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Stability of peakons for the generalized Camassa-Holm equation *

Orlando Lopes

Abstract

We study the existence of minimizers for a constrained variational problems in $H^1(\mathbb{R})$. These minimizers are stable waves solutions for the Generalized Camassa-Holm equation, and their derivative may have a singularity (in which case the travelling wave is called a peakon). The existence result is based on a method developed by the same author in a previous work. By giving examples, we show how our method works.

1 Introduction

In this paper of consider the problem

$$\begin{aligned} \text{minimize} \quad & V(u) = -\int_{\mathbb{R}} (uu_x^2 + F(u) + ku^2)\, dx \\ \text{subject to} \quad & I(u) = \int_{\mathbb{R}} (u_x^2 + u^2)\, dx = \lambda > 0. \end{aligned} \tag{1.1}$$

This problem appears in connection with the existence and stability of solitary waves for the generalized Camassa-Holm equation

$$u_t - u_{txx} = 2u_x u_{xx} + uu_{xxx} - [f(u)/2]_x - ku_x. \tag{1.2}$$

In fact, formally, $V(u)$ and $I(u)$ are conserved quantities for (1.2) and then the set of the minimizers of the problem above is stable with respect to (1.2) (see [2]) and its elements are solitary waves (including their translations in the space variable x). Some information about the Cauchy problem for (1.2) can be found in [4].

If, for a given $\lambda > 0$, the set of the minimizers is finite (with possibly different multipliers), then each individual solitary wave is stable. In particular, if, for a given $\lambda > 0$, the minimizer is unique (which is the case if $k = 0$ and $F(u) = cu^3, c > 0$), then the unique solitary wave is stable.

The derivative of the solutions of the variational problem above may have a singularity at some point (which can be taken as $x = 0$); it may also be an

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everywhere regular solution. The first type will be called a peaked solution (or peakon) and the later smooth solitary wave.

In the integrable case $F(u) = u^3$ and $k = 0$ problem (1.1) has been studied in [3] using the concentration-compactness method and some results about compact sets in some spaces of distributions. The homogeneity of $V(u)$ plays an important role in the proof.

In [4] the local minimization problem has been considered and an elementary proof is given. It is based on very special identities and, probably, the existence of such identities is related to the integrability.

In the case $F(u) = u^3$ and $k > 0$ considered in [5], the solutions are regular solitary waves and their stability is studied using the method of Grilakkis, Shatah and Strauss [7], GSS method for short, for finding local minimizers.

In the integrable case $F(u) = u^3$ (with $k = 0$) the solitary waves are $ce^{-|x|}$ and then they belong to the space $W^{1,\infty}(R)$. The reason for this is that the integrated Euler-Lagrange equation gives the following equation for the solitary waves:

$$u_x^2 = \frac{\alpha u^2 - F(u)}{\alpha - u}$$

where $\alpha > 0$ is the multiplier (velocity). Now for if $F(u) = u^3$ we see that if for some x the denominator $\alpha - u(x)$ vanishes then the numerator $\alpha u^2(x) - F(u(x)) = \alpha u^2(x) - u^3(x) = u^2(x)(\alpha - u(x))$ also vanishes. For more general nonlinearities this does not happen in general; for instance, take $F(u) = cu^3, c \neq 1$. As a consequence, close to the point where the solution $u(x)$ is singular, say, $x = 0$, $u(x)$ behaves as $x^{2/3}$ and $u_x(x)$ as $x^{-1/3}$. So, in this case, the best we can have is that $u \in W^{1,p}_{loc}$ for $1 \leq p < 3$.

Due to the fact that the terms in the constraint have different behavior with respect to dilations, the verification of the strict subadditivity of $V(u)$ for more general nonlinearities could be difficult.

In this paper we show the existence of minimizer for the problem above for a large class of nonlinearities. We use a method that we have developed and applied in several problems. A more abstract version of that method is presented in [10]. The main assumption that has to be verified is that any critical point which is candidate to be minimizer has Morse index ≥ 1 (this implies that it is actually equal to 1). We show that for problem (1.1) this condition is met for very general nonlinearities $F(u)$. In particular, there is no convexivity assumption on $F(u)$.

