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SINGULAR SOLUTIONS TO PROTTER'S PROBLEM FOR THE 3-D WAVE EQUATION INVOLVING LOWER ORDER TERMS

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Abstract. In 1952, at a conference in New York, Protter formulated some boundary value problems for the wave equation, which are three-dimensional analogues of the Darboux problems (or Cauchy-Goursat problems) on the plane. Protter studied these problems in a 3-D domain Ω_0, bounded by two characteristic cones Σ_1 and $\Sigma_{2,0}$, and by a plane region Σ_0. It is well known that, for an infinite number of smooth functions in the right-hand side, these problems do not have classical solutions. Popivanov and Schneider (1995) discovered the reason of this fact for the case of Dirichlet's and Neumann's conditions on Σ_0: the strong power-type singularity appears in the generalized solution on the characteristic cone $\Sigma_{2,0}$. In the present paper we consider the case of third boundary-value problem on Σ_0 and obtain the existence of many singular solutions for the wave equation involving lower order terms. Especifica ally, for Protter's problems in $\mathbb{R}^3$ it is shown here that for any $n \in N$ there exists a $C^n(\bar{\Omega}_0)$-function, for which the corresponding unique generalized solution belongs to $C^n(\bar{\Omega}_0 \backslash O)$ and has a strong power type singularity at the point O. This singularity is isolated at the vertex O of the characteristic cone $\Sigma_{2,0}$ and does not propagate along the cone. For the wave equation without lower order terms, we presented the exact behavior of the singular solutions at the point O.

1. Introduction

Consider the hyperbolic partial differential equation, involving the wave operator in its main part, with lower order terms of the form

$$Lu \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} + b_1u_{x_1} + b_2u_{x_2} + bu_t + cu = f \tag{1.1}$$

expressed in Cartesian coordinates x_1, x_2, t in a simply connected region $\Omega_0 \subset \mathbb{R}^3$. The region

$$\Omega_0 := \{(x_1, x_2, t) : 0 < t < 1/2, t < \sqrt{x_1^2 + x_2^2} < 1 - t\}$$

is bounded by the disk

$$\Sigma_0 := \{(x_1, x_2, t) : t = 0, x_1^2 + x_2^2 < 1\}$$

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with center at the origin $O(0,0,0)$ and the characteristic surfaces of (1.1):

$$\Sigma_1 := \{(x_1,x_2,t) : 0 < t < 1/2, \sqrt{x_1^2+x_2^2} = 1-t\},$$
$$\Sigma_{2,0} := \{(x_1,x_2,t) : 0 < t < 1/2, \sqrt{x_1^2+x_2^2} = t\}.$$

In this work we are interested in finding sufficient conditions for the existence and uniqueness of a generalized solution to the following problem.

Problem $\mathbf{P}_\alpha$. Find a solution to (1.1) in Ω_0 that satisfies the boundary conditions

$$u|_{\Sigma_1} = 0, \quad [u_t + \alpha u]|_{\Sigma_0 \setminus O} = 0, \tag{1.2}$$

where $\alpha \in C^1(\bar{\Sigma}_0 \setminus O)$.

The adjoint problem to $\mathbf{P}_\alpha$ is as follows.

Problem $\mathbf{P}_\alpha^*$. Find a solution of the adjoint equation

$$L^* u \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} - (b_1 u)_{x_1} - (b_2 u)_{x_2} - (bu)_t + cu = g \quad \text{in } \Omega_0$$

with the boundary conditions:

$$u|_{\Sigma_{2,0}} = 0, \quad [u_t + (\alpha + b)u]|_{\Sigma_0} = 0. \tag{1.3}$$

The following problems were introduced by Protter (see [23]).

Protter's Problems. Find a solution of the wave equation

$$\Box u \equiv \Delta_x u - u_{tt} \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} = f \quad \text{in } \Omega_0 \tag{1.4}$$

with one of the following boundary conditions

$$\begin{aligned} P1: &\quad u|_{\Sigma_0 \cup \Sigma_1} = 0, & P1^*: &\quad u|_{\Sigma_0 \cup \Sigma_{2,0}} = 0\,; \\ P2: &\quad u|_{\Sigma_1} = 0, u_t|_{\Sigma_0} = 0, & P2^*: &\quad u|_{\Sigma_{2,0}} = 0, u_t|_{\Sigma_0} = 0\,. \end{aligned} \tag{1.5}$$

Protter [23] formulated and investigated both Problems $P1$ and $P1^*$ in Ω_0 as multi-dimensional analogues of the Darboux problem on the plane. It is well known that the corresponding Darboux problems on $\mathbb{R}^2$ are well posed, which is not true for the Protter's problems in $\mathbb{R}^3$. The uniqueness of a classical solution of Problem $P1$ was proved by Garabedian [8]. For recent results concerning the Protter's problems (1.5) see the work of Popivanov, Schneider [21], Grammatikopoulos, Hristov, Popivanov [9] and references therein. For more publications in this area see, for example: [1, 2, 7, 12, 15, 16, 17, 22]. Some different statements of Darboux type problems can be found in [4, 5, 6, 14, 18] in bounded or unbounded domains, different from Ω_0.

According to the ill-possedness of Protter's Problems $P1$ and $P2$, it is interesting to find some their regularizations. A nonstandard, nonlocal regularization of Problem P1, can be found in [7]. In the present paper we are looking for some other kind of regularization and formulate the following problem.

Open Question 1. Is it possible to find conditions for the coefficients b_1, b_2, b_3, c and α, under which for all smooth functions f Problem P_α has only regular solutions?

Remark 1.1. If the answer to the above question is positive, then, using an operator L_k with lower order perturbations in the wave equation (1.4), we can find possible regularization for Problem $P2$. Solving the equation $L_k u_k = f$, with $L_k \to \Box$ (i.e. $b_{1k}, b_{2k}, b_{3k}, c_k \to 0$) and $\alpha_k \to 0$, we can find an approximated sequence u_k. Due to the fact that in this case the cones Σ_1 and $\Sigma_{2,0}$ are again characteristics for L_k, this process, with respect to our boundary value problem, looks to be natural.

For Problem eq:0p1 with P_α and $\alpha(x) \neq 0$ there are only few publications, while for (1.4) with P_α, we refer the reader to [9]. Some results of this type can be found also in Section 7 of this paper.

In the case of the equation (1.1) , which involves either lower order terms or some other type perturbations, Problem P_α in Ω_0 with $\alpha(x) \equiv 0$ has been studied by Aldashev in [1, 2, 3]. For comments, concerning Aldashev's results, we refer the reader to Remark 6.5. Finally, we point out that in the case of (1.1), with nonzero lower order terms, Karatoprakliev [13] obtained a priori estimates, but only for solutions enough smooth of Problem $P1$ in Ω_0.

Next, we formulate the following well known result [24, 20], presented here in the terms of the polar coordinates $x_1 = \varrho\cos\varphi$, $x_2 = \varrho\sin\varphi$.

Theorem 1.2. *For all $n \in \mathbb{N}$, $n \geq 4$; a_n, b_n arbitrary constants, the functions*
$$v_n(\varrho,\varphi,t) = t\varrho^{-n}\left(\varrho^2 - t^2\right)^{n-\frac{3}{2}}(a_n\cos n\varphi + b_n\sin n\varphi) \tag{1.6}$$
are classical solutions of the homogeneous problem $P1^*$ and the functions
$$w_n(\varrho,\varphi,t) = \varrho^{-n}\left(\varrho^2 - t^2\right)^{n-\frac{1}{2}}(a_n\cos n\varphi + b_n\sin n\varphi) \tag{1.7}$$
are classical solutions of the homogeneous problem $P2^*$.

This theorem shows that for the classical solvability (see [6]) of the problem $P1$ (respectively, $P2$) the function f at least must be orthogonal to all smooth functions (1.6) (respectively, (1.7)). The reason of this fact has been found by Popivanov and Schneider in [20], where they announced for Problems $P1$ and $P2$ that there exist singular solutions for the wave equation (1.4) with power type isolated singularities even for very smooth functions f. Using Theorem 1.2, Popivanov and Schneider [21] proved the existence of *generalized solutions* of Problems $P1$ and $P2$, which have at least power type singularities at the vertex O of the cone $\Sigma_{2,0}$. Considering Problems $P1$ and $P2$, Popivanov and Schneider [20] announced the existence of singular solutions for both wave and degenerate hyperbolic equation. First a priori estimates for singular solutions of Protter's Problems $P1$ and $P2$, concerning the wave equation in $\mathbb{R}^3$, were obtained in [21]. On the other hand, for the case of the wave equation in $\mathbb{R}^{m+1}$, Aldashev [1] shows that there exist solutions of Problem $P1$ (respectively, $P2$) in the domain Ω_ε, which grow up on the cone $\Sigma_{2,\varepsilon}$ like $\varepsilon^{-(n+m-2)}$ (respectively, $\varepsilon^{-(n+m-1)}$), when $\varepsilon \to 0$ and the cone $\Sigma_{2,\varepsilon} := \{\varrho = t + \varepsilon\}$ approximates $\Sigma_{2,0}$. It is obvious that for $m = 2$ this results can be compared with the estimate (1.10) of Theorem 1.4 and with the analogous estimates of Theorems 6.1 and 6.3. For the homogeneous Problem P_α^* (except the case $\alpha \equiv 0$, i.e. except Problem $P2^*$), even for the wave equation, we do not know nontrivial solutions analogous to (1.6) and (1.7). Anyway, in the present paper under appropriate conditions for the coefficients of the general equation (1.1), we derive results which ensure the existence of many singular solutions of Problem P_α. Here we refer also to Khe Kan Cher [16], who gives some nontrivial solutions for the homogeneous Problems $P1^*$ and $P2^*$, but in the case of Euler-Poisson-Darboux equation. These results are closely connected to the such ones of Theorem 1.2.

In order to obtain our results, we give the following definition of a generalized solution of Problem P_α with a possible singularity at the point O.

Definition 1.3. A function $u = u(x_1,x_2,t)$ is called a *generalized solution* of P_α: in Ω_0, if

(1) $u \in C^1(\bar\Omega_0\backslash O)$, $[u_t + \alpha(x)u]\big|_{\Sigma_0\backslash O} = 0$ $u\big|_{\Sigma_1} = 0$,

(2) The equality

$$\begin{aligned}&\int_{\Omega_0}[u_t v_t - u_{x_1}v_{x_1} - u_{x_2}v_{x_2} + (b_1 u_{x_1} + b_2 u_{x_2} + b u_t + cu - f)v]dx_1 dx_2 dt\\ &= \int_{\Sigma_0}\alpha(x)(uv)(x,0)dx_1 dx_2\end{aligned} \tag{1.8}$$

holds for all v in

$$V_0 := \{v \in C^1(\bar\Omega_0) : [v_t + (\alpha + b)v]\big|_{\Sigma_0} = 0,\ v = 0 \text{ in a neighborhood of } \Sigma_{2,0}\}.$$

To deal with difficulties such as singularities of *generalized solutions* on the cone $\Sigma_{2,0}$, we introduce the region

$$\Omega_\varepsilon = \Omega_0 \cap \{\varrho - t > \varepsilon\}, \quad \varepsilon \in [0,1),$$

which in polar coordinates becomes

$$\Omega_\varepsilon = \{(\varrho,\varphi,t) : t > 0,\ 0 \le \varphi < 2\pi, \varepsilon + t < \varrho < 1 - t\}. \tag{1.9}$$

We define *generalized solution* of Problem P_α in $\Omega_\varepsilon, \varepsilon \in (0,1)$, in Definition 2.2 below. Note that, if a generalized solution u belongs to $C^1(\bar\Omega_\varepsilon) \cap C^2(\Omega_\varepsilon)$, it is called a *classical solution* of Problem P_α in Ω_ε, $\varepsilon \in (0,1)$, and it satisfies the equation (1.1) in Ω_ε. It should be pointed out that the case $\varepsilon = 0$ is totally different from the case $\varepsilon \neq 0$.

This paper is a generalization, extension, and improvement of the results obtained in [9]. The paper, besides Introduction, consists of six more sections. In Section 2, using some appropriate techniques, we formulate the 2-D boundary value problems $P_{\alpha,1}$, $P_{\alpha,2}$ and $P_{\alpha,3}$, corresponding to the 3-D Problem P_α. The aim of Section 3 is to treat Problem $P_{\alpha,3}$. For this reason, we construct and study the system of integral equations, assigned to the under consideration equation (1.1). Also, we present results concerning the classical solutions of Problem $P_{\alpha,3}$ in $\Omega_\varepsilon, \varepsilon \in (0,1)$ and give corresponding a priori estimates. We mention also here Lemma 3.2, which is actually a maximum principle for Problem $P_{\alpha,3}$. In Section 4 we prove Theorems 4.1 and 4.2, which ensure the existence and uniqueness of a generalized solution of Problem $P_{\alpha,1}$ in 2-D domain. Using the results of the previous section, in Section 5 we study the existence and uniqueness of a generalized solution of 3-D Problem P_α. More precisely, Theorem 5.1 ensures the uniqueness of a generalized solution for Problem P_α in $\Omega_\varepsilon, \varepsilon \in [0,1)$, while Theorems 5.2, 5.3 and 5.4 ensure the existence of a generalized solution for problem P_α in Ω_0, which is a classical one in each domain $\Omega_\varepsilon, \varepsilon \in [0,1)$ and satisfies some a priori estimates in $C^2(\Omega_\varepsilon)$. Comparing these estimates with such ones of [9], we see that the new estimates are better even in the case of the wave equation (1.4) without lower order terms. In Theorems 6.1, 6.3 and 1.4 under different conditions, imposed on the coefficients of the equation (1.1), we present some singular *generalized solutions* which are smooth enough away from the point O, while at the point O they have power type singularity of the type $(x_1^2 + x_2^2 + t^2)^{-n/2}$. More precisely, we formulate and prove the following theorem.

Theorem 1.4. *Let the coefficients b_1, b_2, b be constants, $c(x_1,x_2,t) = c(|x|,t) \in C^1(\bar\Omega_0)$, $4c \le b_1^2 + b_2^2 - b^2$ in $\bar\Omega_0$ and $\alpha = \alpha(|x|) \in C^1(0,1] \cap C[0,1]$. Then for each $n \in \mathbb{N}$ there exists a function $f_n(x_1,x_2,t) \in C^{n-2}(\bar\Omega_0) \cap C^\infty(\Omega_0)$, for which* *the corresponding generalized solution u_n of problem P_α belongs to $C^2(\bar{\Omega}_0\backslash O)$ and satisfies the estimate*

$$|u_n(x_1,x_2,|x|)| \geq \frac{1}{2}|u_n(2x_1,2x_2,0)| + |x|^{-n}|\cos n(\arctan\frac{x_2}{x_1})|. \tag{1.10}$$

On the other hand, in Theorems 6.1 and 6.3 the coefficients of the equation (1.1) are nonconstant.

Finally, in Theorem 7.1 for the wave equation (1.4) we present two-sided estimates for the singularities of the *generalized solution* of Problem P_α. In particular, the exact behavior for the singular solution $u_n(x_1,x_2,t)$ around O is $(x_1^2+x_2^2+t^2)^{-n/2}\cos n(\arctan\frac{x_2}{x_1})$.

