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Some integral inequalities for functions of two variables *

Sever S. Dragomir & Young-Ho Kim

Abstract

In this article, we establish some integral inequalities for function with two independent variables. Also we show applications of these inequalities for finding bounds of solutions to partial differential equations.

1 Introduction

Let $u : [\alpha, \alpha + h] \to \mathbb{R}$ be a continuous function satisfying the inequality

$$0 \le u(t) \le \int_{\alpha}^{t} [a + bu(s)]\, ds, \quad \text{for } t \in [\alpha, \alpha + h],$$

where a, b are nonnegative constants. Then $u(t) \le ahe^{bh}$ for $t \in [\alpha, \alpha + h]$. This result was proved by Gronwall in 1919, and is the prototype for the study of many integral inequalities of Volterra type, and also for obtaining explicit bounds of the unknown function. Therefore integral inequalities of this type are usually associated with the name of Gronwall. Integral inequalities are a necessary tool in the study of various classes of equations. During the past few years many authors (please, see refeences below and some of the reference cited therein) have established several Gronwall type integral inequalities in two or more independent variables. In [8], Pachpatte considered the finite difference inequality in two independent variables. Many of these are analogues of integral inequalities already known to us. Our main objective here, as an integral version of Pachpatte's finite difference inequalities in [8], is to establish some new integral inequalities involving functions of two independent variables which can be used in the analysis of certain classes of partial differential equations.

2 Results

Throughout this paper, all the functions which appear in the inequalities are assumed to be real valued and all the integrals exist on their domains of definitions. We shall introduce some notation: $\mathbb{R}$ denotes the set of real numbers

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and $\mathbb{R}_+ = [0, \infty)$. The first order partial derivatives of a functions $z(x, y)$ defined for $x, y \in \mathbb{R}$ with respect to x and y are denoted by $z_x(x, y)$ and $z_y(x, y)$ respectively. We need the inequalities in the following lemma, which appear in [6, p. 356].

Lemma 2.1 *Let $u(t)$ and $k(t)$ be continuous and $a(t)$ and $b(t)$ Riemann integrable functions on $J = [\alpha, \beta]$ with $b(t)$ and $k(t)$ nonnegative on J.*

1. *If $u(t) \le a(t) + b(t) \int_\alpha^t k(s)u(s)ds$ for $t \in J$, then*
$$u(t) \le a(t) + b(t) \int_\alpha^t a(s)k(s) \exp\Big(\int_s^t b(r)k(r)dr \Big)\, ds, \quad t \in J.$$
2. *If $u(t) \le a(t) + b(t) \int_t^\beta k(s)u(s)ds$ for $t \in J$, then*
$$u(t) \le a(t) + b(t) \int_t^\beta a(s)k(s) \exp\Big(\int_t^s b(r)k(r)dr \Big)\, ds, \quad t \in J.$$

Also, we need the inequalities in the following lemma which is given in [1, p. 110].

Lemma 2.2 *Let $u(x, y), a(x, y), b(x, y)$ be nonnegative continuous functions defined for $x, y \in \mathbb{R}_+$.*

1. *Assume that $a(x, y)$ is non-decreasing in x and non-increasing in y for $x, y \in \mathbb{R}_+$. If $u(x, y) \le a(x, y) + \int_0^x \int_y^\infty b(s, t)u(s, t)\, dtds$ for all $x, y \in \mathbb{R}_+$, then*
$$u(x, y) \le a(x, y) \exp\Big(\int_0^x \int_y^\infty b(s, t)\, dtds \Big).$$
2. *Assume that $a(x, y)$ is non-increasing in each of the variables $x, y \in \mathbb{R}_+$. If $u(x, y) \le a(x, y) + \int_x^\infty \int_y^\infty b(s, t)u(s, t)\, dtds$ for all $x, y \in \mathbb{R}_+$, then*
$$u(x, y) \le a(x, y) \exp\Big(\int_x^\infty \int_y^\infty b(s, t)\, dtds \Big).$$

The proofs of these inequalities can be completed as in [1, p. 109-111]; thus, we omit the proof.