For simplicity we make assumptions to guarantee that the minimization problem picks positive solutions but we can also consider more general cases. Under minor changes, we can also study the problem of minimizing $I(u)$ under $V(u) = \lambda$ but, for the sake of tradition, we prefer to consider the problem (1.1) above because $V(u)/2$ is the hamiltonian and $I(u)$ is the charge (or energy) for the evolution equation (1.2).

Note that in the case of peaked solutions, zero belongs to the spectrum of the linearized equation but it is not an eigenvalue; so, the spectrum of the linearized equation accumulates at zero (from the right, of course). This fact seems to exclude the possibility of using the GSS method for finding peaked local minimizers because the spectral property of solitons is not satisfied. It seems to exclude also the possibility of using the implicit function theorem in the functional analytic setting to show that the solitary wave is isolated modulo translation. For special $F(u)$, bifurcation diagrams are given in [9].

2 Statement and Proof of the Main Result

We consider the problem (1.1) for $k \geq 0$. The first and second derivatives of V and I are:

$$V'(u)h = -\int_{\mathbb{R}} (2uu_x h_x + u_x^2 h + f(u)h + 2kuh)\, dx$$
$$V''(u)(h,h) = -\int_{\mathbb{R}} (2uh_x^2 + 4u_x h_x h + f'(u)h^2 + 2kh^2)\, dx$$
$$I'(u)h = \int_{\mathbb{R}} (2u_x h_x + 2uh)\, dx$$
$$I''(u)(h,h) = \int_{\mathbb{R}} (2h_x^2 + 2h^2)\, dx.$$

The minimization will be done in the space $H^1(R)$ under the following assumptions:

A1 $F : \mathbb{R} \to \mathbb{R}$ is a C^2 function satisfying the following two conditions:

i) $F(u)$ and its derivative $f(u)$ vanish at $u = 0$, and its second derivative satisfies the smallness condition, at $u = 0$, $|F''(u)| \leq M|u|^\beta$ for some $M > 0$, $\beta > 0$ and all $|u| \leq 1$.

ii) for $u \neq 0$, we have $uF(u) > 0$.

A2 There is an admissible element $u \in H^1(\mathbb{R})$ such that $V(u) < -k\lambda$.

Note that under Assumption A1, part i, the functionals V and I are of class C^2 and their first and second derivative are given by the formulas above and are uniformly continuous on bounded sets of $H^1(\mathbb{R})$. Our main result is the following.

Theorem 2.1 *Under assumptions A1 and A2 and except for translations in the space variable x, any minimizing sequence of the problem* (1.1) *is precompact in $H^1(R)$. Moreover, any minimizer is a positive function decaying exponentially at infinity.*

Before proving Theorem 2.1 we make some comments about our assumptions: Assumption A1 part i simply says that the term ku^2 incorporates the quadratic part. As far as A2 is concerned, the following statements are true:

1. If $F(u) \geq k_1 u^\gamma$ for $u > 0$ and small, where $k_1 > 0$ and $\gamma < 6$, then assumption A2 is satisfied for any $\lambda > 0$

2. If $F(u) \leq k_2 u^\gamma$ for $u > 0$ and small, for some $k_2 > 0$ and $\gamma > 6$, then for $\lambda > 0$ and small, the infimum of $V(u)$ on the admissible set is not achieved

3. Assumption A2 is satisfied if λ is large. In fact, if $u > 0$ is any element in $H^1(\mathbb{R})$ then $\lim_{t\to+\infty} V(tu) = -\infty$ and this implies the statement.

Next, we define the even piecewise linear function

$$u(x) = \begin{cases} c & \text{if } 0 \leq x \leq a \\ c(1 + a/b - x/b) & \text{if } a \leq x \leq a + b \\ 0 & \text{if } a + b \leq x \end{cases}$$

Imposing the constraint $I(u) = \lambda$ we obain $c^2 = \frac{3\lambda b}{2(3ab+b^2+3)}$.

