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Almost periodic solutions of semilinear equations with analytic semigroups in Banach spaces *

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Abstract

We establish the existence and uniqueness of almost periodic solutions of a class of semilinear equations having analytic semigroups. Our basic tool in this paper is the use of fractional powers of operators.

1 Introduction

The existence of almost periodic solutions of abstract differential equations has been considered in several works; see for example [1, 2, 4, 7, 11, 12, 13, 14, 15] and reference listed therein. There is also an extensive literature for the same question in semilinear equations. Most of these works are concerned with equation

$$x'(t) + Ax(t) = f(t, x(t)), \tag{1.1}$$

where f is uniformly almost periodic and $-A$ is the infinitesimal generator of a $\mathcal{C}_0-$semigroup [3, 13, 14, 15]. Ballotti, Goldstein and Parrott [4] gave necessary and sufficient conditions for the existence of almost periodic solutions of the equation

$$x'(t) = A(t)x(t),$$

where $A(t)$ is the generator of a $\mathcal{C}_0$ semigroup on a Banach space. These authors used the mean ergodic theorem. Zaidman [12] proved the existence and uniqueness of an almost periodic mild solution of the inhomogeneous equation

$$x'(t) + Ax(t) = g(t), \tag{1.2}$$

where $-A$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $S(t)$ satisfying the exponential stability, and g is almost periodic function from $\mathbb{R}$ into X. In this case, the solution is $x(t) = \int_{-\infty}^{t} S(t-\sigma)g(\sigma)d\sigma$. When A generates a $\mathcal{C}_0$ semigroup $S(t)$ satisfying the exponential stability and f is uniformly Lipschitz continuous with a Lipschitz constant small enough, existence and uniqueness of an almost periodic mild solution of (1.1) was proved in [13].

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In this paper, we consider the semilinear equation (1.1) when $-A$ is the infinitesimal generator of an analytic $\mathcal{C}_0$ semigroup $S(t)$ satisfying the exponential stability. We investigate whether or not the classical solution inherits uniform almost periodicity from f. We proposed a new method for proving existence whose main component is the use of fractional powers of operators. More precisely, we assume that the function $f : \mathbb{R} \times X_\alpha \to X$ satisfies the hypothesis:

(F) There are numbers $L \geq 0$ and $0 \leq \theta \leq 1$ such that $|f(t_1, x_1) - f(t_2, x_2)| \leq L(|t_1 - t_2|^\theta + |x_1 - x_2|_\alpha)$ for all (t_1, x_1) (t_2, x_2) in $\mathbb{R}\times X_\alpha$,

where X is a real or complex Banach space with norm $|\cdot|$, A^α is the fractional power, and X_α is the Banach space $D(A^\alpha)$ endowed with the norm $|x|_\alpha = |A^\alpha x|$. We prove first that the map

$$T\varphi(t) = \int_{-\infty}^{t} A^\alpha S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma))d\sigma$$

is a strict contraction. Then we prove the existence of an almost periodic classical solution over $\mathbb{R}$ of (1.1). See Theorem 3.1 below. Our main theorem complements the results in [13] by considering almost periodic classical solutions instead of almost periodic mild solutions.

Our work is organized as follows. Section 2 is devoted to a review of some results on fractional powers of operators and almost periodic functions with values in a Banach space. In section 3, we state and prove our main result. The last section is devoted to giving an example of a function satisfying hypothesis (F).

2 Preliminary results

Throughout this work, we use the following notation: X denotes a real or complex Banach space endowed with the norm $|\cdot|$ and $\mathcal{L}(X)$ stands for the Banach algebra of bounded linear operators defined on X. For A a linear operator with domain $D(A)$, we denote by $\mathcal{R}(A)$ the range of A.

Fractional powers of operators

We start by a brief outline of the theory of fractional powers as developed in [6, 10]. Let $-A$ is the infinitesimal generator of an analytic semigroup in a Banach space and $0 \in \rho(A)$. For $\alpha > 0$ we define the fractional power $A^{-\alpha}$ by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t)dt$$

Since $A^{-\alpha}$ is one to one, $A^\alpha = (A^{-\alpha})^{-1}$.