Theorem 2.3 *Let $u(x, y), a(x, y), b(x, y), c(x, y), d(x, y), f(x, y)$ be real-valued non-negative continuous functions defined for $x, y \in \mathbb{R}_+$. Let $W(u(x, y))$ be real-valued, positive, continuous, strictly non-decreasing, subadditive, and submultiplicative function for $u(x, y) \ge 0$ and let $H(u(x, y))$ be a real-valued, continuous, positive, and non-decreasing function defined for $x, y \in \mathbb{R}_+$. Assume that $a(x, y), f(x, y)$ are nondecreasing in x for $x \in \mathbb{R}_+$. If*
$$\begin{aligned} u(x, y) \le& a(x, y) + b(x, y) \int_\alpha^x c(s, y)u(s, y)\, ds \\ &+ f(x, y)H\Big(\int_0^x \int_y^\infty d(s, t)W\big(u(s, t)\big)\, dtds \Big) \end{aligned} \tag{2.1}$$

for $\alpha, x, y \in \mathbb{R}_+$ *with* $\alpha \le x$*, then*

$$\begin{aligned} u(x,y) \le & p(x,y)\Big\{a(x,y) + f(x,y)H\Big[G^{-1}\big(G(A(s,t)) \\ & + \int_0^x \int_y^\infty d(s,t)W\big(p(s,t)f(s,t)\big)\,dtds\Big)\Big]\Big\} \end{aligned} \tag{2.2}$$

for $\alpha, x, y \in \mathbb{R}_+$ *with* $\alpha \le x$*, where*

$$p(x,y) = 1 + b(x,y)\int_\alpha^x c(s,y)\exp\Big(\int_s^x b(r,y)c(r,y)\,dr\Big)\,ds, \tag{2.3}$$

$$A(s,t) = \int_0^\infty \int_0^\infty d(s,t)W\big(p(s,t)a(s,t)\big)\,dtds, \tag{2.4}$$

$$G(r) = \int_{r_0}^r \frac{ds}{W\big((H(s)\big)}, \quad r \ge r_0 > 0. \tag{2.5}$$

Here G^{-1} *is the inverse function of* G *and*

$$G\Big(\int_0^\infty \int_0^\infty d(s,t)W\big(p(s,t)a(s,t)\big)\,dtds\Big) + \int_0^x \int_y^\infty d(s,t)W\big(p(s,t)f(s,t)\big)\,dtds$$

is in the domain of G^{-1} *for* $x, y \in \mathbb{R}_+$*.*

Proof. Define a function $z(x,y)$ by

$$z(x,y) = a(x,y) + f(x,y)H\Big(\int_0^x \int_y^\infty d(s,t)W\big(u(s,t)\big)\,dtds\Big). \tag{2.6}$$

Then (2.1) can be restated as

$$u(x,y) \le z(x,y) + b(x,y)\int_\alpha^x c(s,y)u(s,y)\,ds. \tag{2.7}$$

Clearly $z(x,y)$ is a nonnegative and continuous in $x, x \in \mathbb{R}_+$. Treating $y, y \in \mathbb{R}_+$ fixed in (2.7) and using 1 of Lemma 2.1 to (2.7), we get

$$u(x,y) \le z(x,y) + b(x,y)\int_\alpha^x z(s,y)c(s,y)\exp\Big(\int_s^x b(r,y)c(r,y)\,dr\Big)\,ds.$$

Moreover, $z(x,y)$ is nondecreasing in $x, x \in \mathbb{R}_+$, we obtain

$$u(x,y) \le z(x,y)p(x,y), \tag{2.8}$$

where $p(x,y)$ is defined by (2.3). ¿From (2.6) we have

$$u(x,y) \le p(x,y)\Big(a(x,y) + f(x,y)H\big(v(x,y)\big)\Big), \tag{2.9}$$

where $v(s,y)$ is defined by

$$v(x,y) = \int_0^x \int_y^\infty d(s,t)W\big(u(s,t)\big)\,dtds.$$

¿From (2.9), we observe that

$$\begin{aligned} v(x,y) &\leq \int_0^x \int_y^\infty d(s,t) W\Big(p(s,t)\Big[a(s,t)+f(s,t)H\big(v(s,t)\big)\Big]\Big)\,dtds \\ &\leq \int_0^x \int_y^\infty d(s,t) W\big(p(s,t)a(s,t)\big)\,dtds \\ &\quad + \int_0^x \int_y^\infty d(s,t) W\big(p(s,t)f(s,t)\big) W\Big(H\big(v(s,t)\big)\Big)\,dtds \\ &\leq \int_0^\infty \int_0^\infty d(s,t) W\big(p(s,t)a(s,t)\big)\,dtds \\ &\quad + \int_0^x \int_y^\infty d(s,t) W\big(p(s,t)f(s,t)\big) W\Big(H\big(v(s,t)\big)\Big)\,dtds, \end{aligned} \tag{2.10}$$