Taking $b = 1$, $c > 0$ and letting $a \to \infty$ we see that $\lim V(u) = -k\lambda$. This shows that assumption A2 is a necessary condition for Theorem 2.1. If we take $a = 1$ and let $b \to \infty$ then statement 1 follows. Statement 2 is a consequence of an interpolation inequality.

The Euler-Lagrange equation $V'(u) + \alpha I'(u) = 0$ is

$$-2\frac{d}{dx}[(\alpha - u)u_x] - u_x^2 + 2(\alpha - k)u - f(u) = 0 \tag{2.1}$$

and in the region where $u(x) \neq \alpha$ we have

$$u_x^2 = \frac{(\alpha - k)u^2 - F(u)}{\alpha - u(x)}. \tag{2.2}$$

In general, the description of the set of the minimizers for a general $F(u)$ seems to be a hard task. In section 3 we give example for which we can give this description.

Next we state a few lemmata which will be used in the proof of Theorem 2.1. The first is taken from [10, lemma 2.8, part ii].

Lemma 2.2 *If u is the weak limit of a minimizing sequence and u satisfies the Euler-Lagrange equation (2.1), then $(V''(u) + \alpha I''(u))(h, h) \geq 0$ for any $h \in H^1(R)$ such that $I'(u)h = 0$. In particular, there cannot exist two functions $h_i \in H^1(R), i = 1, 2$ with disjoint supports such that $(V''(u) + \alpha I''(u))(h_i, h_i) < 0, i = 1, 2$.*

Lemma 2.3 *If if $u \in H^1(R), u \not\equiv 0$, is the weak limit of a minimizing sequence and u satisfies the Euler-Lagrange equation (2.1), then $u(x) > 0$, $\alpha - u(x) \geq 0$ everywhere and $\alpha > k$. In particular, $\alpha > 0$.*

Proof. From assumption A1, part ii, and using Fatou's lemma we see that

$$\int_{\mathbb{R}} F(-u_-(x))\, dx = 0$$

and this implies that $u(x) \geq 0$ everywhere. The fact that $u(x) > 0$ everywhere follows from (2.2).

Now, suppose by contradiction that there is an open interval J where $\alpha - u(x) < 0$. Then u is a regular solution of (2.1) on J and for any function $h \in H^1(R)$ with support contained in J we have

$$\begin{aligned} &V''(u)(h,h) + \alpha I''(u)(h,h) \\ &= \int_J (2(\alpha - u)h_x^2 + (2u_{xx} - f'(u(x)) + 2(\alpha - k))h^2\,dx \end{aligned}$$

and then we can construct h_1 and h_2 with disjoint supports contained in J such that $V''(u)(h_i,h_i) + \alpha I''(u)(h_i,h_i) < 0$, $i = 1, 2$, but this is impossible because of Lemma 2.2. If $\alpha \leq k$ then (2.2) is violated as x tends to, say, $+\infty$ and this completes the proof. □

Remark. It is easy to construct examples of regular solutions $u(x)$ of (2.1) such that $u(x) > \alpha$ in some interval. Lemma 2.3 says is that such solutions cannot be local minimizers of problem (1.1).

The next lemma tells us that any solution of the Euler-Lagrange equation (2.1) which is candidate to be a minimizer of problem (1.1) has Morse index ≥ 1. This information will be crucial for applying our method.

Lemma 2.4 . *If $u \in H^1(R), u \not\equiv 0$, is the weak limit of a minimizing sequence and u satisfies (2.1) then there is an element $h \in H^1(R)$ such that*

$$V''(u)(h,h)+\alpha I''(u)(h,h) = \int_{\mathbb{R}} [2(\alpha-u)h_x^2 - 4u_x h h_x + (2\alpha - 2k - f'(u))h^2]\,dx < 0. \tag{2.3}$$

Proof. The proof will be divided in three cases. Note that according to the previous lemma, for such element u we must have $\alpha > k$ and $\alpha - u(x) \geq 0$. In all cases we assume that a translation have been made so that $u(x)$ is an even function.