For $0 < \alpha \leq 1$, A^α is a closed linear operator whose domain $D(A^\alpha) \supset D(A)$ is dense in X. The closedness of A^α implies that $D(A^\alpha)$ endowed with the graph norm

$$|x|_{D(A)} = |x| + |A^\alpha x|, \quad x \in D(A^\alpha)$$

is a Banach space. Since $0 \in \rho(A)$, A^α is invertible, and its graph norm is equivalent to the norm $|x|_\alpha = |A^\alpha x|$. Thus $D(A^\alpha)$ equipped with the norm $|\cdot|_\alpha$ is a Banach space which we denote X_α.

Lemma 2.1 *Let $-A$ be the infinitesimal generator of an analytic semigroup $S(t)$. If $0 \in \rho(A)$ then*

(a) $S(t) : X \to D(A^\alpha)$ for every $t > 0$ and $\alpha \geq 0$

(b) For every $x \in D(A^\alpha)$, we have $S(t)A^\alpha x = A^\alpha S(t)x$

(c) For every $t > 0$ the operator $A^\alpha S(t)$ is bounded and $|A^\alpha S(t)|_{\mathcal{L}(X)} \leq M_\alpha t^{-\alpha} e^{-\delta t}$

(d) For $0 < \alpha \leq 1$ and $x \in D(A^\alpha)$, we have $|S(t)x - x| \leq C_\alpha t^\alpha |A^\alpha x|$.

For more details, see [10, section 2.6].

Almost periodic functions in Banach spaces

The theory of almost periodic functions with values in a Banach space was developed by H. Bohr, S. Bochner, J. von Neumann, and others; cf., e.g., [1, 5]. From their results, we will mention several results which will be used in this work.

Let $C_b(\mathbb{R}, X)$ denote the usual Banach space of bounded continuous functions from $\mathbb{R}$ into X under the supremum norm $|\cdot|_\infty$. Given a function $f : \mathbb{R} \to X$ and $\omega \in \mathbb{R}$, we define the ω-translate of f as $f_\omega(t) = f(t + \omega)$, $t \in \mathbb{R}$. We will denote by $H(f) = \{f_\omega : \omega \in \mathbb{R}\}$ the set of all translates of f.

Definition. (Bochner's characterization of almost periodicity) A function $f \in C_b(\mathbb{R}, X)$ is said to be almost periodic if and only if $H(f)$ is relatively compact in $C_b(\mathbb{R}, X)$.

Of course, almost periodic functions can as well be characterized in terms of relatively dense sets in $\mathbb{R}$ of τ-almost periods.

Definition. A function $f : R \to X$ is called almost periodic if

(i) f is continuous, and

(ii) for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$, such that every interval I of length $l(\varepsilon)$ contains a number τ such that $|f(t + \tau) - f(t)| < \varepsilon$ for all $t \in R$.

Let Y denote a Banach space and Ω an open subset of Y.

Definition. A continuous function $f : \mathbb{R} \times \Omega \to X$ is called uniformly almost periodic if for every $\varepsilon > 0$ and every compact set $K \subset \Omega$ there exists a relatively dense set P_ε in $\mathbb{R}$ such that $|f(t+\tau, x) - f(t, x)| \leq \varepsilon$ for all $t \in \mathbb{R}$, $\tau \in P_\varepsilon$ and all $y \in K$.

The following is essential for our results and is proven in [11, Theorem I.2.7].

Lemma 2.2 *Let $f : R \times \Omega \to X$ be uniformly almost periodic and $y : R \to \Omega$ be an almost periodic function such that $\overline{\mathcal{R}(y)} \subset \Omega$, then the function $t \to f(t, y(t))$ also is almost periodic.*

We are now in position to state and prove the main result of this paper.

3 Main Result

Definition. A function $x : [0, T[\to X$ is a (classical) solution of (1.1) on $[0, T[$ if x is continuous on $[0, T[$, continuously differentiable on $]0, T[$, $x(t) \in D(A)$ for $0 < t < T$ and (1.1) is satisfied.

