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Multiple positive solutions for a class of quasilinear elliptic boundary-value problems *

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Abstract

Using variational arguments we prove some nonexistence and multiplicity results for positive solutions of a class of elliptic boundary-value problems involving the p-Laplacian and a parameter.

1 Introduction

In a recent paper, Maya and Shivaji [4] studied the existence, multiplicity, and non-existence of positive classical solutions of the semilinear elliptic boundary-value problem

$$\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\lambda > 0$ is a parameter, and f is a C^1 function such that

$$f(0) = 0, \quad \lim_{t \to \infty} \frac{f(t)}{t} = 0. \tag{1.2}$$

Assuming

- (f_1) $f'(0) < 0$,
- (f_2) $\exists \beta > 0$ such that $f(t) < 0$ for $0 < t < \beta$ and $f(t) > 0$ for $t > \beta$,
- (f_3) f is eventually increasing,

they showed using sub-super solutions arguments and recent results from semi-positone problems that there are $\underline{\lambda}$ and $\overline{\lambda}$ such that (1.1) has no positive solution for $\lambda < \underline{\lambda}$ and at least two positive solutions for $\lambda \geq \overline{\lambda}$.

In the present paper we consider the corresponding quasilinear problem

$$\begin{aligned} -\Delta_p u &= \lambda f(x, u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.3}$$

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where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian, $1 < p < \infty$, $\lambda > 0$, and f is a Carathéodory function on $\Omega \times [0,\infty)$ satisfying

$$f(x,0) = 0, \quad |f(x,t)| \le Ct^{p-1} \tag{1.4}$$

for some constant $C > 0$. Note that when $p = 2$ and f is C^1 and satisfies (1.2), the existence of the limits $\lim_{t\to 0} f(t)/t = f'(0)$ and $\lim_{t\to\infty} f(t)/t$ imply (1.4). Using variational methods, we shall prove the following theorems.

Theorem 1.1. *There is a* $\underline{\lambda}$ *such that* (1.3) *has no positive solution for* $\lambda < \underline{\lambda}$.

Theorem 1.2. *Set* $F(x,t) = \int_0^t f(x,s)ds$, *and assume*

(F$_1$) $\exists \delta > 0$ *such that* $F(x,t) \le 0$ *for* $0 \le t \le \delta$,

(F$_2$) $\exists t_0 > 0$ *such that* $F(x,t_0) > 0$,

(F$_3$) $\displaystyle\overline{\lim_{t\to\infty}} \frac{F(x,t)}{t^p} \le 0$ *uniformly in* x.

Then there is a $\overline{\lambda}$ *such that* (1.3) *has at least two positive solutions* $u_1 > u_2$ *for* $\lambda \ge \overline{\lambda}$.

Note that we have substantially relaxed the assumptions in [4] and therefore our results seem to be new even in the semilinear case $p = 2$. More specifically, we have let f depend on x and dropped the assumption of differentiability in t, and replaced (f$_1$), (f$_2$), and (f$_3$) with the much weaker assumptions (F$_1$) and (F$_2$) on the primitive F. We emphasize that (F$_1$) follows from (f$_1$), while (f$_2$) and (f$_3$) together imply (F$_2$), and that we make no monotonicity assumptions. The limit in (F$_3$) equals 0 in the p-sublinear case

$$\lim_{t\to\infty} \frac{f(x,t)}{t^{p-1}} = 0 \text{ uniformly in } x, \tag{1.5}$$

in particular, in the special case considered in [4].

2 Proofs of Theorems 1.1 and 1.2

Recall that the first Dirichlet eigenvalue of $-\Delta_p$ is positive and is given by

$$\lambda_1 = \min_{u\ne 0} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \tag{2.1}$$

(see Lindqvist [3]). If (1.3) has a positive solution u, multiplying (1.3) by u, integrating by parts, and using (1.4) gives

$$\int_\Omega |\nabla u|^p = \lambda \int_\Omega f(x,u)u \le C\lambda \int_\Omega u^p, \tag{2.2}$$

and hence $\lambda \ge \lambda_1/C$ by (2.1), proving Theorem 1.1.

We will prove Theorem 1.2 using critical point theory. Set $f(x,t) = 0$ for $t < 0$, and consider the C^1 functional

$$\Phi_\lambda(u) = \int_\Omega |\nabla u|^p - \lambda p F(x,u), \quad u \in W_0^{1,p}(\Omega). \tag{2.3}$$

If u is a critical point of Φ_λ, denoting by u^- the negative part of u,

$$0 = (\Phi_\lambda'(u), u^-) = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla u^- - \lambda f(x,u) u^- = \|u^-\|^p \tag{2.4}$$

shows that $u \geq 0$. Furthermore, $u \in L^\infty(\Omega) \cap C^1(\Omega)$ by Anane [1] and di Benedetto [2], so it follows from the Harnack inequality (Theorem 1.1 of Trudinger [6]) that either $u > 0$ or $u \equiv 0$. Thus, nontrivial critical points of Φ_λ are positive solutions of (1.3).

By (F_3) and (1.4), there is a constant $C_\lambda > 0$ such that

$$\lambda p F(x,t) \leq \frac{\lambda_1}{2} |t|^p + C_\lambda \tag{2.5}$$

and hence

$$\Phi_\lambda(u) \geq \int_\Omega |\nabla u|^p - \frac{\lambda_1}{2} |u|^p - C_\lambda \geq \frac{1}{2} \|u\|^p - C_\lambda \mu(\Omega) \tag{2.6}$$

where μ denotes the Lebesgue measure in $\mathbb{R}^n$, so Φ_λ is bounded from below and coercive. This yields a global minimizer u_1 since Φ_λ is weakly lower semicontinuous.