First Case: u is a regular solution. In this case for $h = u_x$ we have $V''(u)(h,h) + \alpha I''(u)(h,h) = 0$ because u_x satisfies the linearized equation. Since u_x changes sign the first case follows from the maximum principle. To handle the singular case we define the even function

$$h_\epsilon(x) = \begin{cases} u_x(x) & \text{if } x \leq -\epsilon \\ u_x(-\epsilon) & \text{if } -\epsilon \leq x \leq \epsilon \\ -u_x(x) & \text{if } \epsilon \leq x \end{cases}$$

where $\epsilon > 0$ will be small. The function h_ϵ is an element of $H^1(\mathbb{R})$ because u_x is odd. Taking in account that u_x and $-u_x$ satisfy the linearized equation in its region of regularity, an integration by parts gives

$$
\begin{aligned}
&[V''(u)+\alpha I''(u)](h_\epsilon,h_\epsilon)\\
&=2[2(\alpha-u)u_xu_{xx}-2u_x^3](-\epsilon)+u_x^2(-\epsilon)\int_{-\epsilon}^{\epsilon}(2(\alpha-k)-f'(u(x))\,dx\\
&=2\big[-u_x^3+(2(\alpha-k)u-f(u))u_x\big](-\epsilon)+u_x^2(-\epsilon)\int_{-\epsilon}^{\epsilon}(2(\alpha-k)-f'(u(x))\,dx.
\end{aligned}
$$

Second Case: u is peaked but $u\in W^{1,\infty}(R)$. In this case, at the singular point $x=0$ we must have $u(0)=\alpha$. So, $u(x)>0$ everywhere and increasing for $x<0$. From the integrated Euler-Lagrange equation (2.2) we see that we also must have $(\alpha-k)u(0)^2-F(u(0))=0$ and then as a consequence of l'Hopital rule we get $u_x(0)^2=-[2(\alpha-k)u(0)-f(u(0))]$ and this implies that

$$\lim_{\epsilon\to0}V''(u)((h_\epsilon,h_\epsilon)=-4\lim_{x\to0_-}u_x^3(x)<0$$

and then h_ϵ verifies (2.3) for ϵ small.
Third Case: u is peaked and $u\notin W^{1,\infty}(R)$. In this case $(\alpha-k)u(0)^2-F(u(0))\neq0$, hence

$$\lim_{x\to0_-}u_x(x)=+\infty$$

and then

$$\lim_{\epsilon\to0}V''(u)((h_\epsilon,h_\epsilon)=-\infty$$

because the term $-u_x^3$ dominates and Lemma 2.4 is proved in all cases. □

As a preparation for the proof of Theorem 2.1, we start with a lemma that prevents vanishing. This lemma is due to P. Lions [8] and it is related to the so called Lieb's lemma [1] (see also [10, lemma 2.3]).

Lemma 2.5 *Let u_n be a bounded sequence in $H^1(R)$ such that the sequence $u(x+c_n)$ converges to zero weakly in $H^1(R)$ for any sequence c_n of elements of R. Then u_n converges to zero strongly in $L^p(R)$ for any p in the interval $2<p\leq\infty$.*

Our next lemma is also taken from [10, lemmas 2.6 and 2.7]. It says that a minimizing sequence satisfies an approximate Euler-Lagrange equation.

Lemma 2.6 *If u_n is a minimizing sequence of the problem* (1.1) *then there are sequences α_n γ_n of real numbers, α_n bounded and γ_n tending to zero, and a sequence ϕ_n of unitary elements of $H^1(R)$ such that for any $h\in H^1(R)$ we have*

$$
\begin{aligned}
&V'(u_n)h+\alpha_nI'(u_n)h+\gamma_n\langle\phi_n,h\rangle\\
&=\int_{\mathbb{R}}[2(\alpha_n-u_n)u_{n,x}h_x-u_{n,x}^2h+2(\alpha_n-k)u_nh\\
&\quad-f(u_n)h+2\gamma_n\phi_{n,x}h_x+2\gamma_n\phi_nh]\,dx=0.
\end{aligned}
\tag{2.4}
$$

Proof of Theorem 2.1 Let u_n be a minimizing sequence for problem (1.1). We first note that there is a sequence of real numbers c_n such that the sequence $u_n(\cdot + c_n)$ has a subsequence converging weakly in $H^1(R)$ to a nonzero function. In fact, otherwise, according to Lemma 2.5, u_n converges to zero strongly in $L_\infty(R)$ and then

$$\lim_{n\to\infty} V(u_n) = -k\lambda + \lim_{n\to\infty} \int_{\mathbb{R}} (-u_n u_{n,x}^2 - F(u_n) + k u_{n,x}^2)\, dx \geq -k\lambda$$

and this contradicts assumption A2.

