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Generalized solutions to parabolic-hyperbolic equations *

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Abstract

We study boundary-value problems for composite type equations: parabolic-hyperbolic equations. We prove the existence and uniqueness of generalized solutions, using energy inequality and the density of the range of the operator generated by the problem.

1 Introduction

The equations of compsite type, as independent mathematical objects, arose first in the works of Hadamard [10]. Then they were continued by Sjostrand [11], and other [4, 7, 8]. In all these works the equations in question are investigated mainly in the plane and with the model operators in the principal part.

In recent years, special equations of composite type have received attention in several papers. Most of the papers were directed to parabolic-elliptic equations, and to hyperbolic-elliptic equations, see for instance [3, 5, 6]. Motivated by this, we study a boundary-value problem for a class of composite equations of parabolic-hyperbolic type.

Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Points in this space are denoted by $x = (x_1, x_2, \dots, x_n)$. In the cylinder $Q = \Omega \times (0, T)$, we consider the boundary-value problem

$$\begin{gathered} lu := (\frac{\partial}{\partial t} - \Delta)(\frac{\partial^2 u}{\partial t^2} - \Delta u) = f(x,t), \quad \text{on } Q, \\ u(x,0) = \frac{\partial u}{\partial t}(x,0) = \frac{\partial^2 u}{\partial t^2}(x,0) = 0, \quad \text{on } \Omega, \\ \frac{\partial u}{\partial \upsilon} = \frac{\partial^3 u}{\partial \upsilon^3} = 0, \quad \text{on } S \end{gathered} \tag{1.1}$$

where $S = \partial\Omega \times (0,T)$, υ is the unit exterior vector, and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

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The aim is to prove existence and uniqueness of a generalized solution to the above equation. The proof is based on an energy inequality and the density of the range of the operator generated by this problem.

Analogous to problem (1.1), we consider its dual problem. We denote by l^* the formal dual of the operator l, which is defined with respect to the inner product in the space $L_2(Q)$ using

$$(lu, v) = (u, lv) \quad \text{for all } u, v \in C_0^{3,4}(Q), \tag{1.2}$$

where $(,)$ is the inner product in $L_2(Q)$. We consider the dual problem (1.3):

$$\begin{gathered}
l^*v := (-\frac{\partial}{\partial t} - \Delta)(\frac{\partial^2 v}{\partial t^2} - \Delta v) = g(x,t), \quad \text{on } Q, \\
v(x,T) = \frac{\partial v}{\partial t}(x,T) = \frac{\partial^2 v}{\partial t^2}(x,T) = 0, \quad \text{on } \Omega, \\
\frac{\partial v}{\partial v} = \frac{\partial^3 v}{\partial v^3} = 0, \quad \text{on } S
\end{gathered} \tag{1.3}$$

2 Functional Spaces

The domain $D(l)$ of the operator l is $D(l) = H_+^{3,4}(Q)$, the subspace of the Sobolev space $H^{3,4}(Q)$, which consists of all the functions $u \in H^{3,4}(Q)$ satisfying the conditions of (1.1).

The domain of l^* is $D(l^*) = H_-^{3,4}(Q)$, which consists of functions $v \in H^{3,4}(Q)$ satisfying the conditions of (1.3).

Let $H_\sigma^{2,3}(Q)$ be the Sobolev space

$$\begin{aligned}
H_\sigma^{2,3}(Q) = \Big\{ & u \in H_0^1(Q) : \sigma(t)^{1/2} u_{tt} \in L_2(Q),\ \sigma(t)^{1/2} \nabla u_t \in L_2(Q), \\
& \nabla u_t \in L_2(Q),\ \sigma(t) \nabla u_{tt} \in L_2(Q),\ \sigma(t) \Delta u_t \in L_2(Q), \\
& \Delta u \in L_2(Q),\ \sigma(t)^{1/2} \Delta u_t \in L_2(Q),\ \sigma(t)^{1/2} \nabla \Delta u \in L_2(Q) \Big\},
\end{aligned}$$

where $\sigma(t) = (T - t)$. We introduce the function space $H_{0,\sigma}^{2,3}(Q) = \big\{ u \in H_\sigma^{2,3}(Q)$ satisfying the conditions of (1.1)$\big\}$.