Lemma 2.1. *There is a* $\overline{\lambda}$ *such that* $\inf \Phi_\lambda < 0$, *and hence* $u_1 \neq 0$, *for* $\lambda \geq \overline{\lambda}$.

Proof. Taking a sufficiently large compact subset Ω' of Ω and a function $u_0 \in W_0^{1,p}(\Omega)$ such that $u_0(x) = t_0$ on Ω' and $0 \leq u_0(x) \leq t_0$ on $\Omega \setminus \Omega'$, where t_0 is as in (F_2), we have

$$\int_\Omega F(x,u_0) \geq \int_{\Omega'} F(x,t_0) - C t_0^p \mu(\Omega \setminus \Omega') > 0 \tag{2.7}$$

and hence $\Phi_\lambda(u_0) < 0$ for λ large enough. □

Now fix $\lambda \geq \overline{\lambda}$, let

$$\widetilde{f}(x,t) = \begin{cases} f(x,t), & t \leq u_1(x), \\ f(x,u_1(x)), & t > u_1(x), \end{cases} \quad \text{and} \quad \widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)\,ds. \tag{2.8}$$

Then consider

$$\widetilde{\Phi}_\lambda(u) = \int_\Omega |\nabla u|^p - \lambda p \widetilde{F}(x,u). \tag{2.9}$$

If u is a critical point of $\widetilde{\Phi}_\lambda$, then $u \geq 0$ as before, and

$$\begin{aligned} 0 &= \big(\widetilde{\Phi}'_\lambda(u) - \Phi'_\lambda(u_1), (u-u_1)^+\big) \\ &= \int_\Omega \big(|\nabla u|^{p-2}\nabla u - |\nabla u_1|^{p-2}\nabla u_1\big)\cdot\nabla(u-u_1)^+ \\ &\quad - \lambda\big(\widetilde{f}(x,u) - f(x,u_1)\big)(u-u_1)^+ \\ &= \int_{u>u_1} \big(|\nabla u|^{p-2}\nabla u - |\nabla u_1|^{p-2}\nabla u_1\big)\cdot(\nabla u - \nabla u_1) \\ &\geq \int_{u>u_1} \big(|\nabla u|^{p-1} - |\nabla u_1|^{p-1}\big)\,(|\nabla u| - |\nabla u_1|) \geq 0 \end{aligned} \tag{2.10}$$

implies that $u \leq u_1$, so u is a solution of (1.3) in the order interval $[0, u_1]$. We will obtain a critical point u_2 with $\widetilde{\Phi}_\lambda(u_2) > 0$ via the mountain-pass lemma, which would complete the proof since $\widetilde{\Phi}_\lambda(0) = 0 > \widetilde{\Phi}_\lambda(u_1)$.

Lemma 2.2. *The origin is a strict local minimizer of* $\widetilde{\Phi}_\lambda$.

Proof. Setting $\Omega_u = \big\{x \in \Omega : u(x) > \min\big\{u_1(x), \delta\big\}\big\}$, by (2.8) and ($F_1$), $\widetilde{F}(x, u(x)) \leq 0$ on $\Omega \setminus \Omega_u$, so

$$\widetilde{\Phi}_\lambda(u) \geq \|u\|^p - \lambda p \int_{\Omega_u} \widetilde{F}(x,u). \tag{2.11}$$

By (1.4), Hölder's inequality, and Sobolev imbedding,

$$\int_{\Omega_u} \widetilde{F}(x,u) \leq C \int_{\Omega_u} u^p \leq C\mu(\Omega_u)^{1-\frac{p}{q}}\|u\|^p \tag{2.12}$$

where $q = np/(n-p)$ if $p < n$ and $q > p$ if $p \geq n$, so it suffices to show that $\mu(\Omega_u) \to 0$ as $\|u\| \to 0$.

Given $\varepsilon > 0$, take a compact subset Ω_ε of Ω such that $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and let $\Omega_{u,\varepsilon} = \Omega_u \cap \Omega_\varepsilon$. Then

$$\|u\|_p^p \geq \int_{\Omega_{u,\varepsilon}} u^p \geq c^p \mu(\Omega_{u,\varepsilon}) \tag{2.13}$$

where $c = \min\big\{\min u_1(\Omega_\varepsilon), \delta\big\} > 0$, so $\mu(\Omega_{u,\varepsilon}) \to 0$. But, since $\Omega_u \subset \Omega_{u,\varepsilon} \cup (\Omega \setminus \Omega_\varepsilon)$,

$$\mu(\Omega_u) < \mu(\Omega_{u,\varepsilon}) + \varepsilon, \tag{2.14}$$

and ε is arbitrary. □

An argument similar to the one we used for Φ_λ shows that $\widetilde{\Phi}_\lambda$ is also coercive, so every Palais-Smale sequence of $\widetilde{\Phi}_\lambda$ is bounded and hence contains a convergent subsequence as usual. Now the mountain-pass lemma gives a critical point u_2 of $\widetilde{\Phi}_\lambda$ at the level

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} \widetilde{\Phi}_\lambda(u) > 0 \tag{2.15}$$

where $\Gamma = \big\{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = u_1\big\}$ is the class of paths joining the origin to u_1 (see, e.g., Rabinowitz [5]).

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