Passing to a subsequence, for which we keep the same notation, we can assume that α_n converges to α and that u_n converges weakly in $H^1(R)$ to a nonzero element $u \in H^1(R)$. Passing (2.4) to the limit we also have

$$V'(u)h + \alpha I'(u)h = \int_{\mathbb{R}} (2(\alpha - u)u_x h_x - u_x^2 h + 2(\alpha - k)uh - f(u)h)\, dx = 0 \quad (2.5)$$

for any $h \in H^1(R)$.

In view of the assumptions we have made and the lemmata we have proven, we can use [10, Theorem 2.1] to conclude that u_n converges to u in $L_p(R)$, $2 < p \leq \infty$.

We can improve this convergence in the following way: taking $h = u_n - u$ in (2.4) and (2.5), subtracting one from the other and using the convergence we have already proved we get

$$\lim_{n\to\infty} \int_{\mathbb{R}} [2(\alpha - u)(u_{n,x} - u_x)^2 + 2(\alpha - k)(u_n - u)^2]\, dx = 0.$$

From Lemma 2.3, we know that $\alpha > k$ and $\alpha - u(x) \geq 0$ everywhere and then u_n converges to u also in $L_2(R)$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}} (\alpha - u)(u_{n,x} - u_x)^2\, dx = 0. \quad (2.6)$$

This equation allows us to conclude that u_n converges to u in the H^1 norm but only outside the points where $u(x) \neq \alpha$. If we assume that $u(0) = \alpha$, it remains to show that

$$\lim_{n\to\infty} \int_{-1}^{1} u_{n,x}^2\, dx = \int_{-1}^{1} u_x^2\, dx. \quad (2.7)$$

To accomplish this, we consider a smooth function ψ such that

$$\psi(x) = \begin{cases} 1 & \text{for } -1 \leq x \leq 1 \\ 0 & \text{for } |x| \geq 2. \end{cases}$$

With $J = [-2, -1] \cup [1, 2]$ and $h = \psi$ in (2.4) we obtain

$$\begin{aligned} \int_{-1}^{1} u_{n,x}^2\, dx = &\int_{-1}^{1} (-f(u_n) - 2ku_n + 2\alpha u_n + \gamma_n \phi_n)\, dx \\ &+ \int_J (2(\alpha_n - u_n)u_{n,x}\psi_x - u_{n,x}^2 \psi \\ &- f(u_n)\psi + (2\alpha - 2k)u_n \psi + 2\gamma_n \phi_{n,x}\psi_x + 2\gamma_n \phi_n \psi)\, dx. \end{aligned}$$

Taking the limit in the above equation, using the convergence we have already proved, and taking $h = \psi$ in (2.5), we see that (2.7) holds and Theorem 2.1 is proved. $\square$

As we have pointed out in the introduction, in the case of a peaked solution, zero belongs to the spectrum of the linearized operator

$$L(h) = -2\frac{d}{dx}[(\alpha - u)h_x - u_x h] - 2u_x h_x + 2(\alpha - k)h - f'(u)h$$

but is not an eigenvalue. Zero belongs to the spectrum of L because, due to translation invariance, a critical point of $V(u) + \alpha I(u)$ is not isolated. Furthermore, if $L(h) = 0$ then for $x < 0$ and for $x > 0$, h has to a multiple (with possible different factors) of u_x. Since h has to be continuous, it remains to consider the case u_x is bounded and has a nonzero limit at $x = 0$. But, in this case,

$$u_{xx} = \frac{u_x^2 + \alpha u - f(u)}{\alpha - u}$$

does not belong to $L^2(\mathbb{R})$ because the numerator has a nonzero limit, equal to $3\lim_{x\to 0} u_x^2(x)/2$, and the denominator behaves as x for $x = 0$.