Remark 1.5. Actually, all these results state some conditions on the coefficients of the equation (1.1), under which we do not have a positive answer to Open Question 1. For example, Theorem 7.1 ensures that for any parameter $\alpha(x)$, involved to the boundary condition (1.2) on Σ_0, there are infinitely many singular solutions of the wave equation (1.4). That means that, it is impossible to give a positive answer to Open Question 1, by using the wave operator only. Possibly, it is necessary to ask some of the nonzero lower order perturbations of the wave equation to be involved to the more general equation (1.1). This is one of the reasons of the existence of the present paper, where we use and developed further the ideas of [9]. Note also that, each one of the singular solutions has a strong singularity at the vertex O of the cone $\Sigma_{2,0}$. The singularities of the generalized solutions do not propagate in the direction of the bicharacteristics on the characteristic cone. It is traditionally assumed that the wave equation, with right-hand side sufficiently smooth in $\bar{\Omega}_0$, cannot have a solution with an isolated singular point. For results, concerning the propagation of singularities for second order operators, see Hörmander [11, Chapter 24.5].

We conclude this section with the following four more questions.

Open Questions:

(1) Find the exact behavior of all singular solutions at the point O, different from those ones which appear in Theorems 6.1, 6.3 and 7.1.
(2) Find appropriate conditions for the function f under which the Problem P_α, even for the wave equation, has only classical solutions. We do not know any kind of such results even for Problem $P2$.
(3) In all results, concerning the existence of singular solutions (except Theorem 1.4), we assume that $a_2 \equiv 0$. Is it possible to find any singular solution, when $a_2 \neq 0$? Even in the case $a_2 \neq 0$, Theorem 5.3 ensures the existence of a generalized solution for any function f, but we do not know the behavior of a such solution at $(0,0,0)$.
(4) From the a priori estimates, obtained in Theorems 5.2–5.4 for all solutions of Problem P_α, including singular ones, it follows that, as $\rho \to 0$, none of these solutions can grow up faster than exponential one. The arising question is: are there singular solutions of Problem P_α with exponential growth as $\rho \to 0$ or any such solution is of polynomial growth satisfying (1.10)?

In the case of Problem $P1$, for the wave equation (1.1) the answer to Open Questions 1, 2 and 4 above can be found in [22].

2. Preliminaries

In this section we consider (1.1) in polar coordinates $x_1 = \varrho\cos\varphi$, $x_2 = \varrho\sin\varphi$ and t

$$Lu = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} + a_1 u_\varrho + a_2 u_\varphi + bu_t + cu = f \tag{2.1}$$

in a simply connected region

$$\Omega_\varepsilon := \big\{(\varrho,\varphi,t) : 0 < t < (1-\varepsilon)/2,\ 0 \le \varphi < 2\pi,\ \varepsilon + t < \varrho < 1 - t\big\}, \tag{2.2}$$

$0 < \varepsilon < 1$, bounded by the disc $\Sigma_0 := \{(\varrho,\varphi,t) : t = 0, \varrho < 1\}$ and the characteristic surfaces of (2.1)

$$\Sigma_1 := \{(\varrho,\varphi,t) : 0 \le \varphi < 2\pi, \varrho = 1 - t\},$$
$$\Sigma_{2,\varepsilon} := \{(\varrho,\varphi,t) : 0 \le \varphi < 2\pi, \varrho = \varepsilon + t\}.$$

The coefficients a_1, a_2 depend on b_1, b_2 in an obvious way. We seek sufficient conditions for the existence and uniqueness of a generalized solution of the equation (2.1) with $f \in C(\bar\Omega_\varepsilon)$, which satisfies the following boundary conditions

$$P_\alpha : \quad u\big|_{\Sigma_1\cap\partial\Omega_\varepsilon} = 0, \quad [u_t + \alpha u]\big|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0; \tag{2.3}$$

$$P^*_\alpha : \quad u\big|_{\Sigma_{2,\varepsilon}} = 0, \quad [u_t + (\alpha + b)u]\big|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0. \tag{2.4}$$

Here, for the sake of simplicity, we assume that all coefficients of (2.1) depend only on ϱ and t, and we set $\alpha(x) \equiv \alpha(|x|) = \alpha(\varrho) \in C^1(0,1]$. The problem P^*_α is the adjoint one to Problem P_α in Ω_ε.

Remark 2.1. In what follows, we consider the domain Ω_ε and its boundary in Cartesian coordinates. Nevertheless, for convenience we use the polar coordinates in the sense that the intersections $\varphi = 0$ and $\varphi = 2\pi$ do not belong to the boundary of Ω_ε and all the functions, which we use here, are considered as periodical ones.

Now, in order to obtain our results, we define the notion of a generalized solution as follows.

Definition 2.2. A function $u = u(\varrho,\varphi,t)$ is called a *generalized solution* of Problem P_α in Ω_ε, $\varepsilon > 0$, if:

(1) $u \in C^1(\bar\Omega_\varepsilon)$, $u\big|_{\Sigma_1\cap\partial\Omega_\varepsilon} = 0$; $[u_t + \alpha(\varrho)u]\big|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0$;

(2) The equality

$$\begin{aligned}&\int_{\Omega_\varepsilon} [u_t v_t - u_\varrho v_\varrho - \frac{1}{\varrho^2}u_\varphi v_\varphi + (a_1 u_\varrho + a_2 u_\varphi + bu_t + cu - f)v]\varrho\, d\varrho\, d\varphi\, dt \\ &= \int_{\Sigma_0\cap\partial\Omega_\varepsilon} \alpha(\varrho)uv\varrho\, d\varrho d\varphi\end{aligned} \tag{2.5}$$

holds for all

$$v \in V_\varepsilon := \{v \in C^1(\bar\Omega_\varepsilon) : [v_t + (\alpha + b)v]\big|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0, v\big|_{\Sigma_{2,\varepsilon}} = 0\}.$$

The following lemma describes the properties of generalized solutions of Problem P_α in Ω_ε.

Lemma 2.3. *Each generalized solution of Problem P_α in Ω_0 is also a generalized solution of the same problem in Ω_ε for $\varepsilon > 0$.*

The proof of this lemma follows from the proof in [9, Lemma 2.1]. In the special, but main case, when

$$f(\varrho,\varphi,t) = f_n^{(1)}(\varrho,t)\cos n\varphi + f_n^{(2)}(\varrho,t)\sin n\varphi, \tag{2.6}$$

we ask the generalized solution to be of the form

$$u(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi + u_n^{(2)}(\varrho,t)\sin n\varphi.$$

Then, in view of (2.1), we obtain the 2-D system

$$\begin{aligned} &\frac{1}{\varrho}(\varrho u^{(1)}_{n,\varrho})_\varrho - u^{(1)}_{n,tt} + a_1 u^{(1)}_{n,\varrho} + b u^{(1)}_{n,t} + (c - \frac{n^2}{\varrho^2}) u_n^{(1)} + n a_2 u_n^{(2)} = f_n^{(1)},\\ &\frac{1}{\varrho}(\varrho u^{(2)}_{n,\varrho})_\varrho - u^{(2)}_{n,tt} + a_1 u^{(2)}_{n,\varrho} + b u^{(2)}_{n,t} + (c - \frac{n^2}{\varrho^2}) u_n^{(2)} - n a_2 u_n^{(1)} = f_n^{(2)}. \end{aligned} \tag{2.7}$$

We consider this system in the domain

$$G_\varepsilon = \{(\varrho,t) : t > 0, \varepsilon + t < \varrho < 1 - t\}$$

which is bounded by the sets:

$$\begin{aligned} S_0 &= \{(\varrho,t) : t = 0, 0 < \varrho < 1\},\\ S_1 &= \{(\varrho,t) : \varrho = 1 - t\},\\ S_{2,\varepsilon} &= \{(\varrho,t) : \varrho = t + \varepsilon\}. \end{aligned} \tag{2.8}$$

In this case, for $u = (u^{(1)}, u^{(2)})(\varrho,t)$, the 2-D problem corresponding to P_α is $P_{\alpha,1}$:

$$\begin{gathered} \frac{1}{\varrho}(\varrho u^{(1)}_{\varrho})_\varrho - u^{(1)}_{tt} + a_1 u^{(1)}_{\varrho} + b u^{(1)}_{t} + (c - \frac{n^2}{\varrho^2}) u^{(1)} + n a_2 u^{(2)} = f^{(1)} \quad \text{in } G_\varepsilon,\\ \frac{1}{\varrho}(\varrho u^{(2)}_{\varrho})_\varrho - u^{(2)}_{tt} + a_1 u^{(2)}_{\varrho} + b u^{(2)}_{t} + (c - \frac{n^2}{\varrho^2}) u^{(2)} - n a_2 u^{(1)} = f^{(2)} \quad \text{in } G_\varepsilon,\\ u^{(i)}\big|_{S_1\cap\partial G_\varepsilon} = 0, \quad [u^{(i)}_t + \alpha(\varrho) u^{(i)}]\big|_{S_0\cap\partial G_\varepsilon} = 0, \quad i = 1,2. \end{gathered} \tag{2.9}$$

The generalized solution of the Problem $P_{\alpha,1}$ is as follows.

Definition 2.4. A function $u = (u^{(1)}, u^{(2)})(\varrho,t)$ is called a *generalized solution* of Problem $P_{\alpha,1}$ in G_ε, $\varepsilon > 0$, if:

(1) $u \in C^1(\bar G_\varepsilon)$, $[u^{(i)}_t + \alpha(\varrho) u^{(i)}]\big|_{S_0\cap\partial G_\varepsilon} = 0$, $u^{(i)}\big|_{S_1\cap\partial G_\varepsilon} = 0, i = 1,2$;

(2) The equalities

$$\begin{aligned} &\int_{G_\varepsilon} \big[u^{(1)}_t v_{1,t} - u^{(1)}_\varrho v_{1,\varrho} + \big(a_1 u^{(1)}_\varrho + b u^{(1)}_t + (c - \frac{n^2}{\varrho^2}) u^{(1)} + n a_2 u^{(2)} - f^{(1)}\big) v_1\big] \varrho d\varrho\, dt\\ &= \int_{S_0\cap\partial G_\varepsilon} \alpha(\varrho) u^{(1)} v_1 \varrho\, d\varrho,\\ &\int_{G_\varepsilon} [u^{(2)}_t v_{2,t} - u^{(2)}_\varrho v_{2,\varrho} + \big(a_1 u^{(2)}_\rho + b u^{(2)}_t + (c - \frac{n^2}{\varrho^2}) u^{(2)} - n a_2 u^{(1)} - f^{(2)}\big) v_2\big] \varrho d\varrho\, dt\\ &= \int_{S_0\cap\partial G_\varepsilon} \alpha(\varrho) u^{(2)} v_2 \varrho\, d\varrho \end{aligned} \tag{2.10}$$

hold for all

$$v_1, v_2 \in V^{(1)}_\varepsilon = \{v \in C^1(\bar G_\varepsilon) : [v_t + (\alpha + b) v]\big|_{S_0\cap\partial G_\varepsilon} = 0, v\big|_{S_{2,\varepsilon}\cap\partial G_\varepsilon} = 0\}.$$

Introducing a new function

$$z^{(i)}(\varrho,t)=\varrho^{\frac12}u^{(i)}(\varrho,t),\ i=1,2, \tag{2.11}$$

we transform the system (2.9) to the system

$$\begin{aligned}
&z^{(1)}_{\varrho\varrho}-z^{(1)}_{tt}+a_1z^{(1)}_{\varrho}+bz^{(1)}_{t}+\big(c-\frac{1}{2\varrho}a_1-\frac{4n^2-1}{4\varrho^2}\big)z^{(1)}+na_2z^{(2)}=\varrho^{\frac12}f^{(1)},\\
&z^{(2)}_{\varrho\varrho}-z^{(2)}_{tt}+a_1z^{(2)}_{\varrho}+bz^{(2)}_{t}+\big(c-\frac{1}{2\varrho}a_1-\frac{4n^2-1}{4\varrho^2}\big)z^{(2)}-na_2z^{(1)}=\varrho^{\frac12}f^{(2)}.
\end{aligned} \tag{2.12}$$

with the string operator in the main part. Substituting the new coordinates

$$\xi=1-\varrho-t,\quad \eta=1-\varrho+t, \tag{2.13}$$

from (2.12) we derive

$$\begin{aligned}
&U^{(1)}_{\xi\eta}-A_1U^{(1)}_{\xi}-B_1U^{(1)}_{\eta}-C_1U^{(1)}-D_1U^{(2)}=F^1(\xi,\eta)\quad \text{in } D_\varepsilon,\\
&U^{(2)}_{\xi\eta}-A_2U^{(2)}_{\xi}-B_2U^{(2)}_{\eta}-C_2U^{(2)}-D_2U^{(1)}=F^2(\xi,\eta)\quad \text{in } D_\varepsilon,
\end{aligned} \tag{2.14}$$

where $D_\varepsilon=\{(\xi,\eta):0<\xi<\eta<1-\varepsilon\}$ and for $i=1,2$:

$$\begin{aligned}
U^{(i)}(\xi,\eta)&=z^{(i)}(\varrho(\xi,\eta),t(\xi,\eta)),\\
F^{(i)}(\xi,\eta)&=\frac{1}{4\sqrt2}(2-\eta-\xi)^{\frac12}f^{(i)}(\varrho(\xi,\eta),t(\xi,\eta)).
\end{aligned} \tag{2.15}$$

Note that, in the case of the equation (2.1), the corresponding coefficients of the system (2.14) are

$$\begin{aligned}
&A_1=A_2=\frac14(a_1+b),B_1=B_2=\frac14(a_1-b),D_2=-D_1=\frac14na_2,\\
&C_1=C_2=\frac14\big(\frac{4n^2-1}{(2-\xi-\eta)^2}+\frac{a_1}{2-\xi-\eta}-c\big).
\end{aligned} \tag{2.16}$$

As we see, Problem $P_{\alpha,1}$ is reduced to the Darboux-Goursat problem for the system (2.14) with the same boundary conditions. That is, we consider the following question.

Problem $P_{\alpha,2}$. Find a solution (U^1,U^2) of the system (2.14) in D_ε with the boundary conditions

$$U^{(i)}(0,\eta)=0,\quad (U^{(i)}_\eta-U^{(i)}_\xi)(\xi,\xi)+\alpha(1-\xi)U^{(i)}(\xi,\xi)=0,\ i=1,2. \tag{2.17}$$

To investigate the smoothness or the singularity of a solution of the original 3-D problem P_α on $\Sigma_{2,0}$, we are seeking a classical solution of the corresponding 2-D problem $P_{\alpha,2}$ not only in the domain D_ε, but also in the domain

$$D^{(1)}_\varepsilon:=\{(\xi,\eta):0<\xi<\eta<1,0<\xi<1-\varepsilon\},\quad \varepsilon>0. \tag{2.18}$$

Clearly, $D_\varepsilon\subset D^{(1)}_\varepsilon$.

Consider now an appropriate boundary value problem for the system of equations whose coefficients are continuous in $\bar D^{(1)}_\varepsilon$, $\varepsilon>0$:

Problem $P_{\alpha,3}$. Find a solution (U^1,U^2) of the system

$$\begin{aligned}
&U^{(1)}_{\xi\eta}-A_1U^{(1)}_{\xi}-B_1U^{(1)}_{\eta}-C_1U^{(1)}-D_1U^{(2)}=F^1(\xi,\eta)\quad \text{in } \bar D^{(1)}_\varepsilon,\\
&U^{(2)}_{\xi\eta}-A_2U^{(2)}_{\xi}-B_2U^{(2)}_{\eta}-C_2U^{(2)}-D_2U^{(1)}=F^2(\xi,\eta)\quad \text{in } \bar D^{(1)}_\varepsilon,
\end{aligned} \tag{2.19}$$

with the boundary conditions

$$U^{(i)}(0,\eta)=0,(U^{(i)}_\eta-U^{(i)}_\xi)(\xi,\xi)+\alpha(1-\xi)U^{(i)}(\xi,\xi)=0, \tag{2.20}$$

for $i=1,2$ and $\xi\in(0,1-\varepsilon)$.

