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# Solution to a semilinear problem on type II regions determined by the Fucik spectrum *

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**Abstract**

We prove the existence of solutions to the semi-linear problem

$$u''(x)+\alpha u^+(x)+\beta u^-(x)=f(x)\,,\quad x\in(0,\pi)\,,$$
$$u(0)=u(\pi)=0$$

where the point $(\alpha,\beta)$ falls in regions of type (II) between curves of the Fučík spectrum. We use a variational method based on the generalization of the Saddle Point Theorem.

## 1 Introduction

We investigate the existence of solutions for the nonlinear boundary-value problem

$$u''(x)+\alpha u^+(x)-\beta u^-(x)=f(x)\,,\quad x\in(0,\pi)\,, \qquad (1.1)$$
$$x(0)=x(\pi)=0\,.$$

Here $u^\pm=\max\{\pm u,0\}$, $\alpha,\beta\in\mathbb{R}$ and $f\in L^2(0,\pi)$. For $f\equiv 0$, $\alpha=\lambda_+$, and $\beta=\lambda_-$ Problem (1.1) becomes

$$u''(x)+\lambda_+u^+(x)-\lambda_-u^-(x)=0\,,\quad x\in(0,\pi)\,, \qquad (1.2)$$
$$x(0)=x(\pi)=0\,.$$

We define $\Sigma=\{(\lambda_+,\lambda_-)\in\mathbb{R}^2:\text{(1.2) has a nontrivial solution}\}$. This set is called the Fučík spectrum (see [3]), and can be expressed as $\Sigma=\bigcup_{j=1}^\infty\Sigma_j$ where

$$\begin{aligned}\Sigma_1&=\{(\lambda_+,\lambda_-)\in\mathbb{R}^2:(\lambda_+-1)(\lambda_--1)=0\}\,,\\ \Sigma_{2i}&=\{(\lambda_+,\lambda_-)\in\mathbb{R}^2:i\Big(\frac{1}{\sqrt{\lambda_+}}+\frac{1}{\sqrt{\lambda_-}}\Big)=1\}\,,\\ \Sigma_{2i+1}&=\Sigma_{2i+1,1}\cup\Sigma_{2i+1,2}\qquad\text{where}\end{aligned}$$

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$$
\begin{aligned}
\Sigma_{2i+1,1} &= \{(\lambda_+,\lambda_-)\in\mathbb{R}^2 : i\Big(\frac{1}{\sqrt{\lambda_+}}+\frac{1}{\sqrt{\lambda_-}}\Big)+\frac{1}{\sqrt{\lambda_+}}=1\}\,,\\
\Sigma_{2i+1,2} &= \{(\lambda_+,\lambda_-)\in\mathbb{R}^2 : i\Big(\frac{1}{\sqrt{\lambda_+}}+\frac{1}{\sqrt{\lambda_-}}\Big)+\frac{1}{\sqrt{\lambda_-}}=1\}\,.
\end{aligned}
$$

Note that there are two types of regions between the curves of $\Sigma$:

**Type (I)** $\mathcal{R}_1$ which consists of regions between the curves $\Sigma_{2i}$ and $\Sigma_{2i+1}$, $i\in\mathbb{N}$.

**Type (II)** $\mathcal{R}_2$ which consists of regions between the curves $\Sigma_{2i+1,1}$ and $\Sigma_{2i+1,2}$, $i\in\mathbb{N}$.

If $(\alpha,\beta)\in\mathcal{R}_1$ one can solve (1.1) for arbitrary $f\in L^2(0,\pi)$ while this is not so for regions $\mathcal{R}_2$ (see [1]).