Definition. A continuous solution x of the integral equation

$$x(t) = S(t - t_0)x(t_0) + \int_{t_0}^{t} S(t - \sigma)f(\sigma, x(\sigma))d\sigma \tag{3.1}$$

will be called a mild solution of (1.1).

Remark. When A generates a semigroup with negative exponent, it is easy to see that if $x(.)$ is a bounded mild solution of (1.1) on $\mathbb{R}$. Then we can take the limit as $t_0 \to -\infty$ on the right-hand of (3.1) to obtain

$$x(t) = \int_{-\infty}^{t} S(t - \sigma)f(\sigma, x(\sigma))d\sigma\,. \tag{3.2}$$

Conversely, if $x(.)$ is a bounded continuous function and (3.2) is verified, then $x(.)$ is a mild solution of (1.1).

The main result of this paper is the following theorem.

Theorem 3.1 *Let $-A$ be the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ satisfying $|S(t)|_{\mathcal{L}(X)} \leq M\exp(\beta t)$, for all $t > 0$ ($\beta < 0$). If $f : \mathbb{R} \times X \to X$ is uniformly almost periodic and f satisfies the assumption (F) Then for L sufficiently small enough, there exists one and only one almost periodic solution over $\mathbb{R}$ of the semilinear equation* (1.1).

Remark. Assumption (F) is commonly used for this type of equations, as seen in [9, 10]).

In the proof of our main result, we will need the following technical lemma.

Lemma 3.2 *If $g : \mathbb{R} \to X$ is almost periodic and locally Hölder continuous, then there exists one and only one almost periodic (classical) solution over $\mathbb{R}$ of the equation* (1.2). *The solution is $x(t) = \int_{-\infty}^{t} S(t - \sigma)g(\sigma)d\sigma$.*

Proof. In [12], it is proved the existence of the almost periodic mild solution of (1.2). It is known, see [10], that in the case of Hölder continuity of g and if A generates an analytic semigroup, then the mild solution is a classical solution of the differential equation (1.2). $\square$

We define the set

$$AP(X) = \{\varphi : \mathbb{R} \to X,\ \varphi \text{ is almost periodic } \}$$

with the usual supremum norm over $\mathbb{R}$ which we denote by $|\cdot|_\infty$. On the set $AP(X)$, we define a mapping

$$T\varphi(t) = \int_{-\infty}^{t} A^\alpha S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma \tag{3.3}$$

First, we show that T is well defined. Let $\varphi \in AP(X)$, using a standard properties of the almost-periodicity, we have

$$N = \sup_{t\in\mathbb{R}} |f(t, A^{-\alpha}\varphi(t))| < \infty.$$

by Lemma 2.1.c, we have

$$|T\varphi(t)| \le M_\alpha N \int_{-\infty}^{t} (t-\sigma)^{-\alpha} \exp(-\delta(t-\sigma)) d\sigma.$$

With the change variable $s = t - \sigma$, we obtain

$$|T\varphi(t)| \le M_\alpha N \int_{0}^{+\infty} s^{-\alpha} \exp(-\delta s) ds$$

which shows that $T\varphi$ exists.

Lemma 3.3 *The operator T is well defined, and maps $AP(X)$ into itself.*

Proof. For $\varphi \in AP(X)$, it follows from Lemma 2.2 that $t \to f(t, A^{-\alpha}\varphi(t))$ is almost periodic. Hence, for each $\varepsilon > 0$ there exists a set P_ε relatively dense in $\mathbb{R}$ such that

$$|f(t+\tau, A^{-\alpha}\varphi(t+\tau)) - f(t, A^{-\alpha}\varphi(t))| \le \varepsilon$$

for all $t \in \mathbb{R}$ and $\tau \in P_\varepsilon$. Therefore, the map T defined by (3.3) satisfies