since W is subadditive and submultiplicative function. Define $r(x,y)$ as the right side of (2.10), then $r(0,y) = r(x,\infty) = \int_0^\infty \int_0^\infty d(s,t)W\big(p(s,t)a(s,t)\big)\,dtds$, $v(x,y) \leq r(x,y)$, $r(x,y)$ is non-increasing in $y, y \in \mathbb{R}_+$ and

$$\begin{aligned} r_x(x,y) &= \int_y^\infty d(x,t) W\big(p(x,t)f(x,t)\big) W\Big(H\big(v(x,t)\big)\Big)\,dt \\ &\leq \int_y^\infty d(x,t) W\big(p(x,t)f(x,t)\big) W\Big(H\big(r(x,t)\big)\Big)\,dt \\ &\leq W\Big(H\big(r(x,y)\big)\Big) \int_y^\infty d(x,t) W\big(p(x,t)f(x,t)\big)\,dt. \end{aligned} \tag{2.11}$$

Dividing both sides of (2.11) by $W\big(H(r(x,y))\big)$ we get

$$\frac{r_x(x,y)}{W\big(H(r(x,y))\big)} \leq \int_y^\infty d(x,t) W\big(p(x,t)f(x,t)\big)\,dt. \tag{2.12}$$

¿From (2.5) and (2.12) we have

$$G_x\big(r(x,y)\big) \leq \int_y^\infty d(x,t) W\big(p(x,t)f(x,t)\big)\,dt. \tag{2.13}$$

Now setting $x = s$ in (2.13) and then integrating with respect to s from 0 to x, we obtain

$$G\big(r(x,y)\big) \leq G\big(r(0,y)\big) + \int_0^x \int_y^\infty d(s,t) W\big(p(s,t)f(s,t)\big)\,dtds. \tag{2.14}$$

Noting that $r(0,y) = \int_0^\infty \int_0^\infty d(s,t)W\big(p(s,t)a(s,t)\big)\,dtds$, we have

$$\begin{aligned} r(x,y) \leq & G^{-1}\Big[G\Big(\int_0^\infty \int_0^\infty d(s,t) W\big(p(s,t)a(s,t)\big)\,dt\,ds\Big) \\ &+ \int_0^x \int_y^\infty d(s,t) W\big(p(s,t)f(s,t)\big)\,dtds\Big]. \end{aligned} \tag{2.15}$$

The required inequality in (2.2) follows from the fact that $v(x,y) \leq r(x,y)$, (2.7) and (2.15). □

Theorem 2.4 *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, $f(x,y)$, $W(u(x,y))$, and $H(u(x,y))$ be as defined in Theorem 2.3. Assume that $a(x,y), f(x,y)$ are non-increasing in x for $x \in \mathbb{R}_+$. If*

$$\begin{aligned} u(x,y) \le & a(x,y) + b(x,y)\int_x^\beta c(s,y)u(s,y)\,ds \\ & + f(x,y)H\Big(\int_x^\infty\int_y^\infty d(s,t)W\big(u(s,t)\big)\,dtds\Big) \end{aligned}$$

for $\beta, x, y \in \mathbb{R}_+$ with $x \le \beta$, then

$$\begin{aligned} u(x,y) \le & \overline{p}(x,y)\Big\{a(x,y) + f(x,y)H\Big[G^{-1}\Big(G\Big(\overline{A}(s,t)\Big) \\ & + \int_x^\infty\int_y^\infty d(s,t)W\big(\overline{p}(s,t)f(s,t)\big)\,dtds\Big)\Big]\Big\} \end{aligned}$$

for $\beta, x, y \in \mathbb{R}_+$ with $x \le \beta$, where

$$\begin{aligned} \overline{p}(x,y) &= 1 + b(x,y)\int_x^\beta c(s,y)\exp\Big(\int_x^s b(r,y)c(r,y)\,dr\Big)\,ds, \\ \overline{A}(s,t) &= \int_0^\infty\int_0^\infty d(s,t)W\big(\overline{p}(s,t)a(s,t)\big)\,dtds, \\ G(r) &= \int_{r_0}^r \frac{ds}{W\big((H(s)\big)}, \quad r \ge r_0 > 0, \end{aligned}$$