We suppose that the point $(\alpha,\beta)\in\mathcal{R}_2$ is between the curves $\Sigma_{2i+1,1}$ and $\Sigma_{2i+1,2}$, $\alpha>\beta$, and $k>0$ such that $\lambda_+=\alpha+k$, $\lambda_-=\beta+k$, $(\lambda_+,\lambda_-)\in\Sigma_{2i+1,2}$ and $\lambda_+<(2i+2)^2$. We denote $\varphi_2$ the solution of (1.2) belonging to $(\lambda_+,\lambda_-)$. In this work we obtain existence results for (1.1) with right hand side $f=-c\varphi_2+\varphi_2^{\perp}$, $c>0$ where $\int_0^\pi \varphi_2\varphi_2^{\perp}\,dx=0$ and $\big(\frac{1}{\beta-(2i)^2}+\frac{1}{(2i+2)^2-\alpha)}\big)\int_0^\pi(\varphi_2^{\perp})^2\,dx<\big(\frac{1}{\lambda_+-\alpha}-\frac{1}{(2i+2)^2-\alpha)}\big)\int_0^\pi(c\varphi_2)^2\,dx$.

This article is inspired by a result in [7] where the author solves the problem $\Delta u(x)+\alpha u^+(x)+\beta u^-(x)+p(x,u(x))=f(x)$ with nontrivial nonlinearity satisfying $p(x,u(x))\neq 0$.

**Remark 1.1** Assuming that $(2i+2)^2>\lambda_+>\lambda_-$, if $(\lambda_+,\lambda_-)\in\Sigma_{2i+1,1}$, then $\lambda_->(2i)^2$.

Assuming that $(2i+2)^2>\alpha$, if the point $(\alpha,\beta)$ is in $\mathcal{R}_2$ between the curves $\Sigma_{2i+1,1}$ and $\Sigma_{2i+1,2}$, then $\beta>(2i)^2$. See the illustration in figure 1.

## 2 Preliminaries

**Notation:** We shall use the classical spaces $C(0,\pi)$, $L^p(0,\pi)$ of continuous and measurable real-valued functions whose $p$-th power of the absolute value is Lebesgue integrable, respectively. $H$ is the Sobolev space of absolutely continuous functions $u\colon(0,\pi)\to\mathbb{R}$ such that $u'\in L^2(0,\pi)$ and $u(0)=u(\pi)=0$. We denote by the symbols $\|\cdot\|$, and $\|\cdot\|_2$ the norm in $H$, and in $L^2(0,\pi)$, respectively. We denote $\langle .,.\rangle$ the pairing in the space $H$.

By a solution of (1.1) we mean a function $u\in C^1(0,\pi)$ such that $u'$ is absolutely continuous, $u$ satisfies the boundary conditions and the equations (1.1) holds a.e. in $(0,\pi)$.

Let $I:H\to\mathbb{R}$ be a functional such that $I\in C^1(H,\mathbb{R})$ (continuously differentiable). We say that $u$ is a critical point of $I$, if

$$\langle I'(u),v\rangle=0\quad\text{for all}\quad v\in H\,.$$

We say that $\gamma$ is a critical value of $I$, if there is $u_0\in H$ such that $I(u_0)=\gamma$ and $I'(u_0)=0$.

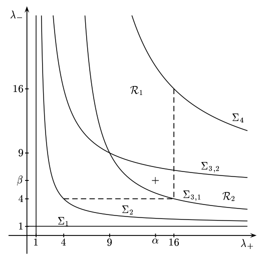


Figure 1: Regions determined by the Fucik spectrum

We say that $I$ satisfies Palais-Smale condition (PS) if every sequence $(u_n)$ for which $I(u_n)$ is bounded in $H$ and $I'(u_n) \to 0$ (as $n \to \infty$) possesses a convergent subsequence.

We study (1.1) by using of varitional methods. More precisely, we look for critical points of the functional $I : H \to \mathbb{R}$, which is defined by

$$I(u) = \frac{1}{2} \int_0^\pi \left[(u')^2 - \alpha(u^+)^2 - \beta(u^-)^2\right] dx + \int_0^\pi f u \, dx \,. \tag{2.1}$$

Every critical point $u \in H$ of the functional $I$ satisfies

$$\langle I'(u), v\rangle = \int_0^\pi \left[u'v' - (\alpha u^+ - \beta u^-)v + fv\right] dx = 0 \qquad \text{for all} \quad v \in H \,.$$

Then $u$ is also a weak solution of (1.1) and vice versa.