$$\begin{aligned}
&|T\varphi(t+\tau) - T\varphi(t)| \\
&= \Big| \int_{-\infty}^{t+\tau} A^\alpha S(t+\tau-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma \\
&\quad - \int_{-\infty}^{t} A^\alpha S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma \Big| \\
&= \Big| \int_{-\infty}^{t} A^\alpha S(t-\sigma) f(\sigma+\tau, A^{-\alpha}\varphi(\sigma+\tau)) d\sigma \\
&\quad - \int_{-\infty}^{t} A^\alpha S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma \Big| \\
&\le \int_{-\infty}^{t} |A^\alpha S(t-\sigma)|_{\mathcal{L}(X)} |f(\sigma+\tau, A^{-\alpha}\varphi(\sigma+\tau)) - f(\sigma, A^{-\alpha}\varphi(\sigma))| d\sigma \\
&\le \varepsilon M_\alpha \int_{-\infty}^{t} (t-\sigma)^{-\alpha} \exp(-\delta(t-\sigma)) d\sigma
\end{aligned}$$

Which shows that the function $T\varphi$ also is almost periodic and that $T : AP(X) \to AP(X)$. □

Proof of Theorem 3.1 Consider the mapping from the Banach space $AP(X)$ into itself defined by

$$T\varphi = \psi(t) = \int_{-\infty}^{t} A^\alpha S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma.$$

We will show that T has a fixed point. Let $\varphi_1, \varphi_2 \in AP(X)$. Then

$$|T\varphi_1(t) - T\varphi_2(t)| \le \int_{-\infty}^{t} |A^\alpha S(t-\sigma)|_{\mathcal{L}(X)} |f(\sigma, A^{-\alpha}\varphi_1(\sigma)) - f(\sigma, A^{-\alpha}\varphi_2(\sigma))| d\sigma.$$

From assumption (F), we have

$$\begin{aligned}
|T\varphi_1(t) - T\varphi_2(t)| &\le L|\varphi_1 - \varphi_2|_\infty \int_{-\infty}^{t} |A^\alpha S(t-\sigma)|_{\mathcal{L}(X)} d\sigma \\
&\le L|\varphi_1 - \varphi_2|_\infty \int_{-\infty}^{t} (t-\sigma)^{-\alpha} \exp(-\delta(t-\sigma)) d\sigma
\end{aligned}$$

and by the change of variable $s = t - \sigma$, we have

$$\begin{aligned}
|T\varphi_1 - T\varphi_2|_\infty &\le L M_\alpha |\varphi_1 - \varphi_2|_\infty \int_0^{+\infty} s^{-\alpha} e^{-\delta s} ds \\
&= L M_\alpha \delta^\alpha \Gamma(1-\alpha) |\varphi_1 - \varphi_2|_\infty,
\end{aligned}$$

where $\Gamma(.)$ is the classical gamma function. We use the well known identity

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha} \quad \text{for } 0 < \alpha < 1.$$

Then, we can deduce that T is a strict contraction, provided L is sufficiently small, $L < \frac{\sin \pi\alpha}{\alpha} \frac{\Gamma(\alpha)}{M_\alpha \delta^\alpha}$. By the contraction mapping theorem there exists $\varphi \in AP(X)$ such that

$$\varphi = \int_{-\infty}^{t} A^\alpha S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma. \tag{3.4}$$

Since A^α is closed,

$$\varphi = A^\alpha \int_{-\infty}^{t} S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma. \tag{3.5}$$

Applying the operator $A^{-\alpha}$ on both sides of (3.5),

$$A^{-\alpha}\varphi = \int_{-\infty}^{t} S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma. \tag{3.6}$$

Next, we show that $t \to f(t, A^{-\alpha}\varphi(t))$ is Hölder continuous on $\mathbb{R}$. To this end we show first that the solution φ of (3.6) is Hölder continuous on $\mathbb{R}$. By Lemma 2.1.d We note that for every β satisfying $0 < \beta < 1-\alpha$ and for every $h > 0$, we have

$$|(S(h) - I)A^\alpha S(t-\sigma)| \le C_\beta h^\beta |A^{\alpha+\beta} S(t-\sigma)| \tag{3.7}$$

and

$$\begin{aligned} |\varphi(t+h) - \varphi(t)| \le & \Big| \int_{-\infty}^{t} (S(h) - I)A^\alpha S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma \Big| \\ & + \Big| \int_{t}^{t+h} A^\alpha S(t+h-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma \Big| \end{aligned} \tag{3.8}$$