3 Examples

First we use a trick presented in [5]: To give a formula for $I(u(\alpha))$ as a function of the multiplier α.

We start by defining the functions $F_1(u) = F(u)/u^2$, $u_1(\alpha)$ as the first zero (if any) of $\alpha - k - F_1(u)$, and $u_0(\alpha) = \min(u_1(\alpha), \alpha)$. Note that $u_0(\alpha) = \alpha$ for peaked solutions and $u_0(\alpha) = u_1(\alpha)$ for regular solitary waves.

For $x < 0$ the solution $u(x)$ is increasing and satisfies $0 \le u(x) \le u_0(\alpha)$. Then

$$u_x = \frac{u\sqrt{\alpha - k - F_1(u)}}{\sqrt{\alpha - u}}u$$

and then

$$u_x^2 + u^2 = \frac{(\alpha - k)u^2 - F(u)}{\alpha - u} + u^2 = \frac{u[(2\alpha - k) - u - F_1(u)]}{\sqrt{(\alpha - u)(\alpha - k - F_1(u)}}u_x\,.$$

This implies

$$\begin{aligned} I(\alpha) = I(u(\alpha)) &= 2\int_{-\infty}^{0} \frac{u[(2\alpha - k) - u - F_1(u)]}{\sqrt{(\alpha - u)(\alpha - k - F_1(u))}}u_x\,dx \\ &= 2\int_0^{u_0(\alpha)} \frac{u[(2\alpha - k) - u - F_1(u)]}{\sqrt{(\alpha - u)(\alpha - k - F_1(u))}}\,du. \end{aligned} \tag{3.1}$$

As a first example we take $F(u) = pu^3 + ku^2$ with $p, k > 0$. According to Theorem 2.1 for any $\lambda > 0$ problem (1.1) has a global minimizer. In this section we study the function $I(\alpha)$ as a function of the multiplier.

For convenience we change notation and replace α by y. Defining the function $R(x) = a + bx + cx^2$ with $c > 0$, and the indefinite integral

$$I_m(x) = \int \frac{x^m}{\sqrt{R(x)}}\,dx,$$

then according to [6, pages 81 and 83], we have

$$\begin{gathered}
I_0(x) = \frac{1}{\sqrt{c}}\log(2\sqrt{cR(x)} + 2cx + b),\\
I_1(x) = \frac{\sqrt{R(x)}}{c} - \frac{b}{2c}I_0(x),\\
I_2(x) = \big(\frac{x}{2c} - \frac{3b}{4c^2}\big)\sqrt{R(x)} + \big(\frac{3b^2}{8c^2} - \frac{a}{2c}\big)I_0(x).
\end{gathered}$$

Using (3.9) we get

$$I(y)/2 = \int_0^{u_0(y)} \frac{(2y-k)u - (p+1)u^2}{\sqrt{(y-u)(y-k-pu)}}\,dx = (2y-k)I_1 - (p+1)I_2.$$

Next we define the following functions and compute their derivatives:

$$\begin{gathered}
f(y) = (-kp + 3k - 3p^2y + 2py - 3y)/4p^2, \quad f'(y) = (-3p^2 + 2p - 3)/(4p^2),\\
r(y) = (p-1)y + k, \quad r'(y) = (p-1),\\
t(y) = \sqrt{y(y-k)}, \quad t'(y) = (2y-k)/(2t(y)),\\
s(y) = 2\sqrt{p}t(y) + k - (p+1)y, \quad s'(y) = \sqrt{p}(2y-k)/t(y) - (p+1),\\
\overline{s}(y) = 2\sqrt{p}t(y) - k + (p+1)y, \quad \overline{s}'(y) = (2y-k)/2t(y) + (p+1),\\
q(y) = (kp - 3k - 3p^2y + 3y)r(y)/(8p^2), \quad q'(y) = (1-p)(3p^2y + pk + 3k - 3y),\\
h(y) = 8(1-p)y^2 + 4k(p-3)y + (p+3)k^2.
\end{gathered}$$