The usual regularity argument for ODE yields immediately (see Fučík [3]) that any weak solution of (1.1) is also the solution in the sense mentioned above.

**Method:** We will use the following variant of Saddle Point Theorem which is proved in Struwe [6, Theorem 8.4].

**Theorem 2.1** *Let $S$ be a closed subset in $H$ and $Q$ a bounded subset in $H$ with relative boundary $\partial Q$. Set $\Gamma = \{h : h \in \mathbf{C}(H,H), h(x) = x \text{ on } \partial Q\}$. Suppose $I \in C^1(H,\mathbb{R})$ and*

*(i) $S \cap \partial Q = \emptyset$,*

*(ii) $S \cap h(Q) \neq \emptyset$, for every $h \in \Gamma$,*

*(iii) there are constants $\mu, \nu$ such that $\mu = \inf_{u\in S} I(u) > \sup_{u\in\partial Q} I(u) = \nu$,*

*(iv) $I$ satisfies Palais-Smale condition.*

*Then the number*

$$\gamma = \inf_{h\in\Gamma} \sup_{u\in Q} I(h(u))$$

*defines a critical value $\gamma > \nu$ of $I$.*

**Remark:** We say that $S$ and $\partial Q$ *link* if they satisfy conditions i), ii) of the theorem above.

Now we present a few results needed later.

**Lemma 2.2** *Let $\varphi$ be a solution of (1.2) with $(\lambda_+, \lambda_-) \in \Sigma$, $\lambda_+ \geq \lambda_-$ and $u = a\varphi + w$, $a \geq 0$, $w \in H$. Then functional $J(u) = \int_0^\pi \big[(u')^2 - \lambda_+(u^+)^2 - \lambda_-(u^-)^2\big]\,dx$ satisfies*

$$\int_0^\pi \big[(w')^2 - \lambda_+ w^2\big]\,dx \leq J(u) \leq \int_0^\pi \big[(w')^2 - \lambda_- w^2\big]\,dx\,. \tag{2.2}$$

**Proof:** We prove the inequality in the right of (2.2), the proof of the inequality in the left is similar. Since $\varphi$ is the solution of (1.2) it holds

$$\int_0^\pi (\varphi')^2\,dx = \int_0^\pi \big[\lambda_+(\varphi^+)^2 + \lambda_-(\varphi^-)^2\big]\,dx \tag{2.3}$$

and

$$\int_0^\pi \varphi' w'\,dx = \int_0^\pi \big[\lambda_+\varphi^+ w - \lambda_-\varphi^- w\big]\,dx \qquad \text{for} \quad w \in H\,. \tag{2.4}$$

By (2.3) and (2.4) we obtain

$$J(u) = \int_0^\pi \Big[((a\varphi + w)')^2 - \lambda_+((a\varphi + w)^+)^2 - \lambda_-((a\varphi + w)^-)^2\Big]\,dx$$

$$
\begin{aligned}
&= \int_0^\pi \Big[(a\varphi')^2 + 2a\varphi' w' + (w')^2 - (\lambda_+ - \lambda_-)((a\varphi + w)^+)^2 \\
&\qquad -\lambda_-(a\varphi + w)^2\Big]\, dx \\
&= \int_0^\pi \Big[(\lambda_+ - \lambda_-)(a\varphi^+)^2 + \lambda_-(a\varphi)^2 + 2a((\lambda_+ - \lambda_-)\varphi^+ + \lambda_-\varphi)w \\
&\qquad +(w')^2 - (\lambda_+ - \lambda_-)((a\varphi + w)^+)^2 - \lambda_-((a\varphi)^2 + 2a\varphi w + w^2)\Big]\, dx \\
&= \int_0^\pi \Big\{(\lambda_+ - \lambda_-)\big[(a\varphi^+)^2 + 2a\varphi^+ w - ((a\varphi + w)^+)^2\big] \\
&\qquad +(w')^2 - \lambda_- w^2\Big\}\, dx\,. \qquad (2.5)
\end{aligned}
$$