G^{-1} is the inverse function of G and

$$G\Big(\int_0^\infty\int_0^\infty d(s,t)W\big(\overline{p}(s,t)a(s,t)\big)\,dtds\Big) + \int_x^\infty\int_y^\infty d(s,t)W\big(\overline{p}(s,t)f(s,t)\big)\,dtds$$

is in the domain of G^{-1} for $x, y \in \mathbb{R}_+$.

The details of the proof of Theorem 2.4 follows by an argument similar to that in the proofs of Theorem 2.3 with suitable changes. We omit the proof.

Theorem 2.5 *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $f(x,y)$ be real-valued nonnegative continuous functions defined for $x, y \in \mathbb{R}_+$ and $L : \mathbb{R}_+^3 \to \mathbb{R}_+$ be a continuous function which satisfies the condition*

$$0 \le L(x,y,u) - L(x,y,v) \le M(x,y,v)\phi^{-1}(u-v) \tag{2.16}$$

for $u \ge v \ge 0$, where $M(x,y,v)$ is a real-valued nonnegative continuous function defined for $x, y, v \in \mathbb{R}_+$. Assume that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly increasing function with $\phi(0) = 0, \phi^{-1}$ is the inverse function of ϕ and

$$\phi^{-1}(uv) \le \phi^{-1}(u)\phi^{-1}(v)$$

for $u, v \in \mathbb{R}_+$. Assume that $a(x,y), f(x,y)$ are nondecreasing in x for $x \in \mathbb{R}_+$. If

$$\begin{aligned} u(x,y) \leq & a(x,y) + b(x,y)\int_\alpha^x c(s,y)u(s,y)\,ds \\ & + f(x,y)\phi\Big(\int_0^x \int_y^\infty L(s,t,u(s,t))\,dtds\Big) \end{aligned} \tag{2.17}$$

for $\alpha, x, y \in \mathbb{R}_+$ with $\alpha \leq x$, then

$$\begin{aligned} u(x,y) \leq p(x,y)\Big\{ & a(x,y) + f(x,y)\phi\Big[e(x,y) \\ & \times \exp\Big(\int_0^x \int_y^\infty M\Big(s,t,p(s,t)a(s,t)\Big)\phi^{-1}\big(p(s,t)f(s,t)\big)\,dtds\Big)\Big]\Big\} \end{aligned} \tag{2.18}$$

for $x, y \in \mathbb{R}_+$, where

$$p(x,y) = 1 + b(x,y)\int_\alpha^x c(s,y)\exp\Big(\int_s^x b(r,y)c(r,y)\,dr\Big)\,ds, \tag{2.19}$$

$$e(x,y) = \int_0^x \int_y^\infty L\Big(s,t,p(s,t)a(s,t)\Big)\,dtds. \tag{2.20}$$

Proof Define the function

$$z(x,y) = a(x,y) + f(x,y)\phi\Big(\int_0^x \int_y^\infty L(s,t,u(s,t))\,dtds\Big). \tag{2.21}$$

Then (2.17) can be restated as

$$u(x,y) \leq z(x,y) + b(x,y)\int_\alpha^x c(s,y)u(s,y)\,ds. \tag{2.22}$$

Clearly $z(x,y)$ is a nonnegative and continuous in $x, x \in \mathbb{R}_+$. Treating $y, y \in \mathbb{R}_+$ fixed in (2.22) and using (i) of Lemma 2.1 to (2.22), we get

$$u(x,y) \leq z(x,y) + b(x,y)\int_\alpha^x z(s,y)c(s,y)\exp\Big(\int_s^x b(r,y)c(r,y)\,dr\Big)\,ds.$$