Let $K = A^{-\alpha}\varphi(\mathbb{R})$ and $N = \sup_{(t,x)\in\mathbb{R}\times K} |f(t,x)|$. Clearly K is compact. Using Lemma 2.1.c and (3.7) we can estimate each of the terms of (3.8) separately:

$$\begin{aligned} & \Big| \int_{-\infty}^{t} (S(h) - I)A^\alpha S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma \Big| \\ & \le M_{\alpha+\beta} N C_\beta h^\beta \int_{-\infty}^{t} (t-\sigma)^{-(\alpha+\beta)} \exp(-\delta(t-\sigma))\, d\sigma. \end{aligned}$$

By Lemma 2.1.c,

$$\begin{aligned} \Big| \int_{t}^{t+h} A^\alpha S(t+h-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma)) d\sigma \Big| & \le M_\alpha N \int_{t}^{t+h} (t+h-\sigma)^{-\alpha} d\sigma \\ & \le M_\alpha N \frac{h^{1-\alpha}}{1-\alpha}. \end{aligned}$$

Combining (3.10) with these estimates, it follows that there is a constant C such that

$$|\varphi(t+h) - \varphi(t)| \le C h^\beta$$

and therefore φ is Hölder continuous on $\mathbb{R}$.

Finally, it remains to proved that $t \to f(t, A^{-\alpha}\varphi(t))$ is Hölder continuous on $\mathbb{R}$. From assumption (F) we have

$$|f(t, A^{-\alpha}\varphi(t)) - f(s, A^{-\alpha}\varphi(s))| \leq L\big(|t-s|^{\theta} + |\varphi(t) - \varphi(s)|\big);$$

therefore, $t \to f(t, A^{-\alpha}\varphi(t))$ is Hölder continuous on $\mathbb{R}$. Let φ be the solution of (3.4) and consider the equation

$$\frac{dx(t)}{dt} + Ax(t) = f(t, A^{-\alpha}\varphi(t)). \tag{3.9}$$

From Lemma 3.2 this equation has a unique solution given by

$$\psi(t) = \int_{-\infty}^{t} S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma))d\sigma. \tag{3.10}$$

Moreover, we have $\psi(t) \in D(A)$ for all $t \in \mathbb{R}$ and a fortiori $\psi(t) \in D(A^{\alpha})$. Applying the operator A^{α} on both sides of (3.10), we have

$$A^{\alpha}\psi(t) = \int_{-\infty}^{t} A^{\alpha} S(t-\sigma) f(\sigma, A^{-\alpha}\varphi(\sigma))d\sigma = \varphi(t) \tag{3.11}$$

From (3.9) and (3.11) we readily see that $\psi(t) = A^{-\alpha}\varphi(t)$ is solution of (1.1). The uniqueness of ψ follows easily from the uniqueness of the solution of (3.4) and (3.9). Therefore, the proof of Theorem 3.1 is complete. □

4 Example

Let $X = L^2((0,1);\mathbb{R})$ and

$$Au = -u'' \quad \text{with} \quad u \in D(A) = \{u \in H_0^1((0,1);\mathbb{R}); u'' \in X\}. \tag{4.1}$$

Then A is self-adjoint, with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$. We take $\alpha = 1/2$, that is $X_{1/2} = (D(A^{1/2}), |\cdot|_{1/2})$. Define the function $f : \mathbb{R} \times X_{1/2} \to X$, by $f(t,u) = h(t)g(u')$ for each $t \in \mathbb{R}$ and $u \in X_{1/2}$, where $h : \mathbb{R} \to \mathbb{R}$ is almost periodic in $\mathbb{R}$ and there exist $k_1 > 0$ and $\theta \in]0,1[$ such that

$$|h(t) - h(s)| \leq k_1 |t-s|^{\theta}, \quad \text{for all } t, s \in \mathbb{R}. \tag{4.2}$$

and $g : X \to X$ is Lipschitz continuous on X. Concrete example of the function g are

$$g(u) = \sin(u), \quad g(u) = ku, \quad g(u) = \arctan(u)$$

We give first some known results for the operators A and $A^{1/2}$ defined by (4.1).