For the function $(a\varphi^+)^2 + 2a\varphi^+ w - ((a\varphi + w)^+)^2$ in the last integral in (2.5) it holds

$$
\begin{aligned}
&(a\varphi^+)^2 + 2a\varphi^+ w - ((a\varphi + w)^+)^2 \\
&= \begin{cases} -((a\varphi + w)^+)^2 \le 0 & \varphi < 0 \\ -w^2 \le 0 & \varphi \ge 0\,, a\varphi + w \ge 0 \\ a\varphi^+(a\varphi^+ + w + w) \le 0 & \varphi \ge 0\,, a\varphi + w < 0\,. \end{cases}
\end{aligned}
$$

Hence and by assumption $\lambda_+ \ge \lambda_-$ we obtain the assertion of the lemma (2.2). □

**Lemma 2.3** *Let $\varphi$ be a solution of (1.2) with $(\lambda_+, \lambda_-) \in \Sigma$ and $\varphi^\perp \in H$ such that $\int_0^\pi \varphi\varphi^\perp\, dx = 0$. Let $d > 0$ and $w \in H$ satisfying $\int_0^\pi [(w')^2 - \lambda_- w^2]\, dx \le -d\int_0^\pi w^2\, dx$. We put $u = a\varphi + w$, $a \ge 0$ and $I(u) = \frac{1}{2}\int_0^\pi [(u')^2 - (\lambda_+ - k)(u^+)^2 - (\lambda_- - k)(u^-)^2 - 2(c\varphi + \varphi^\perp)u]\, dx$ where $0 < k \le d$ and $c > 0$. Then there is constant $\tilde a > 0$ such that for $u = \tilde a\varphi + w$ it holds*

$$
I(u) \quad \le \quad -\frac{1}{2}\int_0^\pi \big[\frac{1}{k}(c\varphi)^2 - \frac{1}{d-k}(\varphi^\perp)^2\big]\, dx \qquad (2.6)
$$

**Proof:** By (2.2) from Lemma 2.2 and the assumptions on $w$, we obtain

$$
\begin{aligned}
&I(u) \\
&= \frac{1}{2}\int_0^\pi \big[(u')^2 - \lambda_+(u^+)^2 - \lambda_-(u^-)^2\big]\, dx + \int_0^\pi \big[\frac{k}{2}u^2 - (c\varphi + \varphi^\perp)u\big]\, dx \\
&\le \frac{1}{2}\int_0^\pi \big[(w')^2 - \lambda_- w^2\big]\, dx + \int_0^\pi \big[\frac{k}{2}u^2 - (c\varphi + \varphi^\perp)u\big]\, dx \le \\
&\le \int_0^\pi \big[-\frac{d}{2}w^2 + \frac{k}{2}u^2 - (c\varphi + \varphi^\perp)u\big]\, dx\,. \qquad (2.7)
\end{aligned}
$$

We substitute $u = a\varphi + w$ and we have for the last integral in (2.7)

$$
\int_0^\pi \big[-\frac{d}{2}w^2 + \frac{k}{2}u^2 - (c\varphi + \varphi^\perp)u\big]\, dx
$$

$$
\begin{aligned}
&= \int_0^\pi \big[-\frac{d}{2}w^2+\frac{k}{2}(a\varphi+w)^2-(c\varphi+\varphi^\perp)(a\varphi+w)\big]\,dx\\
&= \int_0^\pi \big[-\frac{1}{2}(d-k)w^2+a\big(\frac{k}{2}a-c\big)\varphi^2+(ak-c)\varphi w-\varphi^\perp w\big]\,dx\,.
\end{aligned}
$$