3. The system of integral equations corresponding to Problem $P_{\alpha,3}$

First of all, we construct an equivalent system of integral equations in such a way that any solution to Problem $P_{\alpha,3}$ is a solution of the constructed system and vice-verca. For this reason, following the ideas of [9], concerning the representation of the solutions of the Protter's problem for the wave equation, we will try to find a corresponding representation of the solutions in the case, where the equation (1.1) involves lower order terms. In this case, because of the appearance in the equation (2.1) of the term a_2u_φ, we have to deal not with a single scalar equation, but with a system of equations. In order to realize these ideas, for any $(\xi_0,\eta_0)\in D^{(1)}_\varepsilon$, we consider the sets

$$\Pi:=\{(\xi,\eta):0<\xi<\xi_0,\xi_0<\eta<\eta_0\},\quad T:=\{(\xi,\eta):0<\xi<\xi_0,\xi<\eta<\xi_0\}$$

and the following integrals:

$$\begin{aligned}
I^{(i)}_0&:=\iint_\Pi U^{(i)}_{\xi\eta}(\xi,\eta)\,d\xi d\eta=\int_0^{\xi_0}\int_{\xi_0}^{\eta_0}U^{(i)}_{\xi\eta}(\xi,\eta)\,d\eta\,d\xi\,,\\
I^{(i)}_1&:=\iint_T U^{(i)}_{\xi\eta}(\xi,\eta)\,d\xi d\eta=\int_0^{\xi_0}\int_{\xi}^{\xi_0}U^{(i)}_{\xi\eta}(\xi,\eta)\,d\eta\,d\xi\,.
\end{aligned}$$

As it has been shown in [9],

$$I^{(i)}_0+2I^{(i)}_1=U^{(i)}(\xi_0,\eta_0)-\int_0^{\xi_0}\alpha(1-\xi)U^{(i)}(\xi,\xi)\,d\xi.$$

Set $p^{(i)}:=U^{(i)}_\xi$, $q^{(i)}:=U^{(i)}_\eta$. Then, in view of the last relation and (2.19), we obtain

$$\begin{aligned}
U^{(1)}(\xi_0,\eta_0)&=\int_0^{\xi_0}\int_{\xi_0}^{\eta_0}[F^1+A_1p^{(1)}+B_1q^{(1)}+C_1U^{(1)}+D_1U^{(2)}](\xi,\eta)\,d\eta\,d\xi\\
&\quad+2\int_0^{\xi_0}\int_0^{\eta}[F^1+A_1p^{(1)}+B_1q^{(1)}+C_1U^{(1)}+D_1U^{(2)}](\xi,\eta)\,d\xi\,d\eta\\
&\quad+\int_0^{\xi_0}\alpha(1-\xi)U^{(1)}(\xi,\xi)\,d\xi,\quad\text{for }(\xi_0,\eta_0)\in\bar D^{(1)}_\varepsilon,
\end{aligned}\tag{3.1}$$

$$\begin{aligned}
U^{(2)}(\xi_0,\eta_0)&=\int_0^{\xi_0}\int_{\xi_0}^{\eta_0}[F^2+A_2p^{(2)}+B_2q^{(2)}+C_2U^{(2)}+D_2U^{(1)}](\xi,\eta)\,d\eta\,d\xi\\
&\quad+2\int_0^{\xi_0}\int_0^{\eta}[F^2+A_2p^{(2)}+B_2q^{(2)}+C_2U^{(2)}+D_2U^{(1)}](\xi,\eta)\,d\xi\,d\eta\\
&\quad+\int_0^{\xi_0}\alpha(1-\xi)U^{(2)}(\xi,\xi)\,d\xi,\quad\text{for }(\xi_0,\eta_0)\in\bar D^{(1)}_\varepsilon,
\end{aligned}\tag{3.2}$$

which is the first couple of desired integral equations. From (3.1) and (3.2) we derive for the first derivatives $p^{(i)}$ and $q^{(i)}$ $(i=1,2)$ the next four integral equations:

$$\begin{aligned} p^{(1)}(\xi_0,\eta_0) = &\int_0^{\xi_0} [F^1 + A_1p^{(1)} + B_1q^{(1)} + C_1U^{(1)} + D_1U^{(2)}](\xi,\xi_0)\,d\xi \\ &+ \int_{\xi_0}^{\eta_0} [F^1 + A_1p^{(1)} + B_1q^{(1)} + C_1U^{(1)} + D_1U^{(2)}](\xi_0,\eta)\,d\eta \\ &+\alpha(1-\xi_0)U^{(1)}(\xi_0,\xi_0), \end{aligned} \tag{3.3}$$

$$q^{(1)}(\xi_0,\eta_0) = \int_0^{\xi_0} [F^1 + A_1p^{(1)} + B_1q^{(1)} + C_1U^{(1)} + D_1U^{(2)}](\xi,\eta_0)\,d\xi, \tag{3.4}$$

$$\begin{aligned} p^{(2)}(\xi_0,\eta_0) = &\int_0^{\xi_0} [F^2 + A_2p^{(2)} + B_2q^{(2)} + C_2U^{(2)} + D_2U^{(1)}](\xi,\xi_0)\,d\xi \\ &+ \int_{\xi_0}^{\eta_0} [F^2 + A_2p^{(2)} + B_2q^{(2)} + C_2U^{(2)} + D_2U^{(1)}](\xi_0,\eta)\,d\eta \\ &+ \alpha(1-\xi_0)U^{(2)}(\xi_0,\xi_0), \end{aligned} \tag{3.5}$$

$$q^{(2)}(\xi_0,\eta_0) = \int_0^{\xi_0} [F^2 + A_2p^{(2)} + B_2q^{(2)} + C_2U^{(2)} + D_2U^{(1)}](\xi,\eta_0)\,d\xi. \tag{3.6}$$

Now, we set

$$\begin{aligned} &M := \max\Big(\sup_{D_\varepsilon^{(1)}}|F^1|, \sup_{D_\varepsilon^{(1)}}|F^2|\Big), \quad M_\alpha := \sup_{[0,1-\varepsilon]}|\alpha(\xi)| \\ &c(\varepsilon) := \max_{i=1,2}\Big\{\sup_{D_\varepsilon^{(1)}}|A_i|,\ \sup_{D_\varepsilon^{(1)}}|B_i|,\ \sup_{D_\varepsilon^{(1)}}|C_i|,\ \sup_{D_\varepsilon^{(1)}}|D_i|\Big\}, \end{aligned} \tag{3.7}$$

and formulate the following results.

Theorem 3.1. *Let $F^i, A_i, B_i, C_i, D_i \in C(\bar{D}_\varepsilon^{(1)})$, $i=1,2$, $\varepsilon>0$. Then there exists a classical solution $(U^{(1)},U^{(2)}) \in C^1(\bar{D}_\varepsilon^{(1)})$ of the Problem $P_{\alpha,3}$ for which $U_{\xi\eta}^{(i)} \in C(\bar{D}_\varepsilon^{(1)}), i=1,2$ and*

$$\begin{aligned} &|U^{(i)}(\xi_0,\eta_0)| \le M[4c(\varepsilon)+M_\alpha]^{-2}\exp\{8c(\varepsilon)+2M_\alpha\} \quad \text{in } D_\varepsilon^{(1)},\ i=1,2, \\ &\sup_{D_\varepsilon^{(1)}}\{|U_\xi^{(i)}|,|U_\eta^{(i)}|\} \le M[4c(\varepsilon)+M_\alpha]^{-1}\exp\{8c(\varepsilon)+2M_\alpha\},\ i=1,2. \end{aligned} \tag{3.8}$$

Proof. To get our results, we will solve the system of integral equations (3.1)–(3.6). For this reason we use sequence of successive approximations $(U_m^{(i)}, p_m^{(i)}, q_m^{(i)}), m =$

$1, 2, \dots,$ defined by the formulae

$$\begin{aligned} U^{(i)}_{m+1}(\xi_0,\eta_0) &= \int_0^{\xi_0}\int_{\xi_0}^{\eta_0} E^{(i)}_m(\xi,\eta)\,d\eta\,d\xi + 2\int_0^{\xi_0}\int_0^{\eta} E^{(i)}_m(\xi,\eta)\,d\xi\,d\eta \\ &\quad + \int_0^{\xi_0}\alpha(1-\xi)U^{(i)}_m(\xi,\xi)\,d\xi, \quad i=1,2; \quad m=0,1,2\dots \\ p^{(i)}_{m+1}(\xi_0,\eta_0) &= \int_0^{\xi_0} E^{(i)}_m(\xi,\xi_0)\,d\xi + \int_{\xi_0}^{\eta_0} E^{(i)}_m(\xi_0,\eta)\,d\eta \\ &\quad + \alpha(1-\xi_0)U^{(i)}_m(\xi_0,\xi_0), \quad i=1,2; \quad m=0,1,2\dots \\ q^{(i)}_{m+1}(\xi_0,\eta_0) &= \int_0^{\xi_0} E^{(i)}_m(\xi,\eta_0)\,d\xi, \quad i=1,2; \quad m=0,1,2\dots \\ U^{(i)}_0(\xi_0,\eta_0) &= 0, \quad p^{(i)}_0(\xi_0,\eta_0)=0, \\ q^{(i)}_0(\xi_0,\eta_0) &= 0, \quad i=1,2, \quad \text{in } D^1_\varepsilon, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} E^{(1)}_m(\xi,\eta) &:= [F^1 + A_1 p^{(1)}_m + B_1 q^{(1)}_m + C_1 U^{(1)}_m + D_1 U^{(2)}_m](\xi,\eta), \\ E^{(2)}_m(\xi,\eta) &:= [F^2 + A_2 p^{(2)}_m + B_2 q^{(2)}_m + C_2 U^{(2)}_m + D_2 U^{(1)}_m](\xi,\eta). \end{aligned}$$

We will show that each of the functions $U^{(i)}_m, p^{(i)}_m$ and $q^{(i)}_m, i=1,2,$ is continuous in $\bar D^{(1)}_\varepsilon$ and for any $(\xi_0,\eta_0)\in \bar D^{(1)}_\varepsilon$ and $m\in\mathbb{N}$

$$|(U^{(i)}_m - U^{(i)}_{m-1})(\xi_0,\eta_0)| \le M\frac{[4c(\varepsilon)+M_\alpha]^{m-1}}{(m+1)!}\,(\xi_0+\eta_0)^{m+1}, \tag{3.10}$$

$$\begin{aligned} &\max\Big\{|(p^{(i)}_m - p^{(i)}_{m-1})|(\xi_0,\eta_0), |(q^{(i)}_m - q^{(i)}_{m-1})|(\xi_0,\eta_0)\Big\} \\ &\le M\frac{[4c(\varepsilon)+M_\alpha]^{m-1}}{m!}(\xi_0+\eta_0)^m \end{aligned} \tag{3.11}$$

Indeed, by induction: 1) For $m=1$

$$U^{(i)}_1(\xi_0,\eta_0) = \int_0^{\xi_0}\int_\xi^{\eta_0} F^{(i)}(\xi,\eta)\,d\eta\,d\xi + \int_0^{\xi_0}\int_\xi^{\xi_0} F^{(i)}(\xi,\eta)\,d\eta\;d\xi\,,$$

and hence

$$|U^{(1)}(\xi_0,\eta_0)| \le M\xi_0\eta_0 \le M(\xi_0+\eta_0)^2/2.$$

Similarly one can estimate $p^{(i)}_1$ and $q^{(i)}_1$. 2) Let now, by the induction hypothesis, (3.10) and (3.11) be satisfied for some $m\in\mathbb{N}$. Then for $i=1,2$

$$|(U^{(i)}_m - U^{(i)}_{m-1})(\xi_0,\eta_0)| \le M\frac{[4c(\varepsilon)+M_\alpha]^{m-1}}{m!}(\xi_0+\eta_0)^m := Q_m(\xi_0+\eta_0)^m.$$

It follows that

$$\begin{aligned}
&|(U^{(i)}_{m+1} - U^{(i)}_{m})(\xi_0, \eta_0)| \\
&\le Q_m\Big[4c(\varepsilon)\Big(\int_0^{\xi_0}\int_{\xi_0}^{\eta_0}(\xi+\eta)^m\,d\eta\,d\xi + 2\int_0^{\xi_0}\int_0^{\eta}(\xi+\eta)^m\,d\xi\,d\eta\Big) \\
&\quad + \frac{M_\alpha}{m+1}\int_0^{\xi_0}(2\xi)^{m+1}\,d\xi\Big] \\
&\le \frac{Q_m}{(m+1)(m+2)}\big[4c(\varepsilon)\big((\xi_0+\eta_0)^{m+2} - \eta_0^{m+2} - \xi_0^{m+2}\big) + M_\alpha(2\xi_0)^{m+2}\big] \\
&\le \frac{Q_{m+1}}{(m+2)}(\xi_0+\eta_0)^{m+2}.
\end{aligned}$$

By (3.3) and (3.5), we have

$$\begin{aligned}
&|(p^{(i)}_{m+1} - p^{(i)}_{m})|(\xi_0, \eta_0) \\
&\le \frac{Q_m}{m+1}\big[4c(\varepsilon)\big((\xi_0+\eta_0)^{m+1} - \xi_0{}^{m+1}\big) + M_\alpha(2\xi_0)^{m+1}\big] \le Q_{m+1}(\xi_0+\eta_0)^{m+1}.
\end{aligned}$$

A similar estimate holds for $(q^{(i)}_{m+1} - q^{(i)}_{m})$. So that (3.10) and (3.11) hold and hence the uniform convergence of the sequences $\{U^{(i)}_m(\xi,\eta)\}_{m\in\mathbb{N}}$, $\{p^{(i)}_m(\xi,\eta)\}_{m\in\mathbb{N}}$ and $\{q^{(i)}_m(\xi,\eta)\}_{m\in\mathbb{N}}$ in $\bar{D}^{(1)}_\varepsilon$ follows. For the limit functions $U^{(i)}, p^{(i)}, q^{(i)} \in C(\bar{D}^{(1)}_\varepsilon)$ we obtain the integral equalities (3.1)–(3.6) with the obvious condition $U^{(i)}(0,\eta_0) = 0$. From the integral equalities (3.1)–(3.6) it follows that $p^{(i)} = U^{(i)}_\xi$ and $q^{(i)} = U^{(i)}_\eta$ in $\bar{D}^{(1)}_\varepsilon$. Therefore, $U^{(i)} \in C^1(\bar{D}^{(1)}_\varepsilon), i = 1, 2$.