Note that $H_{0,\sigma}^{2,3}(Q)$ is Hilbert space with the inner product:

$$\begin{aligned}
(u, v)_\sigma = & (u, v)_1 + (u_{tt}, v_{tt})_{0,\sigma} + (\nabla u_t, \nabla v_t)_{0,\sigma} + (\nabla u_t, \nabla v_t)_0 \\
& + (\Delta u, \Delta v)_0 + (\Delta u_t, \Delta v_t)_{0,\sigma} + (\nabla \Delta u, \nabla \Delta v)_{0,\sigma}
\end{aligned}$$

where the symbols $(,)_0, (,)_1$, and $(,)_{0,\sigma}$ denote the inner product in $L_2(Q)$, $H^1(Q)$, and $L_{2,\sigma}(Q)$ respectively. This space is equipped with the norm

$$\begin{aligned}
\|u\|_{2,3,\sigma}^2 = & \int_Q [u^2 + u_t^2 + |\nabla u|^2] dx\, dt + \int_Q [|\nabla u_t|^2 + (\Delta u)^2] dx\, dt \\
& + \int_Q (T - t)[u_{tt}^2 + |\nabla u_t|^2 + (\Delta u_t)^2 + (\nabla \Delta u)^2] dx\, dt.
\end{aligned}$$

The dual of this space is denoted by $H_\sigma^{-2,-3}(Q)$ with respect to the canonical bilinear form $\langle u, v\rangle$ for $u \in H_{0,\sigma}^{2,3}(Q)$ and $v \in H_\sigma^{-2,-3}(Q)$, which is the extension by continuity of the bilinear form (u, v), where $u \in L_2(Q)$ and $v \in H_{0,\sigma}^{2,3}(Q)$.

Definition The solution of (1.1) will be seen as a solution of the operational equation

$$lu = f, \quad u \in D(l). \tag{2.1}$$

The solution of (1.3) will be seen as a solution of the operational equation

$$l^*v = g, \quad v \in D(l). \tag{2.2}$$

To solve the equation (2.1) for every $f \in H_\sigma^{-2,-3}(Q)$, we construct, through the bilinear form $v \to a_u(v) = \langle l^*v, u\rangle$ for all $v \in D(l)$, the extension L of the operator l, whose range $R(L)$ coincides with $H_\sigma^{-2,-3}(Q)$, meaning that L is invertible.

Then we have the fundamental relation $\langle l^*v, u\rangle = \langle v, Lu\rangle$ for all $u \in D(l)$ and all $\in H_{0,\sigma}^{2,3}(Q)$, which is obtained by analytic form of Hann-Banach's theorem.

In the same manner, we construct, through the bilinear form: $u \to a_v(u) = \langle v, lu\rangle$ for all $u \in D(l)$, the extension L^* of the operator l^*. We obtain,

$$\langle v, lu\rangle = \langle L^*v, u\rangle, \quad \forall u \in H_{0,\sigma}^{2,3}(Q), \forall v \in D(L^*).$$

We denote the norm of Lu in $H_\sigma^{-2,-3}(Q)$ by $\|Lu\|_{-2,-3,\sigma}$.

Definition The solution of the operational equation

$$Lu = f, \quad u \in D(L),$$

is called generalized solution of (1.1), and the solution of the operational equation

$$L^*v = g, \quad v \in D(L^*),$$

is called generalized solution of (1.3).

3 A priori estimates

Theorem 3.1 *For Problem* (1.1)*, we have the following a priori estimates:*

$$\|u\|_{2,3,\sigma} \le c\|Lu\|_{-2,-3,\sigma}, \quad \forall u \in D(L), \tag{3.1}$$

$$\|v\|_{2,3,\sigma} \le c^*\|L^*v\|_{-2,-3,\sigma}, \quad \forall v \in D(L^*), \tag{3.2}$$

where the positive constants c and c^* are independent of u and v.

Proof. We first prove the inequality (3.1) for the functions $u \in D(l)$. For $u \in D(l)$ define the operator

$$Mu = \Phi(t)u_{tt} - \Phi(t)\Delta u_t,$$

where $\Phi(t) = (t-T)^2$. Consider the scalar product $(lu, Mu)_0$. Employing integration by parts and taking into account of conditions of (1.1), we see that