We put $a=\frac{c}{k}(=\tilde a)$ and we obtain

$$
\begin{aligned}
&\int_0^\pi \big[-\frac{1}{2}(d-k)w^2+\frac{c}{k}\big(\frac{k}{2}\frac{c}{k}-c\big)\varphi^2-\varphi^\perp w\big]\,dx\\
&= -\frac{1}{2}\int_0^\pi \big[\big(\sqrt{(d-k)}w+\frac{1}{\sqrt{(d-k)}}\varphi^\perp\big)^2+\frac{1}{k}(c\varphi)^2-\frac{1}{d-k}(\varphi^\perp)^2\big]\,dx \le\\
&\le -\frac{1}{2}\int_0^\pi \big[\frac{1}{k}(c\varphi)^2-\frac{1}{d-k}(\varphi^\perp)^2\big]\,dx\,. \qquad (2.8)
\end{aligned}
$$

From the above inequality and (2.7) it follows the assertion of Lemma 2.3. □

**Lemma 2.4** *Let $\widehat\varphi$ be a solution of (1.2) with $(\widehat\lambda_+,\widehat\lambda_-)\in\Sigma$. Let $u=a\widehat\varphi+w$, $a\ge 0$, $f\in L_2(0,\pi)$ and $w\in H$ such that $\int_0^\pi[(w')^2-\widehat\lambda_-w^2]\,dx\le 0$. Let $l>0$ then $\forall k_2<0\ \exists K$ such that for $u$ with $\|u\|_2\ge K$ it holds*

$$
I(u)=\frac{1}{2}\int_0^\pi \big[(u')^2-(\widehat\lambda_++l)(u^+)^2-(\widehat\lambda_-+l)(u^-)^2+2fu\big]\,dx\le k_2<0\,. \qquad (2.9)
$$

**Proof:** From inequality (2.2) in Lemma 2.2 and Hölder inequality we obtain

$$
\begin{aligned}
I(u) &= \frac{1}{2}\int_0^\pi \big[(u')^2-\widehat\lambda_+(u^+)^2-\widehat\lambda_-(u^-)^2\big]\,dx+\int_0^\pi\big[-\frac{l}{2}u^2+fu\big]\,dx\\
&\le \frac{1}{2}\int_0^\pi \big[(w')^2-\widehat\lambda_-w^2\big]\,dx-\frac{l}{2}\|u\|_2^2+\|f\|_2\|u\|_2\,.
\end{aligned}
$$

By assumption, the integral in the above inequality is less than zero. Then it is easy to see that for sufficiently large $K$ and $\|u\|_2>K$, it holds $I(u)\le k_2<0$.

# 3 Main result

**Theorem 3.1** *Let $(\alpha,\beta)$ be a point in $\mathcal{R}_2$ between the curves $\Sigma_{2i+1,1}$ and $\Sigma_{2i+1,2}$,. Let $\alpha>\beta$ and $k>0$ be such that $\lambda_+=\alpha+k$, $\lambda_-=\beta+k$, and $(\lambda_+,\lambda_-)\in\Sigma_{2i+1,2}$. Assume that $\lambda_+<(2i+2)^2$. We denote by $\varphi_2$ the solution of (1.2) with $(\lambda_+,\lambda_-)$ and $\varphi_2^\perp$ be the function satisfying $\int_0^\pi\varphi_2\varphi_2^\perp\,dx=0$. We put $f=-c\varphi_2+\varphi_2^\perp$, with $c>0$ and we assume that*

$$
\begin{aligned}
&\big(\frac{1}{\beta-(2i)^2}+\frac{1}{(2i+2)^2-\alpha}\big)\int_0^\pi(\varphi_2^\perp)^2\,dx\\
&< \big(\frac{1}{\lambda_+-\alpha}-\frac{1}{(2i+2)^2-\alpha}\big)\int_0^\pi(c\varphi_2)^2\,dx\,.
\end{aligned}
$$