Also, in view of (3.10), we see that

$$\begin{aligned}
|(U^{(i)}(\xi_0,\eta_0)| &= \Big|\sum_{m=0}^{\infty}(U^{(i)}_{m+1} - U^{(i)}_{m})(\xi_0,\eta_0)\Big| \le M\sum_{m=0}^{\infty}\frac{[4c(\varepsilon)+M_\alpha]^m}{(m+2)!}(\xi_0+\eta_0)^{m+2} \\
&\le M[4c(\varepsilon)+M_\alpha]^{-2}\exp\{8c(\varepsilon)+2M_\alpha\}, \quad i = 1, 2.
\end{aligned}$$

So, using (3.11), for the derivatives $U^{(i)}_{\xi_0}(\xi_0,\eta_0)$ and $U^{(i)}_{\eta_0}(\xi_0,\eta_0)$ we get the estimates

$$\begin{aligned}
|U^{(i)}_{\xi_0}(\xi_0,\eta_0)| &= \Big|\sum_{m=0}^{\infty}(p^{(i)}_{m+1} - p^{(i)}_{m})(\xi_0,\eta_0)\Big| \le M\sum_{m=0}^{\infty}\frac{[4c(\varepsilon)+M_\alpha]^m}{(m+1)!}(\xi_0+\eta_0)^{m+1} \\
&\le M[4c(\varepsilon)+M_\alpha]^{-1}\exp\{8c(\varepsilon)+2M_\alpha\}, \quad i = 1, 2
\end{aligned}$$

and

$$|U^{(i)}_{\eta_0}(\xi_0,\eta_0)| \le M[4c(\varepsilon)+M_\alpha]^{-1}\exp\{8c(\varepsilon)+2M_\alpha\}, \quad i = 1, 2,$$

which shows (3.8). Also, by (3.3)–(3.6), it follows that

$$\begin{aligned}
U^{(1)}_{\xi_0\eta_0} &\equiv U^{(1)}_{\eta_0\xi_0} = F^1 + A_1U^{(1)}_{\xi_0} + B_1U^{(1)}_{\eta_0} + C_1U^{(1)} + D_1U^{(2)}, \\
U^{(2)}_{\xi_0\eta_0} &\equiv U^{(2)}_{\eta_0\xi_0} = F^2 + A_2U^{(2)}_{\xi_0} + B_2U^{(2)}_{\eta_0} + C_2U^{(2)} + D_2U^{(1)}.
\end{aligned}$$

Thus, the vector-valued function $U(\xi_0,\eta_0)$ is a solution to the system (2.19) and $U_{\xi\eta} \in C(\bar{D}^{(1)}_\varepsilon)$. Finally, using representations (3.3)–(3.6) for the first derivatives of $U^{(i)}$, we conclude that each function $U^{(i)}(\xi_0,\eta_0)$ satisfies the boundary condition (2.20) of the Problem $P_{\alpha,3}$ for $\eta = \xi$. □

The next lemma is very important for the investigation of the singularity of a generalized solution of Problem P_α.

Lemma 3.2. *Let* $F^i, A_i, B_i, C_i, D_i \in C(\bar{D}_\varepsilon^{(1)})$*,* $i = 1, 2$*,*

$$A_i \geq 0, B_i \geq 0, C_i \geq 0, D_i \geq 0,\ \alpha(1-\xi) \geq 0 \text{ in } \bar{D}_\varepsilon^{(1)},\ \ i = 1, 2 \tag{3.12}$$

and

$$(\mathbf{a})\ p_1^{(i)} \geq 0 \ \text{ and } \ q_1^{(i)} \geq 0, \quad \text{or} \quad (\mathbf{b})\ F^{(i)} \geq 0 \text{ in } \bar{D}_\varepsilon^{(1)}, \quad i = 1, 2. \tag{3.13}$$

Then for the solution $(U^{(1)}, U^{(2)})$ *of Problem* $P_{\alpha,3}$ *(already found in Theorem 3.1) we have*

$$U^{(i)}(\xi,\eta) \geq 0, \quad U_\eta^{(i)}(\xi,\eta) \geq 0, \quad U_\xi^{(i)}(\xi,\eta) \geq 0 \quad \text{for } (\xi,\eta) \in \bar{D}_\varepsilon^{(1)},\ i = 1, 2. \tag{3.14}$$

Proof. First, suppose that the condition (**b**) is satisfied. Then, in view of (3.9), for $(\xi_0, \eta_0) \in \bar{D}_\varepsilon^{(1)}$ we have

$$U_1^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0}\int_{\xi_0}^{\eta_0} F^{(i)}(\xi,\eta)\,d\eta\,d\xi + 2\int_0^{\xi_0}\int_0^{\eta} F^{(i)}(\xi,\eta)\,d\xi\,d\eta \geq 0, \tag{3.15}$$

$$p_1^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0} F^{(i)}(\xi,\xi_0)\ d\xi + \int_{\xi_0}^{\eta_0} F^{(i)}(\xi_0,\eta)\ d\eta \geq 0, \tag{3.16}$$

$$q_1^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0} F^{(i)}(\xi,\eta_0)\,d\xi \geq 0, \quad i = 1, 2. \tag{3.17}$$

Thus, the condition (**b**) is stronger, than (**a**). Assume now that (**a**) $p_1^{(i)} \geq 0$ and $q_1^{(i)} \geq 0$ in $\bar{D}_\varepsilon^{(1)}$. Then, using (3.17), we find that $U_1^{(i)} \geq 0$. Thus, in both cases (**a**) or (**b**), the inequalities (3.15) - (3.17) hold. Suppose next that for some $m \in \mathbb{N}$

$$(U_m^{(i)} - U_{m-1}^{(i)}) \geq 0,\ (p_m^{(i)} - p_{m-1}^{(i)}) \geq 0,\ (q_m^{(i)} - q_{m-1}^{(i)}) \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)},\ i = 1, 2.$$

Then

$$\begin{aligned} E_m^{(i)} - E_{m-1}^{(i)} = & A_i(p_m^{(i)} - p_{m-1}^{(i)}) + B_i(q_m^{(i)} - q_{m-1}^{(i)}) + C_i(U_m^{(i)} - U_{m-1}^{(i)}) \\ & + D_i(U_m^{(i+1)} - U_{m-1}^{(i+1)}) \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}, \quad i = 1, 2, \end{aligned}$$

where we denote $U_m^{(3)} := U_m^{(1)}$. Therefore, we see that

$$\begin{aligned} (U_{m+1}^{(i)} - U_m^{(i)})(\xi_0,\eta_0) = & \int_0^{\xi_0}\int_{\xi_0}^{\eta_0} (E_m^{(i)} - E_{m-1}^{(i)})(\xi,\eta)\,d\eta\,d\xi \\ & + 2\int_0^{\xi_0}\int_0^{\eta} (E_m^{(i)} - E_{m-1}^{(i)})(\xi,\eta)\,d\xi\,d\eta \\ & + \int_0^{\xi_0} \alpha(1-\xi)(U_m^{(i)} - U_{m-1}^{(i)})(\xi,\xi)\,d\xi \geq 0. \end{aligned}$$

In the same manner,

$$\begin{aligned} (p_{m+1}^{(i)} - p_m^{(i)})(\xi_0,\eta_0) = & \int_0^{\xi_0} (E_m^{(i)} - E_{m-1}^{(i)})(\xi,\xi_0)\,d\xi + \int_{\xi_0}^{\eta_0} (E_m^{(i)} - E_{m-1}^{(i)})(\xi_0,\eta)\ d\eta \\ & + \alpha(1-\xi_0)(U_m^{(i)} - U_{m-1}^{(i)})(\xi_0,\xi_0) \geq 0, \end{aligned}$$

$$(q_{m+1}^{(i)} - q_m^{(i)})(\xi_0,\eta_0) = \int_0^{\xi_0} (E_m^{(i)} - E_{m-1}^{(i)})(\xi,\eta_0)\,d\xi \geq 0.$$

Finally, by induction, we conclude that

$$
\begin{gathered}
U^{(i)}(\xi_0,\eta_0)=\sum_{m=0}^{\infty}(U^{(i)}_{m+1}-U^{(i)}_{m})(\xi_0,\eta_0)\geq 0,\\
p^{(i)}(\xi_0,\eta_0)\geq 0,\quad q^{(i)}(\xi_0,\eta_0)\geq 0,\quad (\xi_0,\eta_0)\in\bar{D}^{(1)}_{\varepsilon}.
\end{gathered}
$$

□

Remark 3.3. Note here that Lemma 3.2, which is actually a maximum principle for Problem $P_{\alpha,3}$, describes the behavior of the system (2.19) around the point $(1,1)$. Thus, this lemma becomes particularly useful in Sections 6 and 7 in finding singular solutions of the equation (2.1). None the less, when the equation (2.1) transforms to the system (2.14), by (2.16), we see that $D_2=-D_1=na_2/4$. Since, in view of Lemma 3.2, $D_1\geq 0$ and $D_2\geq 0$, it should be $a_2\equiv 0$. Because of this fact, we are able to find singular solutions only when $a_2\equiv 0$ (see also Introduction, Open Questions, 3).

As a consequence of Theorem 3.1 and representations (3.3)–(3.6), we have the following smoothness result:

Theorem 3.4. *Let $F^i,A_i,B_i,C_i,D_i\in C^1(\bar{D}^{(1)}_{\varepsilon})$, $i=1,2$, $\varepsilon>0$. Then there exists a classical solution $U\in C^2(\bar{D}^{(1)}_{\varepsilon})$ of Problem $P_{\alpha,3}$.*

Proof. Since we have already shown that

$$
p^{(1)}_{\eta}(\xi_0,\eta_0)\equiv q^{(1)}_{\xi}(\xi_0,\eta_0)=[F^1+A_1p^{(1)}+B_1q^{(1)}+C_1U^{(1)}+D_1U^{(2)}](\xi_0,\eta_0) \tag{3.18}
$$

and that similar representations for $p^{(2)}_{\eta}$ and $q^{(2)}_{\xi}$ hold, we have to prove only the fact that $p^{(i)}_{\xi}$ and $q^{(i)}_{\eta}$ exist and belong to $C(\bar{D}^{(1)}_{\varepsilon})$. Indeed, to do this, we observe the following:

1. For fixed η_0 the equality (3.18) is a linear ODE for the function $q^{(1)}(\xi_0,\eta_0)$. So, using the well known formula for the solution with the initial Cauchy data $q^{(1)}(0,\eta_0)=0$ from (3.4), we find that

$$
q^{(1)}(\xi_0,\eta_0)=\int_0^{\xi_0}[F^1+A_1p^{(1)}+C_1U^{(1)}+D_1U^{(2)}](\xi,\eta_0)\exp\Big(\int_{\xi}^{\xi_0}B_1(\tau,\eta_0)d\tau\Big)d\xi. \tag{3.19}
$$

Since $F^1,A_1,B_1,C_1,D_1,U^{(1)},U^{(2)}\in C^1(\bar{D}^{(1)}_{\varepsilon})$ and $p^{(1)},p^{(1)}_{\eta}\in C(\bar{D}^{(1)}_{\varepsilon})$, by (3.19), we conclude that $q^{(1)}\in C^1(\bar{D}^{(1)}_{\varepsilon})$.

2. For fixed ξ_0 the equality (3.18) is a linear ODE for the function $p^{(1)}(\xi_0,\eta_0)$. So, arguments similar to those above lead to

$$
\begin{aligned}
p^{(1)}(\xi_0,\eta_0)=&G_1(\xi_0)\exp\Big(\int_{\xi_0}^{\eta_0}A_1(\xi_0,\eta)d\eta\Big)\\
&+\int_{\xi_0}^{\eta_0}\Big[F^1+B_1q^{(1)}+C_1U^{(1)}+D_1U^{(2)}\Big](\xi_0,\eta)\\
&\times\exp\Big(\int_{\eta}^{\eta_0}A_1(\xi_0,\tau)d\tau\Big)d\eta.
\end{aligned} \tag{3.20}
$$

The function $G_1(\xi_0)$, which is defined implicitly by (3.3), is of the form

$$G_1(\xi_0) = \int_0^{\xi_0} \big[F^1 + A_1 p^{(1)} + B_1 q^{(1)} + C_1 U^{(1)} + D_1 U^{(2)}\big](\xi, \xi_0) d\xi + \alpha(1-\xi_0) U^{(1)}(\xi_0, \xi_0).$$

Obviously $G_1 \in C^1(\bar{D}_\varepsilon^{(1)})$, because F^1, A_1, B_1, C_1, D_1, α, $U^{(1)}$, $U^{(2)}$, $q^{(1)} \in C^1(\bar{D}_\varepsilon^{(1)})$ and $p^{(1)}, p_\eta^{(1)} \in C(\bar{D}_\varepsilon^{(1)})$. Finally, by (3.20), we see that $p^{(1)} \in C^1(\bar{D}_\varepsilon^{(1)})$. □

Remark 3.5. By studying a solution of the Problem $P_{\alpha,3}$ in the domain $\bar{D}_\varepsilon^{(1)}$, we are actually investigating the behavior of the solution of Problem $P_{\alpha,2}$ in the domain $\bar{D}_\delta$, when $\delta \to 0$, around the line $\eta = 1$. It is easy to show that, for any $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, the solutions of these two problems coincide in their common domain $\bar{D}_\varepsilon^{(1)} \cap \bar{D}_\delta$.

4. Existence and uniqueness theorems for the 2-D Problem $\mathbf{P}_{\alpha,1}$

Consider the 2-D problem $P_{\alpha,1}$:

$$\begin{gathered}
\frac{1}{\varrho}(\varrho u_\varrho^{(1)})_\varrho - u_{tt}^{(1)} + a_1 u_\varrho^{(1)} + b u_t^{(1)} + (c - \frac{n^2}{\varrho^2}) u^{(1)} + n a_2 u^{(2)} = f^{(1)} \text{ in } G_\varepsilon, \\
\frac{1}{\varrho}(\varrho u_\varrho^{(2)})_\varrho - u_{tt}^{(2)} + a_1 u_\varrho^{(2)} + b u_t^{(2)} + (c - \frac{n^2}{\varrho^2}) u^{(2)} - n a_2 u^{(1)} = f^{(2)} \text{ in } G_\varepsilon, \\
u^{(i)}\big|_{S_1 \cap \partial G_\varepsilon} = 0, \quad [u_t^{(i)} + \alpha(\varrho) u^{(i)}]\big|_{S_0 \cap \partial G_\varepsilon} = 0, \quad i = 1,2.
\end{gathered} \tag{4.1}$$

Note that, the generalized solution of the problem $P_{\alpha,1}$ in the domain G_ε , $\varepsilon \in (0,1)$, was defined by Definition 2.4.

Theorem 4.1. *Let $a_1, a_2, b, c, f^{(1)}, f^{(2)} \in C^1(\bar{G}_0 \setminus (0,0))$. Then there exists a generalized solution $u = (u^{(1)}, u^{(2)}) \in C^2(\bar{G}_0 \setminus (0,0))$ of problem $P_{\alpha,1}$ in G_0, which is a classical solution of the problem $P_{\alpha,1}$ in any domain G_ε, $\varepsilon \in (0,1)$.*

Proof. In view of (2.11) and (2.13), i.e. $z(\varrho,t) = \varrho^{1/2} u(\varrho,t)$ and $\xi = 1 - \varrho - t$, $\eta = 1 - \varrho + t$, we introduce the function

$$U^{(i)}(\xi,\eta) = z^{(i)}(\varrho(\xi,\eta), t(\xi,\eta)).$$

Then Problem $P_{\alpha,1}$, in the new terms, becomes $P_{\alpha,2}$, i.e.

$$\begin{gathered}
U_{\xi\eta}^{(1)} - A_1 U_\xi^{(1)} - B_1 U_\eta^{(1)} - C_1 U^{(1)} - D_1 U^{(2)} = F^1(\xi,\eta) \text{ in } D_\varepsilon, \\
U_{\xi\eta}^{(2)} - A_2 U_\xi^{(2)} - B_2 U_\eta^{(2)} - C_2 U^{(2)} - D_2 U^{(1)} = F^2(\xi,\eta) \text{ in } D_\varepsilon,
\end{gathered} \tag{4.2}$$

$$U^{(i)}(0,\eta) = 0, (U_\eta^{(i)} - U_\xi^{(i)})(\xi,\xi) + \alpha(1-\xi) U^{(i)}(\xi,\xi) = 0, \quad i = 1,2, \tag{4.3}$$

where the connection between the coefficients is given by (2.16). For each fixed $\varepsilon \in (0,1)$ Theorem 3.4 ensures the existence of a classical solution $(U^{(1)}, U^{(2)}) \in C^2(\bar{D}_\varepsilon^{(1)})$ of the problem $P_{\alpha,3}$. More precisely, for any fixed $\varepsilon_1, \varepsilon_2$ with $0 < \varepsilon_1 < \varepsilon_2 < 1$ the corresponding vector-valued solution U_{ε_2} is a restriction of U_{ε_1} in the region D_{ε_2}. So, essentially we have a function of class $C^2\left(\bar{D}_0 \backslash (0,0)\right)$, which in any region D_ε coincides with the corresponding solution U_ε and is a classical solution of Problem $P_{\alpha,3}$. We remark that the inverse transformations (2.11) and (2.13) lead to a vector-valued function $(u^{(1)}, u^{(2)}) \in C^2\left(\bar{G}_0 \backslash (0,0)\right)$, which is a classical solution of Problem $P_{\alpha,1}$ in each G_ε. This solution is also a generalized solution of the same problem in G_0, because for each concrete test function $v \in V_0$ there is an $\varepsilon_v > 0$ for which $v \equiv 0$ in $G_0 \setminus G_{\varepsilon_v}$ and (1.8) coincides with (2.5). The proof of the theorem is complete. □