$$\begin{aligned}(lu,(t-T)^2u_{tt})_0 &= \int_Q (T-t)(u_{tt})^2 dxdt + \int_Q (T-t)|\nabla u_t|^2 dx\,dt \\ &\quad + \int_Q (T-t)^2|\nabla u_{tt}|^2 dx\,dt + \int_Q (\Delta u)^2 dx\,dt \\ &\quad - \int_Q (T-t)^2(\Delta u_t)^2 dx\,dt\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&(lu, -(t-T)^2\Delta u_t)_0 \\ &= -\int_Q (T-t)^2|\nabla u_{tt}|^2 dx\,dt + \int_Q |\nabla u_t|^2 dx\,dt + \int_Q (T-t)^2(\Delta u_t)^2 dx\,dt \\ &\quad + \int_Q (T-t)(\Delta u_t)^2 dxdt + \int_Q (T-t)(\nabla\Delta u)^2 dx\,dt\,.\end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned}&(lu,(t-T)^2u_{tt} - (t-T)^2\Delta u_t)_0 \\ &= \int_Q (T-t)(u_{tt})^2 dx\,dt + \int_Q (T-t)|\nabla u_t|^2 dx\,dt + \int_Q (\Delta u)^2 dx\,dt \\ &\quad + \int_Q |\nabla u_t|^2 dx\,dt + \int_Q (T-t)(\Delta u_t)^2 dx\,dt + \int_Q (T-t)(\nabla\Delta u)^2 dx\,dt\end{aligned} \tag{3.5}$$

For the function $u \in D(l)$, we have the following Poincaré estimates

$$\begin{gathered}\int_Q u^2 dx\,dt \le 4T^2\int_Q u_t^2 dx\,dt, \quad \forall u \in D(l), \\ \int_Q u_t^2 dx\,dt \le 4T\int_Q (T-t)u_{tt}^2 dx\,dt, \quad \forall u \in D(l) \\ \int_Q |\nabla u|^2 dx\,dt \le 4T\int_Q (T-t)|\nabla u_t|^2 dx\,dt, \quad \forall u \in D(l).\end{gathered} \tag{3.6}$$

We now apply the ε-inequality to the left hand side of (3.5). Using inequalities (3.6), we obtain (3.1).

For $u \in D(L)$, we use the regularization operators of Freidrich [2, 9] to conclude (3.1). This completes the proof. □

4 Solvability Problem

Theorem 4.1 *For each function $f \in H_\sigma^{-2,-3}(Q)$ (resp. $g \in H_\sigma^{-2,-3}(Q)$) there exists a unique solution of* (1.1) *(resp.*(1.3) *).*

Proof. The uniqueness of the solution follows immediately from inequality (3.1). This inequality also ensures the closure of the range $R(L)$ of the operator L. To prove that $\overline{R(L)}$ equals the space $H_\sigma^{-2,-3}(Q)$, we obtain the inclusion $\overline{R(L)} \subseteq R(L)$, and $R(L) = H_\sigma^{-2,-3}(Q)$. Indeed, let $\{f_k\}_{k\in N}$ be a Cauchy sequence in the space $H_\sigma^{-2,-3}(Q)$, which consists of elements of set$R(L)$. Then it corresponds to a sequence $\{u_k\}_{k\in N} \subseteq D(L)$ such that: $Lu_k = f_k$, $k \in \mathbb{N}$.

From the inequality (3.1), we conclude that the sequence $\{u_k\}$ is also a Cauchy sequence in the space $H_\sigma^{-2,-3}(Q$ and converges to an element u in $H_{0,\sigma}^{2,3}(Q)$.

It remains to obtain the density of the set$R(L)$ in the space $H_\sigma^{-2,-3}(Q)$ when u belongs to $D(L)$. Therefore, we establish an equivalent result which amounts to proving that $R(L)^\perp = \{0\}$.

Indeed, let $v \in H_\sigma^{-2,-3}(Q)$ be such that $\langle Lu, v\rangle = 0$ for all $u \in D(L)$, that is $\langle l^*v, u\rangle = 0$ for all $u \in D(L)$. By virtue of the equality $\langle l^*v, u\rangle = (v, Lu)$ for all $u \in D(L)$, we have $\langle v, Lu\rangle = 0$ for all $u \in D(L)$ and $v \in H_\sigma^{-2,-3}(Q)$. From the last equality, by virtue of the estimate (3.2), we conclude that $v = 0$ in the space $H_\sigma^{-2,-3}(Q)$ when u belongs to $D(L)$.

The second part of the theorem can be proved in a similar way by using the operator $M^*v = t^2 v_{tt} - t^2 \Delta v_t$. $\square$

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