Moreover, $z(x,y)$ is nondecreasing in $x, x \in \mathbb{R}_+$, we obtain

$$u(x,y) \leq z(x,y)p(x,y), \tag{2.23}$$

where $p(x,y)$ is defined by (2.19). ¿From (2.21) and (2.23) we have

$$u(x,y) \leq p(x,y)\Big(a(x,y) + f(x,y)\phi\big(v(x,y)\big)\Big), \tag{2.24}$$

where $v(s,y)$ is defined by

$$v(x,y) = \int_0^x \int_y^\infty L(s,t,u(s,t))\,dtds.$$

¿From (2.24) and the hypotheses on L and ϕ, it is to observe that

$$\begin{aligned} v(x,y) \le& \int_0^x \int_y^\infty \Big(L\Big((s,t,p(s,t)\Big[a(s,t)+f(s,t)\phi\big(v(s,t)\big)\Big]\Big) \\ &- L\Big((s,t,p(s,t)a(s,t)\Big) + L\Big((s,t,p(s,t)a(s,t)\Big)\Big)\,dtds \\ \le& \int_0^x \int_y^\infty L\Big((s,t,p(s,t)a(s,t)\Big)\,dtds \\ &+ \int_0^x \int_y^\infty M\Big((s,t,p(s,t)a(s,t)\Big)\phi^{-1}\Big(p(s,t)f(s,t)\phi\big(v(s,t)\big)\Big)\,dtds \\ \le& e(x,y) + \int_0^x \int_y^\infty M\Big((s,t,p(s,t)a(s,t)\Big)\phi^{-1}\big(p(s,t)f(s,t)\big)v(s,t)\,dtds, \end{aligned} \tag{2.25}$$

where $e(x,y)$ is defined by (2.20). Clearly, $e(x,y)$ is nonnegative, continuous, nondecreasing in $x, x \in \mathbb{R}_+$ and non-increasing in $y, y \in \mathbb{R}_+$. Now, by 1 of Lemma 2.2, we obtain

$$v(x,y) \le e(x,y)\exp\Big(\int_0^x \int_y^\infty M\Big((s,t,p(s,t)a(s,t)\Big)\phi^{-1}\big(p(s,t)f(s,t)\big)\,dtds,\Big). \tag{2.26}$$

Using (2.24) in (2.26) we get the required inequality in (2.18). □

Theorem 2.6 *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $f(x,y)$, L, M, ϕ, and ϕ^{-1} be as defined in Theorem 2.5. Assume that $a(x,y), f(x,y)$ are non-increasing in x for $x \in \mathbb{R}_+$. If*

$$\begin{aligned} u(x,y) \le& a(x,y) + b(x,y)\int_x^\beta c(s,y)u(s,y)\,ds \\ &+ f(x,y)\phi\Big(\int_x^\infty \int_y^\infty L(s,t,u(s,t))\,dtds\Big) \end{aligned}$$

for $\beta, x, y \in \mathbb{R}_+$ with $x \le \beta$, then

$$\begin{aligned} u(x,y) \le \overline{p}(x,y)\Big\{& a(x,y) + f(x,y)\phi\Big[\overline{e}(x,y) \\ &\times \exp\Big(\int_x^\infty \int_y^\infty M\Big(s,t,\overline{p}(s,t)a(s,t)\Big)\phi^{-1}\big(\overline{p}(s,t)f(s,t)\big)\,dtds\Big)\Big]\Big\} \end{aligned}$$

for $x, y \in \mathbb{R}_+$, where

$$\begin{aligned} \overline{p}(s,t) &= 1 + b(x,y)\int_x^\beta c(s,y)\exp\Big(\int_x^s b(r,y)c(r,y)\,dr\Big)\,ds, \\ \overline{e}(x,y) &= \int_x^\infty \int_y^\infty L\Big(s,t,\overline{p}(s,t)a(s,t)\Big)\,dtds. \end{aligned} \tag{2.27}$$

The proof of Theorem 2.5 follows by an argument similar to that in the proofs of Theorem 2.4 with suitable changes. We omit the details.

3 Further Inequalities

To establish some of our results in this section, we require the class of functions S as defined in [2]. A function $g : [0,\infty) \to [0,\infty)$ is said to belong to the class S if
(i) $g(u)$ is positive, nondecreasing and continuous for $u \geq 0$, and
(ii) $(1/v)g(u) \leq g(u/v), u > 0, v \geq 1$.