Then $I(y)/2 = -t(y)f(y) + q(y)I_0(y)$ and

$$I'(y)/2 = -\frac{(2y-k)}{t(y)}f(y) - t(y)f'(y) + q'(y)I_0(y) + q(y)I_0'(y).$$

In order to analyze the behavior of $I'(y)$ we define the functions

$$g(y) = \big(\frac{2y-k}{t(y)}f(y) + t(y)f'(y) - q(y)I_0'(y)\big)/q'(y)$$

and $w(y) = g(y) - I_0(y)$.

To give the formula for $I_0(y)$ we have to consider two cases.

First case (smooth solitary wave) $k \le y \le k/(1-p)$: In this case, $u_0(y) = (y-k)/p$ and

$$I_0(y) = \frac{1}{\sqrt{p}}\log(-r(y)/s(y)) = \frac{1}{\sqrt{p}}\log(\overline{s}(y)/r(y))$$

because $s(y)\overline{s}(y) = -r^2(y)$. Defining $y_0 = \frac{k(p+3)}{3(1-p)(1+p)}$, we have $k < y_0 < k/(1-p)$. Now we collect the following elementary facts: Regarding q, we have $q'(y) > 0$ for $k < y < y_0$, $q'(y_0) = 0$ and $q'(y) < 0$ for $y_0 < y < k/(1-p)$. Regarding g, we have

$$g(y) = \frac{-2(3p^2 - 2p + 3)t(y)}{q'(y)}$$

then $g(k) = 0$, $g(y) < 0$ for $k < y < y_0$, $\lim_{y \to y_0^-} g(y) = -\infty$, $\lim_{y \to y_0^+} g(y) = +\infty$ and $g(y) > 0$ for $y_0 < y \le k/(1-p)$. Regarding I_0, we have $I_0(k) = 0$ and $\lim_{y \to (k/(1-p))^-} I_0(y) = +\infty$. Regarding w, we have $w(k) = 0$, $\lim_{y \to y_0^-} w(y) = -\infty$, $\lim_{y \to y_0^+} w(y) = +\infty$, $\lim_{y \to (k/(1-p))^-} w(y) = -\infty$, and

$$w'(y) = \frac{-4p^2(p-1)h(y)}{r(y)q'^2(y)} < 0.$$

This last inequality holds because the interval $[k, k/(1-p)]$ is interior to the interval whose ends are the roots of the quadratic equation $h(y) = 0$.

Putting the above facts together we see that there is a point y_1, $y_0 < y_1 < k/(1-p)$ such that $I'(y) > 0$ for $k < y < y_1$, $I'(y_1) = 0$ and $I'(y) < 0$ for $y_1 < y < k/(1-p)$.

Second Case (peakon): $k/(1-p) \le y$. In this case, $u_0(y) = y$ and

$$I_0(y) = \frac{1}{\sqrt{p}} \log(r(y)/s(y)) = \frac{1}{\sqrt{p}} \log(-\overline{s}(y)/r(y))\,.$$

As elementary facts we have: Regarding q: $q'(y) < 0$. Regarding, g we have $g(y) > 0$ and

$$\lim_{y \to +\infty} g(y) = \frac{2(3p^2 - 2p + 3)}{3(1-p)^2(p+1)}\,.$$

Regarding I_0, we have: $\lim_{y \to (k/(1-p))^+} I_0(y) = +\infty$ and

$$\lim_{y \to +\infty} I_0(y) = \frac{1}{\sqrt{p}} \log\Big(\frac{(1-p)}{p - 2\sqrt{p} + 1}\Big)\,.$$