*Then there exists a solution of (1.1).*

**Proof:** We take $l > 0$ such that $(\alpha - l, \beta - l) \in \Sigma_{2i+1,1}$ and we denote by $\varphi_1$ the nontrivial solution of (1.2) with $(\alpha - l, \beta - l)$. Let $H^-$ be the subspace of $H$ spanned by function $\sin x, \sin 2x, \dots, \sin 2ix$. We define $V \equiv V_1 \cup V_2$ where

$$\begin{aligned} V_1 &= \{u \in H : u = a_1\varphi_1 + w,\ 0 \le a_1,\ w \in H^-\},\\ V_2 &= \{u \in H : u = a_2\varphi_2 + w,\ 0 \le a_2,\ w \in H^-\}. \end{aligned}$$

Let $\tilde a > 0$, $L > 0$ then we define $Q \equiv Q_1 \cup Q_2$ where

$$\begin{aligned} Q_1 &= \{u \in V_1 : 0 \le a_1 \le \tilde a,\ \|w\| \le L\},\\ Q_2 &= \{u \in V_2 : 0 \le a_2 \le \tilde a,\ \|w\| \le L\}. \end{aligned}$$

Let $S$ be the subspace of $H$ spanned by function $\sin(2i+2)x, \sin(2i+3)x, \dots$. Next, we verify the assumptions of Theorem 2.1. We see that $S$ is the closed subset of $H$ and $Q$ is the bounded subset in $H$.

**i)** For $u \in \partial Q_1$ we have $u = a_1\varphi_1 + w$ with $a_1 > 0$ and $\langle a_1\varphi_1 + w, \sin(2i+1)x\rangle = a_1\langle \varphi_1, \sin(2i+1)x\rangle > 0$. Similarly for $u \in \partial Q_2$ we obtain $\langle u, \sin(2i+1)x\rangle < 0$. For $z \in S$ it holds $\langle z, \sin(2i+1)x\rangle = 0$. Hence it follows $\partial Q \cap S = \emptyset$.

**ii)** The proof of the assumption $S \cap h(Q) \ne \emptyset \quad \forall h \in \Gamma$ is similar to the proof in [7, example 8.2]. We see that $H = S \oplus V$ and let $\pi\colon H \to V$ be the continuous projection of $H$ onto $V$. We have to show that $0 \in \pi(h(Q))$. For $t \in [0,1]$, $u \in Q$ we define

$$h_t = t\pi(h(u)) + (1-t)u\,.$$

Function $h_t$ defines a homotopy of $h_0 = id$ with $h_1 = \pi \circ h$. Moreover, $h_t|\partial Q = \mathrm{id}$ for all $t \in [0,1]$. Hence the topological degree $\deg(h_t, Q, 0)$ is well-defined and by homotopy invariance we have

$$\deg(\pi \circ h, Q, 0) = \deg(id, Q, 0) = 1\,.$$

Hence $0 \in \pi(h(Q))$, as was to be shown.

**iii)** First we find the infimum of functional $I$ on the set $S$. We have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_0^\pi \big[(u')^2 - \alpha(u^+)^2 - \beta(u^-)^2 + 2fu\big]\,dx\\ &= \frac{1}{2}\int_0^\pi \big[(u')^2 - \alpha u^2 + (\alpha - \beta)(u^-)^2 - 2fu\big]\,dx\,. \end{aligned}$$

For $u \in S$ it holds $\int_0^\pi (u')^2\,dx \ge (2i+2)^2 \int_0^\pi u^2\,dx$. We denote $b = (2i+2)^2 - \alpha$ (which is positive by assumption) and we obtain

$$\begin{aligned} I(u) &\ge \frac{1}{2}\Big\{\int_0^\pi \big[bu^2 - 2fu\big]\,dx + \int_0^\pi \big[(\alpha-\beta)(u^-)^2\big]\,dx\Big\}\\ &= \frac{1}{2}\Big\{\int_0^\pi \big(\sqrt{b}u - \frac{1}{\sqrt{b}}f\big)^2\,dx - \frac{1}{b}\int_0^\pi f^2\,dx + \int_0^\pi (\alpha-\beta)(u^-)^2\,dx\Big\}\,. \end{aligned}$$