Theorem 3.1 *Let $u(x,y)$, $a(x,y)$, $c(x,y)$, $d(x,y)$, $f(x,y)$ be real-valued non-negative continuous functions defined for $x,y \in \mathbb{R}_+$ and let $g \in S$. Also let $W(u(x,y))$ be real-valued, positive, continuous, strictly nondecreasing, subadditive, and submultiplicative function for $u(x,y) \geq 0$ and let $H(u(x,y))$ be a real-valued, continuous, positive, and nondecreasing function defined for $x,y \in \mathbb{R}_+$. Assume that a function $m(x,y)$ is nondecreasing in x and $m(x,y) \geq 1$, which is defined by*

$$m(x,y) = a(x,y) + f(x,y)H\Big(\int_0^x \int_y^\infty d(s,t)u(s,t)\,dtds\Big)$$

for $x,y \in \mathbb{R}_+$. If

$$u(x,y) \leq m(x,y) + \int_\alpha^x c(s,y)g\big(u(s,y)\big)\,ds \tag{3.1}$$

for $\alpha, x, y \in \mathbb{R}_+$ and $\alpha \leq x$, then

$$\begin{aligned} u(x,y) \leq{}& F(x,y)\Big\{a(x,y) + f(x,y)H\Big[G^{-1}\Big(G\Big(B(s,t)\Big) \\ &+ \int_0^x \int_y^\infty d(s,t)W\big(F(s,t)f(s,t)\big)\,dtds\Big)\Big]\Big\} \end{aligned} \tag{3.2}$$

for $x,y \in \mathbb{R}_+$, where

$$F(x,y) = \Omega^{-1}\Big(\Omega(1) + \int_\alpha^x b(s,y)\,ds\Big), \tag{3.3}$$

$$B(s,t) = \int_0^\infty \int_y^\infty d(s,t)W\big(F(s,t)a(s,t)\big)\,dtds, \tag{3.4}$$

$$\Omega(u) = \int_{u_0}^u \frac{ds}{g(s)}, \quad u \geq u_0 > 0, \tag{3.5}$$

where Ω^{-1} is the inverse function of Ω; G, G^{-1} are defined in Theorem 2.3, $\Omega(1) + \int_\alpha^x b(s,y)\,ds$ is in the domain of Ω^{-1}, and

$$G\Big(\int_0^\infty \int_0^\infty d(s,t)W\big(F(s,t)a(s,t)\big)\,dtds\Big) + \int_0^x \int_y^\infty d(s,t)W\big(F(s,t)f(s,t)\big)\,dtds$$

is in the domain of G^{-1} for $x,y \in \mathbb{R}_+$.

Proof Let $m(x,y)$ be a positive, continuous, nondecreasing in x and let $g \in S$. Then (3.1) can be restated as

$$\frac{u(x,y)}{m(x,y)} \leq 1 + \int_\alpha^x b(s,y) g\Big(\frac{u(s,y)}{m(s,y)}\Big)\, ds. \tag{3.6}$$

The inequality (3.6) may be treated as a one-dimensional Bihari inequality [1] for any fixed $y, y \in \mathbb{R}_+$, which implies that

$$u(x,y) \leq F(x,y) m(x,y),$$

where $F(x,y)$ is defined by (3.3). Now, by following the last argument as in the proof of Theorem 2.3, we obtain desired inequality in (3.2). □

Theorem 3.2 *Let $u(x,y)$, $a(x,y)$, $c(x,y)$, $d(x,y)$, $f(x,y)$, $W(u(x,y))$, and $H(u(x,y))$ be as defined in Theorem 3.1 and $g \in S$. Assume that a function $\overline{m}(x,y)$ is non-increasing in x and $\overline{m}(x,y) \geq 1$, which is defined by*

$$\overline{m}(x,y) = a(x,y) + f(x,y) H\Big(\int_x^\infty \int_y^\infty d(s,t) u(s,t)\, dt ds\Big)$$

for $x, y \in \mathbb{R}_+$. If

$$u(x,y) \leq \overline{m}(x,y) + \int_x^\beta c(s,y) g\big(u(s,y)\big)\, ds \tag{3.7}$$