Regarding w, we have:

$$\lim_{y \to +\infty} w(y) = \frac{2(3p^2 - 2p + 3)}{3(1-p)^2(p+1)} - \frac{1}{\sqrt{p}} \log\Big(\frac{(1-p)}{p - 2\sqrt{p} + 1}\Big) > 0,$$

this inequality because by defining

$$\phi(p) = \sqrt{p}\Big(\frac{2(3p^2 - 2p + 3)}{3(1-p)^2(p+1)} - \frac{1}{\sqrt{p}} \log\Big(\frac{(1-p)}{p - 2\sqrt{p} + 1}\Big)\Big)\,,$$

we have $\phi(0) = 0$ and $\phi'(p) = \frac{-32p^2}{3\sqrt{p}(p+1)^3(p-1)^3} > 0$. Also, there exists $y_2 > k/(1-p)$ such that $w'(y) = \frac{-4p^2(p-1)h(y)}{r(y)q'^2(y)} > 0$ for $k/(1-p) < y < y_2$, $w(y_2) = 0$,

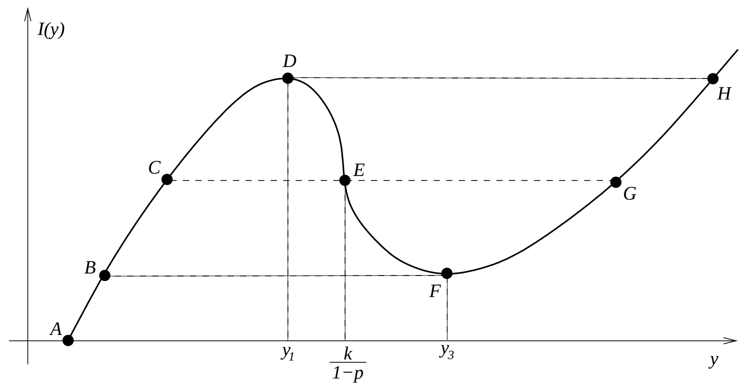

Figure 1: Behavior of $I(y)$ for $y < k/(1-p)$

and $w'(y) < 0$ for $y_2 < y$. From the inequalities above, we see that there is a point $y_3 > y_2$ such that $I'(y) < 0$ for $k/(1-p) < y < y_3$, $I'(y_3) = 0$ and $I'(y) > 0$ for $y_3 < y$.

Figure 1 shows the qualitative behavior of $I(y)$. For $y < k/(1-p)$ we are in the region of solitary waves. The region AB corresponds to global minimizers whose existence is given by Theorem 2.1. Numerical experiments indicate that the branch of global minimizer goes up to the point C which is in the level of the transition point E. The branch ABCDE, except for the point D, falls in the GSS method. So, from A to D the solitary waves are stable (as local minimizers) and from D to E they are linearly unstable.

For $y > k/(1-p)$ we are in the region of peakons. There the GSS method as it is proved in [7] does not apply because the spectrum of the linearized operator does not have the spectral property of solitons.

The upper branch starting at the point H corresponds to global minimizers (hence stable peakons) whose existence is given by Theorem 2.1. As in the region of solitary waves, numerical experiments indicate that the branch of global minimizers goes further to the point G which is in the level of the transition point E. Probably the branch EF corresponds to unstable peakons and FG to stable ones (local minimizers) but so far there are no results that support this conclusion.

We can also consider the case $p \geq 1$ and $k > 0$. In this case $u_0(y) = (y-k)/p$ (solitary wave) and elementary calculations give the following facts: $I_0(y) = \frac{1}{\sqrt{p}} \log(\overline{s}(y)/r(y))$ is regular everywhere, $q'(y) > 0$, and $w'(y) > 0$. Then $I'(y) > 0$ for $y > k$ So, these solitary waves are stable global minimizers. The fact that they are local minimizers follows from the GSS method. The case $p = 1$ has been considered in [5]; but their trick to verify that $I'(y) > 0$ does not seem to work in the case $p > 1$.

When $k = 0$ it is easy to see that $I(y) = yI(1)$ and then our minimization result gives the existence of a stable solitary wave if $p > 1$ and of a stable peakon if $p \leq 1$ (the case $p = 1$ has been considered in [3]).

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Orlando Lopes
Departamento de Matematica-IMECC-UNICAMP-
C.P. 6065 Campinas, SP 13083-859, Brasil
e-mail: lopes@ime.unicamp.br