Theorem 4.2. *Let $a_1, a_2, b, c \in C^1(\bar{G}_0 \setminus (0,0))$. Then for each fixed $\varepsilon \in (0,1)$ there exists at most one generalized solution of the problem $P_{\alpha,1}$ in G_ε.*

Proof. Let $(u_1^{(1)}, u_1^{(2)})$ and $(u_2^{(1)}, u_2^{(2)})$ be two generalized solutions of $P_{\alpha,1}$ in G_ε. Then for $u^{(i)} := u_1^{(i)} - u_2^{(i)}$, $i = 1, 2$, we see that

(1) $u^{(i)} \in C^1(\bar{G}_\varepsilon)$, $[u_t^{(i)} + \alpha(\varrho)u^{(i)}]\big|_{S_0 \cap \partial G_\varepsilon} = 0$, $u^{(i)}\big|_{S_1 \cap \partial G_\varepsilon} = 0$, $i = 1, 2$;

(2) The equalities

$$\begin{aligned}
&\int_{G_\varepsilon} \Big[u_t^{(1)} v_t^{(1)} - u_\varrho^{(1)} v_\varrho^{(1)} + \big(a_1 u_\varrho^{(1)} + b u_t^{(1)} + (c - \frac{n^2}{\varrho^2}) u^{(1)} + n a_2 u^{(2)}\big) v^{(1)}\Big] \varrho d\varrho\, dt \\
&= \int_{S_0 \cap \partial G_\varepsilon} \alpha(\varrho) u^{(1)} v^{(1)} \varrho\, d\varrho, \\
&\int_{G_\varepsilon} \Big[u_t^{(2)} v_t^{(2)} - u_\varrho^{(2)} v_\varrho^{(2)} + \big(a_1 u_\varrho^{(2)} + b u_t^{(2)} + (c - \frac{n^2}{\varrho^2}) u^{(2)} - n a_2 u^{(1)}\big) v^{(2)}\Big] \varrho d\varrho\, dt \\
&= \int_{S_0 \cap \partial G_\varepsilon} \alpha(\varrho) u^{(2)} v^{(2)} \varrho\, d\varrho
\end{aligned} \tag{4.4}$$

hold for all functions $v^{(1)}, v^{(2)} \in V_\varepsilon^{(1)}$.

If the functions $v^{(i)} \in C^2(\bar{G}_\varepsilon)$, then from (4.4) we conclude that

$$\begin{aligned}
&\int_{G_\varepsilon} \Big[\big(\frac{1}{\varrho}(\rho v_\varrho^{(1)})_\varrho - v_{tt}^{(1)} - \frac{1}{\varrho}(\varrho a_1 v^{(1)})_\varrho - (b v^{(1)})_t \\
&\quad + (c - \frac{n^2}{\varrho^2}) v^{(1)}\big) u^{(1)} + n a_2 v^{(1)} u^{(2)}\Big] \varrho d\varrho\, dt = 0, \\
&\int_{G_\varepsilon} \Big[\big(\frac{1}{\rho}(\rho v_\varrho^{(2)})_\varrho - v_{tt}^{(2)} - \frac{1}{\varrho}(\varrho a_1 v^{(2)})_\varrho - (b v^{(2)})_t \\
&\quad + (c - \frac{n^2}{\varrho^2}) v^{(2)}\big) u^{(2)} - n a_2 v^{(2)} u^{(1)}\Big] \varrho d\varrho\, dt = 0.
\end{aligned} \tag{4.5}$$

For $h^{(1)}, h^{(2)} \in C^1\big(\bar{G}_0 \setminus (0,0)\big)$ we state the following problem.

Problem $P_{\alpha,1}^*$. Find a solution $v^{(1)}, v^{(2)} \in V_\varepsilon^{(1)} \cap C^2(\bar{G}_\varepsilon)$ of the system

$$\begin{aligned}
&\frac{1}{\varrho}(\rho v_\varrho^{(1)})_\varrho - v_{tt}^{(1)} - \frac{1}{\varrho}(\varrho a_1 v^{(1)})_\varrho - (b v^{(1)})_t + (c - \frac{n^2}{\varrho^2}) v^{(1)} - n a_2 v^{(2)} = h^{(1)}, \\
&\frac{1}{\varrho}(\varrho v_\varrho^{(2)})_\varrho - v_{tt}^{(2)} - \frac{1}{\varrho}(\varrho a_1 v^{(2)})_\varrho - (b v^{(2)})_t + (c - \frac{n^2}{\varrho^2}) v^{(2)} + n a_2 v^{(1)} = h^{(2)}.
\end{aligned}$$

For $z^{(i)} = \varrho^{1/2} v^{(i)}$, $\xi_1 = 1 - \varepsilon - \eta$, $\eta_1 = 1 - \varepsilon - \xi$, and

$$V^{(i)}(\xi_1, \eta_1) = z^{(i)}(1 - \varepsilon - \eta_1, 1 - \varepsilon - \xi_1), \tag{4.6}$$

the domain G_ε maps into D_ε, and for appropriate coefficients A_i, B_i, C_i, D_i and $\beta = \alpha + b$ the above Problem $P_{\alpha,1}^*$ transforms to the Darboux–Goursat Problem $P_{\beta,3}$. But for this problem Theorem 3.4 ensures the solvability in $C^2(\bar{D}_\varepsilon)$. Consequently, there exists a classical solution $(V^{(1)}, V^{(2)}) \in C^2(\bar{D}_\varepsilon)$ and so the inverse transformations (2.11) and (2.13) lead to a classical solution $(v^{(1)}, v^{(2)}) \in C^2(\bar{G}_\varepsilon)$ of Problem $P^*_{\alpha,1}$. Moreover, with these functions $v^{(1)}, v^{(2)}$ the system (4.5) becomes

$$\begin{aligned} &\int_{G_\varepsilon} \Big[\Big(h^{(1)} + na_2 v^{(2)}\Big) u^{(1)} + na_2 v^{(1)} u^{(2)}\Big] \varrho d\varrho\, dt = 0, \\ &\int_{G_\varepsilon} \Big[\Big(h^{(2)} - na_2 v^{(1)}\Big) u^{(2)} - na_2 v^{(2)} u^{(1)}\Big] \varrho d\varrho\, dt = 0. \end{aligned} \tag{4.7}$$

Since the functions $h^{(1)}(\varrho, t), h^{(2)}(\varrho, t) \in C^1\left(\bar{G}_0 \backslash (0,0)\right)$ are arbitrary, (4.7) gives $u^{(1)}(\varrho, t) = u^{(2)}(\varrho, t) = 0$ in G_ε, i.e. $(u^{(1)}_1, u^{(2)}_1) \equiv (u^{(1)}_2, u^{(2)}_2)$. The proof is complete. □

5. Existence and uniqueness theorems for the 3-D Problem P_α

In this section we consider the following 3-D boundary value problem.
Problem P_α. Find a solution to the equation

$$Lu = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2} u_{\varphi\varphi} - u_{tt} + a_1 u_\varrho + a_2 u_\varphi + b u_t + c u = f(\varrho, \varphi, t) \text{ in } \Omega_\varepsilon, \tag{5.1}$$

which satisfies the boundary conditions

$$u\big|_{\Sigma_1 \cap \partial\Omega_\varepsilon} = 0, \quad [u_t + \alpha(\varrho) u]\big|_{\Sigma_0 \cap \partial\Omega_\varepsilon} = 0. \tag{5.2}$$

For this problem we formulate the following theorems.

Theorem 5.1. *Let $a_1, a_2, b, c \in C^1(\bar{\Omega}_0 \backslash O)$. Then for $0 \le \varepsilon < 1$ there exists at most one generalized solution of Problem P_α in Ω_ε.*

Proof. *Case* $0 < \varepsilon < 1$. If u_1, u_2 are two generalized solutions of P_α in Ω_ε, then $u := u_1 - u_2 \in C^1(\bar{\Omega}_\varepsilon)$ satisfies (5.2) and

$$\begin{aligned} &\int_{\Omega_\varepsilon} [u_t v_t - u_\varrho v_\varrho - \frac{1}{\varrho^2} u_\varphi v_\varphi + (a_1 u_\varrho + a_2 u_\varphi + b u_t + c u) v] \varrho\, d\varrho\, d\varphi\, dt \\ &= \int_{\Sigma_0 \cap \partial\Omega_\varepsilon} \alpha(\varrho) u v \varrho\, d\varrho d\varphi \end{aligned} \tag{5.3}$$

holds for all $v \in V_\varepsilon$. We will show that in the Fourier expansion

$$u(\varrho, \varphi, t) = \sum_{n=0}^{\infty} \Big\{ u^{(1)}_n(\varrho, t) \cos n\varphi + u^{(2)}_n(\varrho, t) \sin n\varphi \Big\} \tag{5.4}$$

the coefficients satisfy $u^{(i)}_n(\varrho, t) \equiv 0$ in Ω_ε, $i = 1, 2$, i.e. $u \equiv 0$ in Ω_ε. Since $u \in C^1(\bar{\Omega}_\varepsilon)$, using the substitution

$$v_1(\varrho, \varphi, t) = w_1(\varrho, t) \cos n\varphi \in V_\varepsilon \quad \text{or} \quad v_2(\varrho, \varphi, t) = w_2(\varrho, t) \sin n\varphi \in V_\varepsilon$$

in (5.3), we derive the system

$$\begin{aligned}
&\int_{G_\varepsilon}\Big[u^{(1)}_{n,t}w_{1,t}-u^{(1)}_{n,\varrho}w_{1,\varrho}+\big(a_1u^{(1)}_{n,\varrho}+bu^{(1)}_{n,t}+(c-\frac{n^2}{\varrho^2})u^{(1)}_n+na_2u^{(2)}_n\big)w_1\Big]\varrho d\varrho\,dt\\
&=\int_{S_0\cap\partial G_\varepsilon}\alpha(\varrho)u^{(1)}_n w_1\varrho\,d\varrho,\\
&\int_{G_\varepsilon}\Big[u^{(2)}_{n,t}w_{2,t}-u^{(2)}_{n,\varrho}w_{2,\varrho}+\Big(a_1u^{(2)}_{n,\varrho}+bu^{(2)}_{n,t}+(c-\frac{n^2}{\varrho^2})u^{(2)}_n-na_2u^{(1)}_n\Big)w_2\Big]\varrho\,d\varrho dt\\
&=\int_{S_0\cap\partial G_\varepsilon}\alpha(\varrho)u^{(2)}_n w_2\varrho\,d\varrho
\end{aligned} \tag{5.5}$$

for all $w_1,w_2\in V^{(1)}_\varepsilon$ and $n\in\mathbb{N}\cup\{0\}$. By Definition 2.4, the function $(u^{(1)}_n,u^{(2)}_n)(\varrho,t)$ is a generalized solution of the homogeneous problem $P_{\alpha,1}$. Clearly, Theorem 4.2 implies $u^{(1)}_n(\varrho,t)\equiv u^{(2)}_n(\varrho,t)\equiv 0$ in Ω_ε for $n\in\mathbb{N}\cup\{0\}$ and so $u^{(1)}=u_1-u_2\equiv 0$ in Ω_ε. *Case* $\varepsilon=0$. Let ε_0 be an arbitrary fixed number of $(0,1)$. Then, by Lemma 2.3, it follows that the generalized solution $u\in C^1(\bar\Omega_0\setminus(0,0,0))$ of Problem P_α in Ω_0 is also a generalized solution of the homogeneous problem P_α in Ω_{ε_0}. Since, by the previous case, $u\equiv 0$ in Ω_{ε_0} and $\varepsilon_0>0$ is arbitrary, we see that $u=u_1-u_2\equiv 0$ in Ω_0. This completes the proof of the theorem. □

Theorem 5.2. *Let* $a_1,a_2,b,c\in C^1\left(\bar\Omega_0\backslash O\right)$ *and the function* $f\in C(\bar\Omega_0)\cap C^1(\bar\Omega_0\backslash O)$ *be of the form*

$$f(\varrho,\varphi,t)=\sum_{n=0}^{k}\big\{f^{(1)}_n(\varrho,t)\cos n\varphi+f^{(2)}_n(\varrho,t)\sin n\varphi\big\},\ k\in\mathbb{N}\cup\{0\}. \tag{5.6}$$

Then there exists one and only one generalized solution

$$u(\varrho,\varphi,t)=\sum_{n=0}^{k}\Big\{u^{(1)}_n(\varrho,t)\cos n\varphi+u^{(2)}_n(\varrho,t)\sin n\varphi\Big\} \tag{5.7}$$

of the problem P_α *in* Ω_0. *This solution* $u\in C^2\left(\bar\Omega_0\backslash O\right)$ *is a classical solution of the problem* P_α *in each domain* $\Omega_\varepsilon,\varepsilon\in(0,1)$. *Moreover, if*

$$|a_1|\le d\varrho^{-1},\quad |a_2|\le d\varrho^{-2},\quad |b|\le d\varrho^{-2},\quad |c|\le d\varrho^{-2},\quad |\alpha|\le d\varrho^{-2}\quad in\ \bar\Omega_0\backslash O,$$

then, in view of (5.7), *for a fixed* n, *the corresponding trigonometric polynomial* u_n *of degree* n, *satisfies the following a priori estimates: For* $n=0$,

$$\|u_0(x_1,x_2,t)\|_{C^1(\Omega^{(1)}_\varepsilon)}=\sum_{|\alpha|\le 1}\sup_{\Omega^{(1)}_\varepsilon}|D^\alpha u_0|\le 6\varepsilon^{3/2}\exp\big(\frac{32d+2}{\varepsilon^2}\big)\|f^{(1)}_0\|_{C^0(\bar G_0)}; \tag{5.8}$$

while for $n\in\mathbb{N}$,

$$\|u_n(x_1,x_2,t)\|_{C^1(\Omega^{(1)}_\varepsilon)}\le\frac{6\varepsilon^{3/2}}{n(n+2d)}\exp\big(\frac{8n(n+3d)}{\varepsilon^2}\big)\big(\|f^{(1)}_n\|_{C^0(\bar G_0)}+\|f^{(2)}_n\|_{C^0(\bar G_0)}\big), \tag{5.9}$$

where $\Omega^{(1)}_\varepsilon=\Omega_0\cap\{(\varrho,t):\varrho+t>\varepsilon\}$.

Proof. It suffices to consider the case of a fixed number n. As in Section 2, we make the substitutions

$$\xi=1-\varrho-t,\quad \eta=1-\varrho+t, \tag{5.10}$$

and introduce the new function

$$U^{(i)}(\xi,\eta)=\varrho^{1/2}u^{(i)}(\varrho(\xi,\eta),t(\xi,\eta)). \tag{5.11}$$

Denote

$$F^{(i)}(\xi,\eta):=\frac{1}{4\sqrt{2}}(2-\eta-\xi)^{1/2}f_n^{(i)}(\xi,\eta)\in C^1\left(\bar{D}_0\backslash(1,1)\right),$$

and use the notation of (2.16). Then the problem reduces to Problem $P_{\alpha,3}$. Thus, we can use Theorems 3.1 and 3.4 to ensure the existence of a classical solution $(U^{(1)},U^{(2)})(\xi,\eta)$ of this problem with the estimates (3.8).