Let $u \in D(A)$ and $\lambda \in \mathbb{R}$, such that $Au = -u'' = \lambda u$; that is,

$$u'' + \lambda u = 0 \tag{4.3}$$

We have $\langle Au, u\rangle = \langle \lambda u, u\rangle$; that is,

$$\langle -u'', u\rangle = |u'|^2_{L^2} = \lambda |u|^2_{L^2}$$

so $\lambda \in \mathbb{R}^*_+$. The solutions of (4.3) have the form

$$u(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

we have $u(0) = u(1)$, so, $C = 0$ and $\sqrt{\lambda} = n\pi$, $n \in \mathbb{N}^*$. Put $\lambda_n = n^2\pi^2$. The solutions of equation (4.3) are

$$u_n(x) = D\sin(\sqrt{\lambda_n}x), \quad n \in \mathbb{N}^*.$$

We have $\langle u_n, u_m\rangle = 0$, for $n \neq m$ and $\langle u_n, u_n\rangle = 1$. So $D = \sqrt{2}$ and

$$u_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x).$$

For $u \in D(A)$, there exists a sequence of reals (α_n) such that

$$u(x) = \sum_{n\in\mathbb{N}^*} \alpha_n u_n(x),$$

$$\sum_{n\in\mathbb{N}^*} (\alpha_n)^2 < +\infty, \quad \sum_{n\in\mathbb{N}^*} (\lambda_n)^2(\alpha_n)^2 < +\infty$$

We have

$$A^{1/2}u(x) = \sum_{n\in\mathbb{N}^*} \sqrt{\lambda_n}\alpha_n u_n(x)$$

with $u \in D(A^{1/2})$; that is, $\sum_{n\in\mathbb{N}^*}(\alpha_n)^2 < +\infty$ and $\sum_{n\in\mathbb{N}^*} \lambda_n(\alpha_n)^2 < +\infty$.

We show now that f satisfies the hypothesis (F). In fact, for $t_1, t_2 \in \mathbb{R}$ and $u_1, u_2 \in X_{1/2}$, we have

$$\begin{aligned} f(t_1, u_1) - f(t_2, u_2) &= h(t_1)g(u_1') - h(t_2)g(u_2') \\ &= [h(t_1) - h(t_2)]g(u_1') + h(t_2)[g(u_1') - g(u_2')] \end{aligned}$$

So,

$$\begin{aligned} |f(t_1, u_1) - f(t_2, u_2)|_{L^2} &\leq |h(t_1) - h(t_2)||g(u_1')|_{L^2} + |h(t_2)||g(u_1') - g(u_2')|_{L^2} \\ &\leq |g|_\infty|h(t_1) - h(t_2)| + |g|_{\mathrm{Lip}}|h(t_2)||u_1' - u_2'|_{L^2}. \end{aligned} \tag{4.4}$$

Since h is almost periodic, there exists $k_2 > 0$, such that

$$|h(t_2)| \leq k_2 \tag{4.5}$$

Therefore, from (4.2), (4.4), (4.5), and the fact that $g(u')$ is Lipschitz on $X_{1/2}$ (see for instance [8, p. 75]), we have

$$\begin{aligned} |f(t_1, u_1) - f(t_2, u_2)|_X &\leq k_1|g|_\infty|t_1 - t_2|^\theta + k_2|g|_{Lip}|u_1 - u_2|_{1/2} \\ &\leq L(|t_1 - t_2|^\theta + |u_1 - u_2|_{1/2}). \end{aligned}$$

Therefore, f satisfies the hypothesis (F), with $L = \max(k_1|g|_\infty, k_2|g|_{\mathrm{Lip}})$.

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