We note that $\alpha > \beta$. Hence and from previous inequality it follows

$$\inf_{u \in S} I(u) \ge -\frac{1}{2b}\int_0^\pi f^2\,dx = -\frac{1}{2((2i+2)^2 - \alpha)}\int_0^\pi (c\varphi_2)^2 + (\varphi_2^\perp)^2\,dx\,. \quad (3.1)$$

Second we estimate the value $I(u)$ for $u \in \partial Q$. For a function $w \in H_-$, we have

$$\|w\|^2 = \int_0^\pi (w')^2\,dx \le (2i)^2 \int_0^\pi w^2\,dx = (2i)^2\|w\|_2^2\,.$$

For $u \in Q_2$ ($u = a_2\varphi_2 + w$) we put $d = \lambda_- - (2i)^2 > 0$ then the function $w$ satisfies the assumption $\int_0^\pi (w')^2 - \lambda_- w^2\,dx \le -d\int_0^\pi w^2\,dx$ of lemma 2.3.

It follows from the Remark 1.1 that $d - k = (\lambda_- - (2i)^2) - (\lambda_- - \beta) = \beta - (2i)^2 > 0$. We can use the inequality (2.6) from the lemma 2.3 and for $u = \frac{c}{k}\varphi_2 + w$ we obtain

$$\begin{aligned} I(u) &\le -\frac{1}{2}\int_0^\pi \Big[\frac{1}{k}(c\varphi)^2 - \frac{1}{d-k}(\varphi^\perp)^2\Big]\,dx \\ &= -\frac{1}{2}\int_0^\pi \Big[\frac{1}{\lambda_+ - \alpha}(c\varphi)^2 - \frac{1}{\beta - (2i)^2}(\varphi^\perp)^2\Big]\,dx\,. \end{aligned} \tag{3.2}$$

For $u \in Q_1$ we put $\widehat{\lambda}_+ = \alpha - l$, $\widehat{\lambda}_- = \beta - l$, $l > 0$ and $(\widehat{\lambda}_+, \widehat{\lambda}_-) \in \Sigma_{2i+1,1}$. It follows from remark 1.1 that $\widehat{\lambda}_- \ge (2i)^2$ then the function $u \in \partial Q_1$ ($u = a_1\varphi_1 + w$) satisfies assumptions of lemma 2.4 with $f = -c\varphi_2 + \varphi_2^\perp$. We put in (2.9) $k_2 = -\frac{1}{2}\int_0^\pi \Big[\frac{1}{\lambda_+ - \alpha}(c\varphi)^2 - \frac{1}{\beta-(2i)^2}(\varphi^\perp)^2\Big]\,dx$ and we obtain that the inequality (3.2) holds for $u \in \partial Q_1$ too. Hence

$$\sup_{u\in\partial Q} I(u) \le -\frac{1}{2}\int_0^\pi \Big[\frac{1}{\lambda_+ - \alpha}(c\varphi)^2 - \frac{1}{\beta - (2i)^2}(\varphi^\perp)^2\Big]\,dx\,. \tag{3.3}$$

By (3.1), (3.3) and from the assumption of theorem 3.1 we have

$$\begin{aligned} \inf_{u\in S} I(u) &\ge -\frac{1}{2((2i+2)^2 - \alpha)}\int_0^\pi (c\varphi_2)^2 + (\varphi_2^\perp)^2\,dx \\ &> -\frac{1}{2}\int_0^\pi \Big[\frac{1}{\lambda_+ - \alpha}(c\varphi)^2 - \frac{1}{\beta - (2i)^2}(\varphi^\perp)^2\Big]\,dx \ge \sup_{u\in\partial Q} I(u)\,. \end{aligned}$$