Case $n\in\mathbb{N}$. In view of (3.7), (2.16), it is easy to see that we can chose

$$c(\varepsilon):=\frac{n(n+2d)}{\varepsilon^2},\quad M_\alpha:=\frac{4d}{\varepsilon^2}$$

$$M\le\frac{1}{4}\max\left\{\|f_n^{(1)}\|_{C^0(\bar{G}_0)},\|f_n^{(2)}\|_{C^0(\bar{G}_0)}\right\}:=M_n,$$

Hence, Theorems 3.1 and 3.4 ensure the smoothness of the solution U of Problem $P_{\alpha,3}$ in $D_\varepsilon^{(1)}=\{(\xi,\eta):0<\xi<\eta<1,\ 0<\xi<1-\varepsilon\}$, $\varepsilon>0$, i.e.

$$(U_n^{(1)},U_n^{(2)})(\xi,\eta):=U(\xi,\eta)\in C^2(\bar{D}_\varepsilon^{(1)})\,. \tag{5.12}$$

On the other hand, these theorems ensure the a priori estimates

$$\begin{aligned}\sup_{D_\varepsilon^{(1)}}|U_n^{(i)}(\xi,\eta)|&\le M_n\left[4c(\varepsilon)+M_\alpha\right]^{-2}\exp\left\{8c(\varepsilon)+2M_\alpha\right\}\\&\le M_n\varepsilon^4[4n(n+2d)]^{-2}\exp\left\{(8n(n+3d)\varepsilon^{-2}\right\},\\ \sup_{D_\varepsilon^{(1)}}\{|U_{n,\xi}^{(i)}|,\ |U_{n,\eta}^{(i)}|\}&\le M_n\varepsilon^2[4n(n+2d)]^{-1}\exp\left\{(8n(n+3d)\varepsilon^{-2}\right\}.\end{aligned}$$

Also, by (5.10) and (5.11), we have $u_n^{(i)}(\varrho,t)=\varrho^{-\frac{1}{2}}U_n^{(i)}(\xi,\eta)$. Since $\varrho\ge\varepsilon/2$ for $(\xi,\eta)\in D_\varepsilon^{(1)}$, by the inverse transformation we see that

$$\begin{aligned}|u_n^{(i)}(\varrho,t)|&\le M_n\frac{\varepsilon^{7/2}}{8n^2(n+2d)^2}\exp\big(\frac{8n(n+3d)}{\varepsilon^2}\big),\\ |u_{n,t}^{(i)}(\varrho,t)|&\le M_n\frac{\varepsilon^{3/2}}{n(n+2d)}\exp\big(\frac{8n(n+3d)}{\varepsilon^2}\big),\\ |u_{n,\varrho}^{(i)}(\varrho,t)|&\le M_n\frac{\varepsilon^{3/2}}{n(n+2d)}\exp\big(\frac{8n(n+3d)}{\varepsilon^2}\big).\end{aligned} \tag{5.13}$$

Therefore, in view of (5.7) and (5.13), for the trigonometrical polynomial

$$u_n(\varrho,\varphi,t)=u_n^{(1)}(\varrho,t)\cos n\varphi+u_n^{(2)}(\varrho,t)\sin n\varphi \tag{5.14}$$

we derive

$$\|\frac{1}{\varrho}u_{n,\varphi}(\varrho,\varphi,t)\|_{C(\Omega_\varepsilon^{(1)})}\le M_n\frac{\varepsilon^{5/2}}{4n(n+2d)^2}\exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right). \tag{5.15}$$

Since $u_n(\varrho\cos\varphi,\varrho\sin\varphi,t)=u_n^{(1)}(\varrho,\varphi,t)$, obviously one has

$$|u_{n,x_i}(x_1,x_2,t)|\le 2M_n\frac{\varepsilon^{3/2}}{n(n+2d)}\exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right),\quad i=1,2.$$

So, the estimate (5.9) holds in $\Omega_\varepsilon^{(1)}$.

Case $n = 0$. By (5.6) and (5.7), we have $f_0(\varrho, \varphi, t) = f_0^{(1)}(\varrho, t)$ and $u_0(x_1, x_2, t) = u_0(\varrho, \varphi, t) = u_0^{(1)}(\varrho, t)$. Take

$$c(\varepsilon) := \frac{8d+1}{4\varepsilon^2}, \quad M_\alpha := \frac{4d}{\varepsilon^2}, \quad M := \frac{1}{4}\|f_0^{(1)}\|_{C^0(\bar{G}_0)} .$$

Then, as in the previous case, we obtain (5.8). □

Theorem 5.3. *Let the conditions of Theorem 5.2 be fulfilled. Also, for the sake of simplicity, suppose that $a_1, a_2, b, c \in C^1(\bar{\Omega}_0)$ and $|\alpha'(\varrho)| \le d_1/\varrho^{-3}$. Then for a fixed $n \in \mathbb{N}$ the corresponding trigonometric polynomial u_n of degree n from* (5.14) *satisfies the following a priori estimate*

$$\|u_n(x_1, x_2, t)\|_{C^2(\bar{\Omega}_\varepsilon^{(1)})} \le C_1 \varepsilon^{-1/2} \exp\big(\frac{8n(n+3d)}{\varepsilon^2}\big)\big(\|f_n^{(1)}\|_{C^0(\bar{G}_0)} + \|f_n^{(2)}\|_{C^0(\bar{G}_0)}\big), \tag{5.16}$$

where the constant C_1 does not depend on n and ε.

Proof. We will use the estimates of Theorem 5.2 and the representations of the second derivatives of Theorem 3.4. Following the same arguments, as in Theorem 5.2, we obtain the estimates

$$\sup_{D_\varepsilon^{(1)}}\{|U_{n,\xi\eta}^{(i)}|, |U_{n,\eta\eta}^{(i)}|,\ |U_{n,\xi\xi}^{(i)}|\} \le C_1 M_n \exp\big\{8n(n+3d)\varepsilon^{-2}\big\}\,,\ i = 1, 2$$

and conclude that

$$\sup_{D_\varepsilon^{(1)}}\{|u_{n,x_i x_j}|, |u_{n,t x_i}|\} \le C_1 M_n \varepsilon^{-1/2} \exp\big(\frac{8n(n+3d)}{\varepsilon^2}\big).$$

□

The next theorem is an immediate consequence of Theorems 5.1, 5.2 and 5.3.

Theorem 5.4. *Let the conditions of Theorem 5.3 be fulfilled and let $f \in C^1(\bar{\Omega}_0)$ be of the form*

$$f(\varrho, \varphi, t) = \sum_{n=0}^{\infty}\{f_n^{(1)}(\varrho, t)\cos n\varphi + f_n^{(2)}(\varrho, t)\sin n\varphi\}. \tag{5.17}$$

Suppose that the Fourier coefficients $f_n^{(1)}(\varrho, t)$ and $f_n^{(2)}(\varrho, t)$ satisfy

$$\begin{aligned}\|f\|_{\exp(\varepsilon)} &:= \exp\big(\frac{32d+2}{\varepsilon^2}\big)\|f_0^{(1)}\|_{C^0(\bar{G}_0)} + \sum_{n=1}^{\infty}\frac{1}{n(n+2d)}\exp\big(\frac{8n(n+3d)}{\varepsilon^2}\big) \\ &\quad \times \big(\|f_n^{(1)}\|_{C^0(\bar{G}_0)} + \|f_n^{(2)}\|_{C^0(\bar{G}_0)}\big) < \infty\,.\end{aligned} \tag{5.18}$$

Then there exists one and only one generalized solution $u \in C^1\left(\bar{\Omega}_\varepsilon^{(1)}\right)$ of the problem P_α in Ω_ε and satisfies the a priori estimate

$$\|u\|_{C^1(\Omega_\varepsilon^{(1)})} \le 6\varepsilon^{3/2}\|f\|_{\exp(\varepsilon)}. \tag{5.19}$$

Moreover, if

$$\begin{aligned}\|f\|_{\exp_1(\varepsilon)} &:= \exp\big(\frac{32d+2}{\varepsilon^2}\big)\|f_0^{(1)}\|_{C^0(\bar{G}_0)} + \sum_{n=1}^{\infty}\exp\big(\frac{8n(n+3d)}{\varepsilon^2}\big) \\ &\quad \times \big(\|f_n^{(1)}\|_{C^0(\bar{G}_0)} + \|f_n^{(2)}\|_{C^0(\bar{G}_0)}\big) < \infty,\end{aligned} \tag{5.20}$$

then $u \in C^2\left(\bar{\Omega}_\varepsilon^{(1)}\right)$, $u(x,t)$ *is a classical solution of the problem* P_α *in* Ω_ε *and satisfies the a priori estimate*

$$\|u\|_{C^2(\Omega_\varepsilon^{(1)})} \le C_2 \varepsilon^{-1/2} \|f\|_{\exp_1(\varepsilon)}. \tag{5.21}$$

Remark 5.5. It is obvious that the estimates (5.19) or (5.21) hold, if the series (5.18) and (5.20) are finite. In this case we have a solution, which is of class $C^1(\bar{\Omega}_0 \backslash O)$ or of class $C^2(\bar{\Omega}_0 \backslash O)$. For example, the condition (5.20) is valid for each $\varepsilon \in (0,1)$, if there exists a sequence $a_n \to +\infty$ as $n \to +\infty$ such that

$$\sum_{n=1}^{\infty} \exp\left(n^2 a_n\right) \left(\|f_n^{(1)}\|_{C^0(\bar{G}_0)} + \|f_n^{(2)}\|_{C^0(\bar{G}_0)} \right) < \infty. \tag{5.22}$$

To show this, it is enough to see that the inequality $n(\varepsilon^2 a_n - 8) \ge 24d$ holds, for all large enough $n \in \mathbb{N}$.

6. On the singularity of solutions of Problem P_α

For the the equation

$$Lu = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2} u_{\varphi\varphi} - u_{tt} + a_1 u_\varrho + a_2 u_\varphi + b u_t + cu = f(\varrho, \varphi, t) \text{ in } \Omega_0, \tag{6.1}$$

we consider the boundary conditions of Problem P_α, i.e.

$$P_\alpha: \quad u\big|_{\Sigma_1} = 0, \quad [u_t + \alpha(\varrho) u]\big|_{\Sigma_0 \backslash O} = 0 \tag{6.2}$$

and prove the following result.

Theorem 6.1. *Let* $\alpha(\varrho) \ge 0$; a_1, b, $c \in C^1(\bar{\Omega}_0 \backslash O)$, $a_2 \equiv 0$ *and*

$$a_1(\varrho, t) \ge |b|(\varrho, t), \quad a_1(\varrho, t) \ge 2\varrho c(\varrho, t), \quad (\varrho, t) \in \Omega_0. \tag{6.3}$$

Then for each function

$$f_n(\varrho, \varphi, t) = \varrho^{-n}(\varrho^2 - t^2)^{n-1/2} \cos n\varphi \in C^{n-2}(\bar{\Omega}_0) \cap C^\infty(\Omega_0), \quad n \in \mathbb{N},$$

the corresponding generalized solution u_n *of the problem* P_α *belongs to* $C^2(\bar{\Omega}_0 \backslash O)$ *and satisfies the estimate*

$$|u_n(\varrho, \varphi, \varrho)| \ge \frac{1}{2} |u_n(2\varrho, \varphi, 0)| + \varrho^{-n} |\cos n\varphi| \ge \varrho^{-n} |\cos n\varphi|, \quad 0 < \varrho < 1\,. \tag{6.4}$$

Proof. Note that, by Theorem 1.2, the functions

$$w_n(\varrho, \varphi, t) = \varrho^{-n}(\varrho^2 - t^2)^{n-1/2}(a_n \cos n\varphi + b_n \sin n\varphi), \quad n \ge 4,$$

are classical solutions of the homogeneous Problem P_α^* for the wave equation, when $\alpha \equiv 0$.

Now consider the special case of Problem P_α:

$$Lu = f_n \equiv \varrho^{-n}(\varrho^2 - t^2)^{n-1/2} \cos n\varphi \quad \text{in } \Omega_0. \tag{6.5}$$

Observe also that

$$f_n(x_1, x_2, t) = (x_1^2 + x_2^2)^{-n}(x_1^2 + x_2^2 - t^2)^{n-1/2} \operatorname{Re}(x_1 + i x_2)^n$$

and obviously $f_n \in C^{n-2}(\bar{\Omega}_0) \cap C^\infty(\Omega_0)$, $n \in \mathbb{N}$. Theorem 5.1 states that the equation (6.5) with boundary conditions (6.2) has at most one generalized solution. On the other hand, from Theorem 5.2 it is known that, for the above right-hand side, there exists a generalized solution in Ω_0 of the form

$$u_n(\varrho, \varphi, t) = u_n^{(1)}(\varrho, t) \cos n\varphi \in C^2(\bar{\Omega}_0 \backslash O),$$

which is classical solution in Ω_ε, $\varepsilon \in (0,1)$. By setting $u_n^{(2)}(\varrho,t) = \varrho^{\frac{1}{2}} u_n^{(1)}(\varrho,t)$ and substituting

$$\xi = 1-\varrho-t, \quad \eta = 1-\varrho+t, \tag{6.6}$$

the equation (6.5), with boundary conditions (6.2), in view of

$$U(\xi,\eta) = u_n^{(2)}(\varrho(\xi,\eta), t(\xi,\eta)), \tag{6.7}$$

becomes a Darboux-Goursat problem $P_{\alpha,3}$:

$$U_{\xi\eta} - AU_\xi - BU_\eta - CU = F(\xi,\eta), \tag{6.8}$$

$$U(0,\eta) = 0, \quad (U_\eta - U_\xi)(\xi,\xi) + \alpha(1-\xi)U(\xi,\xi) = 0. \tag{6.9}$$

Note that, because of the condition $a_2 \equiv 0$ and the special right-hand side of (6.5), we do not obtain a system as in the general case of Section 3, but a single equation (6.8). According to (2.16), the coefficients of (6.8) are defined as follows:

$$\begin{gathered} A = \frac{1}{4}(a_1+b) \ge 0, \quad B = \frac{1}{4}(a_1-b) \ge 0, \\ C(\xi,\eta) = \frac{1}{4}\Big(\frac{4n^2-1}{(2-\eta-\xi)^2} + \frac{a_1(\xi,\eta)}{2-\eta-\xi} - c(\xi,\eta)\Big) \ge 0, \ n \in \mathbb{N}, \end{gathered} \tag{6.10}$$

$$F(\xi,\eta) = 2^{n-\frac{5}{2}}\Big[\frac{(1-\xi)(1-\eta)}{2-\eta-\xi}\Big]^{n-\frac{1}{2}} \in C^{n-1}(\bar{D}_\varepsilon^{(1)}), \quad F(\xi,\eta) \ge 0, \tag{6.11}$$

where we preserve the same notations for a_1, b and c in the new coordinates (ξ,η). Next, in view of Theorem 3.4 and Lemma 3.2, we formulate the following result.

