References

[1] Chae, S. B.: Holomorphy and Calculus in Normed Spaces, volume 92 of Monographs and textbooks in pure and applied Mathematics, Marcel Dekker, (1985)

[2] Kreiss, G., Lundbladh, A., Henningson, D.S.: Bounds for threshold amplitudes in subcritical shear flows, J. Fluid Mech., **270**, 175-198 (1994)

[3] Liefvendahl, M., Kreiss, G.: Analytical and numerical investigation of the resolvent for plane Couette flow, SIAM J. Appl. Math. *Submitted*

[4] Romanov, V.A.: Stability of plane-parallel Couette flow, Functional Anal. Applics., **7**, 137-146 (1973)

[5] Reddy, S. C., Henningson, D. S.: Energy growth in viscous channel flows, J. Fluid Mech., **252**, 209-238 (1993)

[6] Trefethen, L. N., Trefethen, A. E., Reddy, S. C., Driscoll, T. A.: Hydrodynamic Stability Without Eigenvalues, Science, **261**, 578-584 (1993)

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