Then assumption iii) of theorem 3.1 holds.
**iv)** For this assumption, we will show that $I$ satisfies the Palais-Smale condition. First, we suppose that the sequence $(u_n)$ is unbounded and there exists a constant $c$ such that

$$\Big|\frac{1}{2}\int_0^\pi \big[(u_n')^2 - \alpha(u_n^+)^2 - \beta(u_n^-)^2\big]\,dx + \int_0^\pi f u_n\,dx\Big| \le c \tag{3.4}$$

and

$$\lim_{n\to\infty} \|I'(u_n)\| = 0\,. \tag{3.5}$$

Let $(w_k)$ be an arbitrary sequence bounded in $H$. It follows from (3.5) and the Schwarz inequality that

$$\begin{aligned} &\Big|\lim_{\substack{n\to\infty\\ k\to\infty}} \int_0^\pi \big[u_n' w_k' - (\alpha u_n^+ - \beta u_n^-)w_k\big]\,dx + \int_0^\pi f w_k\,dx\Big| \\ &= \Big|\lim_{\substack{n\to\infty\\ k\to\infty}} \langle I'(u_n), w_k\rangle\Big| \le \lim_{\substack{n\to\infty\\ k\to\infty}} \|I'(u_n)\|\cdot\|w_k\| = 0\,. \end{aligned} \tag{3.6}$$

Put $v_n = u_n/\|u_n\|$. Due to compact imbedding $H \subset L^2(0,\pi)$ there is $v_0 \in H$ such that (up to subsequence) $v_n \rightharpoonup v_0$ weakly in H, $v_n \to v_0$ strongly in $L^2(0,\pi)$. We divide (3.6) by $\|u_n\|$ and we obtain

$$\lim_{\substack{n\to\infty\\k\to\infty}} \int_0^\pi \big[v_n' w_k' - (\alpha v_n^+ - \beta v_n^-) w_k\big]\, dx = 0 \tag{3.7}$$

and also

$$\lim_{\substack{n\to\infty\\m\to\infty\\k\to\infty}} \int_0^\pi \big[(v_n' - v_m')w_k' - [\alpha(v_n^+ - v_m^+) - \beta(v_n^- - v_m^-)]w_k\big]\, dx = 0\,. \tag{3.8}$$

We set $w_k = v_n - v_m$ in (3.8) and we get

$$\lim_{\substack{n\to\infty\\m\to\infty}} \Big[\|v_n - v_m\|^2 - \int_0^\pi \big[[\alpha(v_n^+ - v_m^+) - \beta(v_n^- - v_m^-)](v_n - v_m)\big]\, dx\Big] = 0\,. \tag{3.9}$$

Since $v_n \to v_0$ strongly in $L^2(0,\pi)$ the integral in (3.9) convergent to 0 and then $v_n$ is a Cauchy sequence in $H$ and $v_n \to v_0$ strongly in $H$ and $\|v_0\| = 1$.

It follows from (3.7) and the usual regularity argument for ordinary differential equations (see Fučík [3]) that $v_0$ is the solution of the equation $v_0'' + \alpha v_0^+ - \beta v_0^- = 0$. From the assumption $(\alpha,\beta) \notin \Sigma$ it follows that $v_0 = 0$. This is contradiction to $\|v_0\| = 1$.

This implies that the sequence $(u_n)$ is bounded. Then there exists $u_0 \in H$ such that $u_n \rightharpoonup u_0$ in $H$, $u_n \to u_0$ in $L^2(0,\pi)$ (up to subsequence). It follows from the equality (3.6) that

$$\lim_{\substack{n\to\infty\\m\to\infty\\k\to\infty}} \int_0^\pi \big[(u_n - u_m)' w_k' - [\alpha(u_n^+ - u_m^+) - \beta(u_n^- - u_m^-)]w_k\big]\, dx = 0\,. \tag{3.10}$$

We put $w_k = u_n - u_m$ in (3.10) and the strong convergence $u_n \to u_0$ in $L^2(0,\pi)$ and (3.10) imply the strong convergence $u_n \to u_0$ in $H$. This shows that the functional $I$ satisfies Palais-Smale condition and the proof of Theorem 2 is complete.

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