Proposition 6.2. *There exists a classical solution $U(\xi,\eta) \in C^2(\bar{D}_0 \setminus (1,1))$ for the problem* (6.8), (6.9) *for which*

$$U(\xi,\eta) \ge 0, \quad U_\xi(\xi,\eta) \ge 0, \quad U_\eta(\xi,\eta) \ge 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}.$$

Set

$$K = \int_{D^{(1)}_{\frac{1}{2}}} F^2(\xi,\eta)\, d\xi d\eta > 0. \tag{6.12}$$

Then from (6.8) for $0 < \varepsilon < 1/2$ it follows that

$$\begin{aligned} 0 < K &\le \int_{D_\varepsilon^{(1)}} F^2(\xi,\eta)\, d\xi d\eta \\ &= \int_{D_\varepsilon^{(1)}} (U_{\xi\eta}F)(\xi,\eta) d\xi\, d\eta - \int_{D_\varepsilon^{(1)}} [(AU_\xi + BU_\eta + CU)F](\xi,\eta)\, d\xi d\eta \\ &=: I_1 - I_2. \end{aligned} \tag{6.13}$$

Using the properties of $F(\xi,\eta)$ from (6.11) and following [9], we find that

$$\begin{aligned} I_1 :&= \int_0^{1-\varepsilon}\int_\xi^1 (U_{\xi\eta}F)(\xi,\eta)\, d\eta\, d\xi \\ &= \int_{D_\varepsilon^{(1)}} (UF_{\xi\eta})(\xi,\eta)\, d\xi\, d\eta - \int_0^{1-\varepsilon} [U_\xi(\xi,\xi)F(\xi,\xi) + U(\xi,\xi)F_\eta(\xi,\xi)]\, d\xi \\ &\quad - \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\, d\eta. \end{aligned} \tag{6.14}$$

An elementary calculation shows that

$$F_\xi(\xi,\eta) \le 0, F_\eta(\xi,\eta) \le 0, \tag{6.15}$$

which actually follows from Lemma 3.2, and

$$F_\xi(\xi,\xi) = F_\eta(\xi,\xi) = \frac{1}{16}(1-2n)(1-\xi)^{n-\frac{3}{2}}. \tag{6.16}$$

From (6.13) and (6.14) it follows that

$$\begin{aligned} 0 < K \le I_1 - I_2 = &- \int_0^{1-\varepsilon} [U_\xi(\xi,\xi)F(\xi,\xi) + U(\xi,\xi)F_\xi(\xi,\xi)]\, d\xi \\ &- \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\, d\eta \\ &+ \int_{D_\varepsilon^{(1)}} \{(F_{\xi\eta} - CF)U - F(AU_\xi + BU_\eta)\}\, d\xi\, d\eta. \end{aligned} \tag{6.17}$$

Also, it is easy to check that

$$F_{\xi\eta}(\xi,\eta) - \frac{4n^2-1}{4(2-\eta-\xi)^2}F(\xi,\eta) = 0 \tag{6.18}$$

and so, because of (6.3), (6.10) and Proposition 6.2,

$$(F_{\xi\eta} - CF)U - F(AU_\xi + BU_\eta) = -\big(\frac{a_1(\xi,\eta)}{2-\eta-\xi} - c(\xi,\eta)\big)\frac{FU}{4} - F(AU_\xi + BU_\eta) \le 0.$$

Thus, we find

$$\begin{aligned} 0 < K \le I_1 - I_2 \le &- \int_0^{1-\varepsilon} [U_\xi(\xi,\xi)F(\xi,\xi) + U(\xi,\xi)F_\xi(\xi,\xi)]\, d\xi \\ &- \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta, \end{aligned} \tag{6.19}$$

where, as it is easy to check,

$$F_\xi(\xi,\xi) = \frac{1}{2}[F(\xi,\xi)]_\xi. \tag{6.20}$$

The function $U(\xi,\eta)$ is a classical solution of (6.8), (6.9) in $\bar{D}_\varepsilon$, $\varepsilon \in (0,1)$ with

$$U_\xi(\xi,\xi) = \frac{1}{2}[U(\xi,\xi)]_\xi + \frac{1}{2}\alpha(1-\xi)U(\xi,\xi). \tag{6.21}$$

If we substitute (6.20) and (6.21) into (6.19), we get

$$\begin{aligned} K \le & I_1 - I_2 = -\frac{1}{2}\int_0^{1-\varepsilon} [F(\xi,\xi)U(\xi,\xi)]_\xi\, d\xi \\ &- \frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U(\xi,\xi)F(\xi,\xi)\, d\xi - \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\, d\eta \\ = &- \frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon) - \frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U(\xi,\xi)F(\xi,\xi)\, d\xi \\ &- \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\, d\eta. \end{aligned} \tag{6.22}$$

Next, in view of Proposition 6.2 and the properties of the function $F(\xi,\eta)$, we find

$$U(\xi,\eta) \ge 0, \quad U_\eta(\xi,\eta) \ge 0, \quad \alpha(\xi) \ge 0, \quad F(\xi,\eta) \ge 0, \quad F_\eta(\xi,\eta) \le 0 \text{ in } \bar{D}_\varepsilon^{(1)},$$

which together with (6.22) and because of $F(1-\varepsilon,1)=0$ implies

$$\begin{aligned}K \le I_1+I_2 &\le -\int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\,d\eta - \frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon)\\ &= \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)|F_\eta(1-\varepsilon,\eta)|\,d\eta - \frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon)\\ &\le \int_{1-\varepsilon}^{1} U(1-\varepsilon,1)|F_\eta(1-\varepsilon,\eta)|\,d\eta - \frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon)\\ &= \big[U(1-\varepsilon,1)-\frac{1}{2}U(1-\varepsilon,1-\varepsilon)\big]F(1-\varepsilon,1-\varepsilon).\end{aligned}$$

Since $F(1-\varepsilon,1-\varepsilon)=\frac{1}{4}\varepsilon^{n-\frac{1}{2}}$, we have

$$0<K\le \big[U(1-\varepsilon,1)-\frac{1}{2}U(1-\varepsilon,1-\varepsilon)\big]\frac{1}{4}\varepsilon^{n-\frac{1}{2}}.$$

For $\xi=1-\varepsilon$, $\eta=1$ we have $\varrho=t=\varepsilon/2$ and so

$$0<4K\varepsilon^{\frac{1}{2}-n}\le u_n^{(2)}\Big(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\Big)-\frac{1}{2}u_n^{(2)}(\varepsilon,0). \tag{6.23}$$

Finally, the inverse transformation gives

$$u_n^{(1)}\Big(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\Big)\ge \frac{1}{2}u_n^{(1)}(\varepsilon,0)+\tilde{C}_1\varepsilon^{-n}\ge \tilde{C}_1\varepsilon^{-n},\quad 0<\varepsilon<\frac{1}{2},$$

where $\tilde{C}_1=2^{\frac{5}{2}}K$. Multiplying the function u_n by $\tilde{C}_1^{-1}$, we see that (6.4) holds. The proof is complete. □

Note that the conditions of Theorem 6.1 are only sufficient, and not invariant with respect to change of variables. Now, we use this fact in order to find some new singular solutions. For this reason, consider the special form of the equation (6.1)

$$Lu=\frac{1}{\varrho}(\varrho u_\varrho)_\varrho+\frac{1}{\varrho^2}u_{\varphi\varphi}-u_{tt}+a_1(u_\varrho-u_t)+cu=f(\varrho,\varphi,t)\text{ in }\Omega_\varepsilon, \tag{6.24}$$

with the boundary conditions (6.2).

Theorem 6.3. *Let a_1, $c\in C^1(\bar{\Omega}_0\backslash O)$; $a_1\in C(\bar{\Omega}_0)$ and $\alpha(\varrho)\in C^1(0,1]\cap C[0,1]$, without any conditions imposed on the sign. Suppose that $a_1(\varrho,t)\ge 2\varrho c(\varrho,t)$ for $(\varrho,t)\in\Omega_0$. Then there exists appropriate constant Λ_0, such that for any other constant $\Lambda\ge\Lambda_0$ and function*

$$f_n(\varrho,\varphi,t)=\exp\{\Lambda(\varrho+t)\}\,\varrho^{-n}(\varrho^2-t^2)^{n-1/2}\cos n\varphi, n\in\mathbb{N},$$

the corresponding singular generalized solution u_n of the problem P_α belongs to $C^2(\bar{\Omega}_0\backslash O)$ and the estimate (6.4) holds.

Proof. We are looking for appropriate right-hand side functions f_n of the equation (6.24), for which singular solutions exist. Set

$$u(\varrho,\varphi,t)=\exp\{\lambda(\varrho+t)\}\,w(\varrho,\varphi,t),$$

where the function $\lambda(s)$ will be chosen later. Then the equation (6.24) becomes

$$\begin{aligned}L_1w&=\frac{1}{\varrho}(\varrho w_\varrho)_\varrho+\frac{1}{\varrho^2}w_{\varphi\varphi}-w_{tt}+(a_1+2\lambda')(w_\varrho-w_t)+(c+\lambda'\varrho^{-1})w\\ &=\exp\{-\lambda(\varrho+t)\}\,f(\varrho,\varphi,t)\quad\text{in }\Omega_\varepsilon\end{aligned} \tag{6.25}$$

and so we lead to the following boundary value problem:

$$P_\beta : \begin{cases} L_1 w = g := \exp\{-\lambda(\varrho+t)\} f & \text{in } \Omega_0, \\ w\big|_{\Sigma_1} = 0, [w_t + \beta(\varrho)w]\big|_{\Sigma_0\setminus O} = 0 \end{cases} \tag{6.26}$$

with $\beta(\varrho) = \alpha(\varrho) + \lambda'(\varrho)$. In order to apply Theorem 6.1 to Problem P_β, we need the following conditions

$$\alpha(\varrho) + \lambda'(\varrho) \geq 0, \quad a_1(\varrho,t) + 2\lambda'(\varrho) \geq 0, \quad a_1(\varrho,t) - 2\varrho c(\varrho,t) \geq 0,$$

which are satisfied, for example, for $\lambda(\varrho) = \Lambda\varrho$ and $\Lambda > 0$ large enough. Following the proof of Theorem 6.1 and using the transformations (6.6) and (6.7), we lead to the function $W(\xi,\eta)$ of (6.7), for which the equation (6.8) reduces to

$$W_{\xi\eta} - \frac{1}{2}(a_1 + 2\lambda')W_\eta - CW = F(\xi,\eta). \tag{6.27}$$

Here $C(\xi,\eta)$ and $F(\xi,\eta)$ are functions from (6.11) and (6.10).

We formulate now a result to be used in the proof of Theorem 7.1.

Proposition 6.4. *There exists a classical solution* $W(\xi,\eta) \in C^2(\bar{D}_0\setminus(1,1))$ *of the problem* (6.27), (6.9) *for which*

$$W(\xi,\eta) \geq 0, \quad W_\xi(\xi,\eta) \geq 0, \quad W_\eta(\xi,\eta) \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}. \tag{6.28}$$

For the function

$$g_n(\varrho,\varphi,t) = \varrho^{-n}(\varrho^2 - t^2)^{n-1/2}\cos n\varphi, \tag{6.29}$$

as the right-hand side of the equation $L_1 w = g$, Theorem 6.1 gives a singular solution w_n satisfying (6.4); that is,

$$|w_n(\varrho,\varphi,\varrho)| \geq \frac{1}{2}|w_n(2\varrho,\varphi,0)| + \varrho^{-n}|\cos n\varphi| \geq \varrho^{-n}|\cos n\varphi|,\ 0<\varrho<1. \tag{6.30}$$

Now, the inverse transform $u_n = \exp\{\Lambda(\varrho+t)\}\, w_n$ gives

$$|u_n(\varrho,\varphi,\varrho)| \geq \frac{1}{2}|u_n(2\varrho,\varphi,0)| + \varrho^{-n}|\cos n\varphi| \geq \varrho^{-n}|\cos n\varphi|,\ 0<\varrho<1, \tag{6.31}$$

where the function $u_n(\varrho,\varphi,t)$ is a solution of the problem (6.24), (6.2) with

$$f(\varrho,\varphi,t) = \exp\{\Lambda(\varrho+t)\}\ \varrho^{-n}(\varrho^2-t^2)^{n-1/2}\cos n\varphi.$$

The proof is complete. □

Next, we find singular solutions for the original Problem P_α, formulated in Section 1.

Proof of Theorem 1.4 Recall that we are looking for a suitable right-hand side function f_n of (1.1) for which singular solutions exist. Set

$$u(x_1,x_2,t) = \exp\{(bt - b_1x_1 - b_2x_2)/2\}\, v(x_1,x_2,t)\,. \tag{6.32}$$

Then equation (2.1) becomes

$$v_{x_1x_1} + v_{x_2x_2} - v_{tt} + c_1 v = h \equiv \exp\{(b_1x_1 + b_2x_2 - bt)/2\} f \quad \text{in } \Omega_0, \tag{6.33}$$

where $c_1 := c + (b^2 - b_1^2 - b_2^2)/4$. In order to apply Theorem 6.3, we rewrite (6.33) in polar coordinates and obtain the problem P_γ:

$$\frac{1}{\varrho}(\varrho v_\varrho)_\varrho + \frac{1}{\varrho^2}v_{\varphi\varphi} - v_{tt} + c_1 v = h(\varrho,\varphi,t) \quad \text{in } \Omega_0,$$
$$v\big|_{\Sigma_1} = 0, \quad [v_t + \gamma(\varrho)v]\big|_{\Sigma_0\setminus O} = 0$$

with $\gamma(\varrho) := \alpha(\varrho) + b/2$. In order to apply Theorem 6.3 to the Problem P_γ, the only condition we need is $c_1 \le 0$, i.e. $c + (b^2 - b_1^2 - b_2^2)/4 \le 0$, which is satisfied. If we now choose

$$h_n(\varrho, \varphi, t) = \exp\{\Lambda(\varrho + t)\}\, \varrho^{-n}(\varrho^2 - t^2)^{n-1/2} \cos n\varphi$$

according to Theorem 6.3 with the constant Λ large enough, then the Problem P_γ has a corresponding singular solution $v_n \in C^2(\bar{\Omega}_0 \backslash O)$ with the estimate

$$|v_n(\varrho, \varphi, \varrho)| \ge \frac{1}{2}|v_n(2\varrho, \varphi, 0)| + \varrho^{-n}|\cos n\varphi|, \quad 0 < \varrho < 1.$$

Now the inverse transform of (6.32) gives a singular *generalized solution* $u_n \in C^2(\bar{\Omega}_0 \backslash O)$ of the Problem P_α with the right-hand side

$$\begin{aligned} f_n =& \exp\{(bt - b_1x_1 - b_2x_2)/2 + \Lambda(\varrho + t)\} \\ & \times |x|^{-n}(x_1^2 + x_2^2 - t^2)^{n-1/2} \cos n(\arctan \frac{x_2}{x_1}). \end{aligned}$$

The proof is complete. □

Remark 6.5. *Aldashev in* [2] *considered* (2.1) *and studied the homogeneous Problems P_α and P_α^*. Unfortunately, as it is easy to check, the procedure which he follows leads to a correct conclusion only in the case of the wave equation, i.e. only in the case where all the lower order terms in* (2.1) *are identically zero. Otherwise, this procedure leads to systems of differential equations which are not equivalent to those which should be solved (see* (2.7)*). This is due to the fact that, in the systems obtained in* [2] *by integration with respect to φ, the Fourier coefficients u_k of degree k depend on the coefficients u_{k-1} of degree $k-1$.*

7. Applications to the wave equation, singular solutions

In this section we consider the wave equation

$$\Box u = u_{x_1x_1} + u_{x_2x_2} - u_{tt} = f(x_1, x_2, t) \tag{7.1}$$

subject to the boundary-value problem P_α, i.e.

$$\Box u = f \text{ in } \Omega_0, \quad u\big|_{\Sigma_1} = 0, \quad [u_t + \alpha(|x|)u]\big|_{\Sigma_0\backslash O} = 0. \tag{7.2}$$

As an application to the wave equation of the results of the previous section, we have the following statement.

Theorem 7.1. *Let $\alpha \in C^\infty(0,1] \cap C[0,1]$ be an arbitrary function. Then:*

(i) *For each $n \in \mathbb{N}$, $n \ge 4$, there exists a function $f_n \in C^{n-2}(\bar{\Omega}_0) \cap C^\infty(\Omega_0)$, for which the corresponding generalized solution u_n of the problem P_α belongs to $C^n(\bar{\Omega}_0 \backslash O)$ and satisfies the estimate*

$$|u_n(x_1, x_2, |x|)| \ge \frac{1}{2}|u_n(2x_1, 2x_2, 0)| + |x|^{-n}|\cos n(\arctan \frac{x_2}{x_1})|. \tag{7.3}$$

(ii) *In the case $\alpha(\varrho) \le 0$ an upper estimate of the singular solution u_n is*

$$|u_n(x_1, x_2, t)| \le C_\mu |x|^{-1/2}\Big(\frac{|x|}{x_1^2 + x_2^2 - t^2}\Big)^{n-\frac{1}{2}} |\cos n(\arctan \frac{x_2}{x_1})|, \quad (|x|, t) \in D_1^\mu, \tag{7.4}$$

where C_μ is a constant. and

$$D_1^\mu := \{(\varrho, t) : 0 < \varrho - t \le \varrho + t \le \mu(\varrho - t)\}, \mu < 2^{\frac{2n+1}{2n-1}} - 1.$$

Thus, for $\alpha(\varrho) \le 0$ we have two-sided estimates, which in the limit cases $t = |x|$ and $t = 0$ are:

$$
\begin{gathered}
|x|^{-n}|\cos n(\arctan \frac{x_2}{x_1})| \le |u_n(x_1, x_2, |x|)|,\\
|u_n(x_1, x_2, 0)| \le C|x|^{-n}|\cos n(\arctan \frac{x_2}{x_1})|,
\end{gathered} \tag{7.5}
$$

with C a constant. That is, in the case of $\alpha(\varrho) \le 0$ the exact behavior of $u_n(x_1, x_2, t)$ around O is $(x_1^2 + x_2^2 + t^2)^{-n/2} \cos n(\arctan \frac{x_2}{x_1})$.

Proof. (i) Note that, the wave equation (7.1) is of the form (6.24) and so the first part of Theorem 7.1 follows from Theorem 6.3 and [9]. Actually, according to this theorem we choose the function f_n to be of the special form

$$
\Box u = f_n = \exp\{\Lambda(\varrho + t)\}\ \varrho^{-n}(\varrho^2 - t^2)^{n-1/2} \cos n\varphi \quad \text{in } \Omega_0, \tag{7.6}
$$

where $\Lambda > 0$ is large enough and such that $\Lambda + \alpha(\varrho) \ge 0, \varrho \in [0, 1]$. Then by Theorems 5.1 and 5.2 there exists a unique generalized solution $u_n(\varrho, \varphi, t)$ of the equation (7.6), satisfying the boundary conditions (7.2) and the estimates (7.3) (see Theorem 6.3). On the other hand, by [9, Theorem 5.2], for the equation (7.6) there exists a generalized solution in Ω_0 of the form

$$
u_n(\varrho, \varphi, t) = u_n^{(1)}(\varrho, t) \cos n\varphi \in C^n(\bar{\Omega}_0 \backslash O),
$$

which is a classical solution of Problem P_α in Ω_ε, $\varepsilon \in (0, 1)$.

(ii) By setting $u_n^{(2)}(\varrho, t) = \varrho^{\frac{1}{2}} u_n^{(1)}(\varrho, t)$ and substituting

$$
\xi = 1 - \varrho - t, \quad \eta = 1 - \varrho + t, \tag{7.7}
$$

the problem (7.6), (7.2), in view of

$$
U_n(\xi, \eta) = u_n^{(2)}(\varrho(\xi, \eta), t(\xi, \eta)), \tag{7.8}
$$

becomes a Darboux-Goursat problem $P_{\alpha,3}$:

$$
U_{n,\xi\eta} - C(\xi, \eta)U_n = G(\xi, \eta) \equiv \exp\{\Lambda(1 - \xi)\} F(\xi, \eta), \tag{7.9}
$$

$$
U_n(0, \eta) = 0, \quad (U_{n,\eta} - U_{n,\xi})(\xi, \xi) + \alpha(1 - \xi)U_n(\xi, \xi) = 0. \tag{7.10}
$$

Here, the coefficients

$$
C(\xi, \eta) = \frac{4n^2 - 1}{4(2 - \eta - \xi)^2} \in C^\infty(\bar{D}_\varepsilon^{(1)}), \quad n \ge 4, \tag{7.11}
$$

and

$$
F(\xi, \eta) = 2^{n-\frac{5}{2}} \Big[\frac{(1 - \xi)(1 - \eta)}{2 - \eta - \xi}\Big]^{n-\frac{1}{2}} \in C^{n-1}(\bar{D}_\varepsilon^{(1)}) \tag{7.12}
$$

are defined by (2.16) and (2.15). Now, we need some information about the behavior of the function $U_n(\xi, \eta)$. Since, by Theorem 6.3,

$$
U_n(\xi, \eta) = \exp\{\Lambda(\varrho + t)\}\ W(\xi, \eta) = \exp\{\Lambda(1 - \xi)\}\ W(\xi, \eta),
$$

$W(\xi, \eta) \ge 0$ and $W_\eta(\xi, \eta) \ge 0$ in $\bar{D}_\varepsilon^{(1)}$, in view of Proposition 6.4, we formulate the following result.

Proposition 7.2. *There exists a classical solution $U_n(\xi, \eta) \in C^n(\bar{D}_0 \setminus (1, 1))$ for (7.9), (7.10) for which*

$$
U_n(\xi, \eta) \ge 0, \quad U_{n,\eta}(\xi, \eta) \ge 0, \quad (\xi, \eta) \in \bar{D}_\varepsilon^{(1)}. \tag{7.13}
$$

Put

$$K_1 = \int_{D_0} G^2(\xi,\eta)\,d\xi\,d\eta > 0. \tag{7.14}$$

Then, by (7.9), for $0 < \varepsilon < 1$ it follows that

$$\begin{aligned} K_1 &\geq \int_{D_\varepsilon^{(1)}} G^2(\xi,\eta)\,d\xi\,d\eta \geq \int_{D_\varepsilon^{(1)}} G(\xi,\eta)\,F(\xi,\eta)\,d\xi\,d\eta \\ &= \int_{D_\varepsilon^{(1)}} U_{n,\xi\eta}F(\xi,\eta)\,d\xi\,d\eta - \int_{D_\varepsilon^{(1)}} C(\xi,\eta)U_n(\xi,\eta)F(\xi,\eta)\,d\xi\,d\eta \\ &=: I_1 - I_2. \end{aligned} \tag{7.15}$$

Then

$$\begin{aligned} I_1 &= \int_0^{1-\varepsilon}\int_\xi^1 (U_{n,\xi\eta}F)(\xi,\eta)\,d\eta\,d\xi \\ &= -\int_0^{1-\varepsilon} U_{n,\xi}(\xi,\xi)F(\xi,\xi)\,d\xi - \int_{D_\varepsilon^{(1)}} (U_{n,\xi}F_\eta)(\xi,\eta)\,d\xi\,d\eta, \end{aligned} \tag{7.16}$$

and, by (7.12), $F(\xi,1) = 0$. Since

$$\begin{aligned} &\int_{D_\varepsilon^{(1)}} (U_{n,\xi}F_\eta)(\xi,\eta)\,d\xi\,d\eta \\ &= \int_0^{1-\varepsilon} (U_nF_\eta)(\eta,\eta)\,d\eta + \int_{1-\varepsilon}^1 (U_nF_\eta)(1-\varepsilon,\eta)\,d\eta - \int_{D_\varepsilon^{(1)}} (U_nF_{\xi\eta})(\xi,\eta)\,d\xi\,d\eta, \end{aligned} \tag{7.17}$$

equation (7.16) becomes

$$\begin{aligned} I_1 = &-\int_0^{1-\varepsilon} [U_{n,\xi}(\xi,\xi)F(\xi,\xi) + U_n(\xi,\xi)F_\eta(\xi,\xi)]\,d\xi \\ &- \int_{1-\varepsilon}^1 (U_nF_\eta)(1-\varepsilon,\eta)\,d\eta + \int_{D_\varepsilon^{(1)}} (U_nF_{\xi\eta})(\xi,\eta)\,d\xi\,d\eta. \end{aligned} \tag{7.18}$$

From (7.18) and (7.15) it follows that

$$\begin{aligned} K_1 \geq & I_1 - I_2 = -\int_0^{1-\varepsilon} [U_{n,\xi}(\xi,\xi)F(\xi,\xi) + U_n(\xi,\xi)F_\xi(\xi,\xi)]\,d\xi \\ &- \int_{1-\varepsilon}^1 (U_nF_\eta)(1-\varepsilon,\eta)\,d\eta + \int_{D_\varepsilon^{(1)}} U_n[F_{\xi\eta} - CF](\xi,\eta)\,d\xi\,d\eta. \end{aligned} \tag{7.19}$$

Because of (6.18), the last integral vanishes. Thus, using the boundary conditions for the functions U_n and F, when $\eta = \xi$, we see that

$$\begin{aligned} K_1 &\geq I_1 - I_2 \\ &= -\int_0^{1-\varepsilon} [U_{n,\xi}(\xi,\xi)F(\xi,\xi) + U_n(\xi,\xi)F_\xi(\xi,\xi)]\,d\xi - \int_{1-\varepsilon}^1 (U_nF_\eta)(1-\varepsilon,\eta)\,d\eta \\ &= -\frac{1}{2}(FU_n)(1-\varepsilon,1-\varepsilon) - \frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U_n(\xi,\xi)F(\xi,\xi)\,d\xi \\ &\quad - \int_{1-\varepsilon}^1 (U_nF_\eta)(1-\varepsilon,\eta)\,d\eta. \end{aligned} \tag{7.20}$$

Since $\alpha(\xi)$, $F_\eta \le 0$ and $F, U_n, U_{n,\eta} \ge 0$, for $0 < \delta < \varepsilon < 1$, we have

$$
\begin{aligned}
K_1 &\ge I_1 - I_2 \\
&\ge -\frac{1}{2}(U_n F)(1-\varepsilon, 1-\varepsilon) + \int_{1-\varepsilon}^1 U_n(1-\varepsilon,\eta)|F_\eta(1-\varepsilon,\eta)|\,d\eta \\
&\ge -\frac{1}{2}(U_n F)(1-\varepsilon, 1-\varepsilon) + \int_{1-\delta}^1 U_n(1-\varepsilon,\eta)|F_\eta(1-\varepsilon,\eta)|\,d\eta \\
&\ge -\frac{1}{2}(U_n F)(1-\varepsilon, 1-\varepsilon) + \int_{1-\delta}^1 U_n(1-\varepsilon,1-\delta)|F_\eta(1-\varepsilon,\eta)|\,d\eta \\
&\ge -\frac{1}{2}(U_n F)(1-\varepsilon, 1-\varepsilon) + (U_n F)(1-\varepsilon, 1-\delta) \\
&\ge U_n(1-\varepsilon,1-\delta)\Big[F(1-\varepsilon,1-\delta) - \frac{1}{2}F(1-\varepsilon,1-\varepsilon)\Big] \\
&\ge \nu(U_n F)(1-\varepsilon,1-\delta),
\end{aligned}
\tag{7.21}
$$

provided taht the constant $v > 0$ satisfies

$$2(1-\nu)F(1-\varepsilon,1-\delta) \ge F(1-\varepsilon,1-\varepsilon). \tag{7.22}$$

Using the explicit formula (7.12) for the function $F(\xi,\eta)$, we see that the above inequality is equivalent to

$$2(1-\nu)\Big(\frac{\delta}{\varepsilon+\delta}\Big)^{n-\frac{1}{2}} \ge 2^{-n+\frac{1}{2}}, \tag{7.23}$$

which implies

$$0 < \nu \le 1 - \frac{1}{2}\Big(\frac{\varepsilon+\delta}{2\delta}\Big)^{n-\frac{1}{2}}. \tag{7.24}$$

A necessary condition, for (7.24) to be satisfied is that

$$1 \le \frac{\varepsilon}{\delta} < 2^{\frac{2n+1}{2n-1}} - 1. \tag{7.25}$$

In this concrete case, using (7.25), we can find an upper estimate for the generalized solution u_n. To do this, we consider the domain

$$D^\mu := \{(\xi,\eta) : 1-\eta \le 1-\xi \le \mu(1-\eta)\}, \tag{7.26}$$

where $1 \le \mu < 2^{\frac{2n+1}{2n-1}} - 1$. Observe that

$$\inf_{D^\mu}\Big\{1 - \frac{1}{2}\Big(\frac{1-\xi+1-\eta}{2(1-\eta)}\Big)^{n-\frac{1}{2}}\Big\} = 1 - \frac{1}{2}\Big(\frac{1+\mu}{2}\Big)^{n-\frac{1}{2}} =: C_\mu > 0.$$

For $\nu = C_\mu$ the inequalities (7.23) and (7.22) are satisfied and so, by (7.21), we see that

$$U(\xi,\eta) \le 2^{-n+5/2}K_1 C_\mu^{-1}\Big(\frac{2-\xi-\eta}{(1-\xi)(1-\eta)}\Big)^{n-\frac{1}{2}}, \quad (\xi,\eta) \in D^\mu. \tag{7.27}$$

By (6.7) and (6.6), the inequality (7.27) transforms to

$$u_n^{(2)}(\varrho,t) \le 4K_1 C_\mu^{-1}\Big(\frac{\varrho}{\varrho^2-t^2}\Big)^{n-\frac{1}{2}}, \tag{7.28}$$

which is satisfied for $(\varrho,t) \in D_1^\mu := \big\{0 < \varrho - t \le \varrho + t \le \mu(\varrho - t)\big\}$. Finally, (7.28) implies

$$u_n^{(1)}(\varrho,t) \le 4K_1 C_\mu^{-1}\varrho^{-1/2}\Big(\frac{\varrho}{\varrho^2-t^2}\Big)^{n-\frac{1}{2}} \quad \text{for } (\varrho,t) \in D_1^\mu, \tag{7.29}$$

which coincides with the estimate (7.4). Note that $C_\mu = 1/2$ on $\{t = 0\}$ and so

$$u_n^{(1)}(\varrho, 0) \le 8K_1\varrho^{-n}, \quad 0 < \varrho < 1, \tag{7.30}$$

which is the upper estimate in (7.5). The proof of Theorem 7.1 is complete. □

Remark 7.3. Since in Theorems 6.1, 6.3 and 7.1 the conditions imposed upon lower terms of (6.1) are not invariant with respect to substitution of the independent variables

$$v(\varrho, \varphi, t) = u(\varrho, \varphi, t)\ \exp\lambda(\varrho, t), \tag{7.31}$$

for various functions $\lambda(\varrho, t)$, we can find a series of singular solutions of Problem P_α for different classes of equations of the form (6.1). This procedure is interesting by itself and is demonstrated by the following simple example

Example. Consider the special form of the equation (6.1), with constant lower order terms, that is

$$Lu \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} + b_1u_{x_1} + b_2u_{x_2} + bu_t + \frac{1}{4}(b_1^2 + b_2^2 - b^2)u = f, \text{ in } \Omega_0, \tag{7.32}$$

with the boundary conditions (6.2). Obviously, the equation (7.32) satisfies the conditions of Theorems 1.4 for $\alpha \in C^1([0, 1])$ and we obtain a singular solution u_n By using the transform (6.32), the equation (7.32) becomes the wave equation (7.1). Then Theorem 7.1 becomes useful and in the case, when $\alpha(|x|) \le -b/2$, in addition, we have two-sided estimates (7.5) of the generalized solution $u_n(x_1, x_2, t)$, whose exact behavior around the point O is $(x_1^2 + x_2^2 + t^2)^{-n/2}\cos n(\arctan\frac{x_2}{x_1})$.

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