for $\beta, x, y \in \mathbb{R}_+$ and $x \leq \beta$, then

$$\begin{aligned} u(x,y) \leq & \overline{F}(x,y)\Big\{a(x,y) + f(x,y) H\Big[G^{-1}\Big(G\Big(\overline{B}(s,t)\Big) \\ & + \int_x^\infty \int_y^\infty d(s,t) W\big(\overline{F}(s,t) f(s,t)\big)\, dt ds\Big)\Big]\Big\} \end{aligned} \tag{3.8}$$

for $x, y \in \mathbb{R}_+$, where

$$\begin{aligned} \overline{F}(x,y) &= \Omega^{-1}\Big(\Omega(1) + \int_x^\beta b(s,y)\, ds\Big), \\ \overline{B}(s,t) &= \int_0^\infty \int_0^\infty d(s,t) W\big(\overline{F}(s,t) a(s,t)\big)\, dt ds, \end{aligned} \tag{3.9}$$

Ω is defined in (3.5), Ω^{-1} is the inverse function of Ω; G, G^{-1} are defined in Theorem 2.3, $\Omega(1) + \int_x^\beta b(s,y)\, ds$ is in the domain of Ω^{-1}, and

$$G\Big(\int_0^\infty \int_0^\infty d(s,t) W\big(\overline{F}(s,t) a(s,t)\big)\, dt ds\Big) + \int_x^\infty \int_y^\infty d(s,t) W\big(\overline{F}(s,t) f(s,t)\big)\, dt ds$$

is in the domain of G^{-1}, for $x, y \in \mathbb{R}_+$.

Proof Let $\overline{m}(x,y)$ be a positive, continuous, nondecreasing in x and let $g \in S$. Then (3.7) can be restated as

$$\frac{u(x,y)}{\overline{m}(x,y)} \le 1 + \int_\alpha^x b(s,y) g\Big(\frac{u(s,y)}{\overline{m}(s,y)}\Big)\, ds. \tag{3.10}$$

The inequality (3.10) may be treated as a one-dimensional Bihari inequality [1] for any fixed $y, y \in \mathbb{R}_+$, which implies that

$$u(x,y) \le \overline{F}(x,y)\overline{m}(x,y),$$

where $\overline{F}(x,y)$ is defined by (3.9). Now, by following the last argument as in the proof of Theorem 2.4, we obtain desired inequality in (3.8). □

Theorem 3.3 *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $f(x,y)$, L, M, ϕ, and ϕ^{-1} be as defined in Theorem 2.5, and let $g \in S$. Assume that a function $n(x,y)$ is nondecreasing in x and $n(x,y) \ge 1$, which is defined by*

$$n(x,y) = a(x,y) + f(x,y)\phi\Big(\int_0^x \int_y^\infty F(s,t,u(s,t))\, dt ds\Big)$$

for $x, y \in \mathbb{R}_+$. If

$$u(x,y) \le n(x,y) + \int_\alpha^x b(s,y) b\big(u(s,y)\big)\, ds \tag{3.11}$$

for $\alpha, x, y \in \mathbb{R}_+$ and $\alpha \le x$, then

$$\begin{aligned} u(x,y) \le F(x,y)\Big\{&a(x,y) + f(x,y)\phi\Big[e(x,y) \\ &\times \exp\Big(\int_0^x \int_y^\infty M\Big(s,t,F(s,t)a(s,t)\Big)\phi^{-1}\big(F(s,t)f(s,t)\big)\, dt ds\Big)\Big]\Big\} \end{aligned} \tag{3.12}$$

for $x, y \in \mathbb{R}_+$, where F is defined in (3.3), $e(x,y)$ is defined in (2.20), Ω is defined in (3.5), Ω^{-1} is the inverse function of Ω and $\Omega(1) + \int_\alpha^x b(s,y)\, ds$ is in the domain of Ω^{-1} for $x, y \in \mathbb{R}_+$.

Proof The proof of this theorem follows by an argument similar to that of Theorem 3.1. Let $n(x,y)$ is a positive, continuous, nondecreasing in x and let $g \in S$. Then (3.11) can be restated as

$$\frac{u(x,y)}{n(x,y)} \le 1 + \int_\alpha^x b(s,y) g\Big(\frac{u(s,y)}{n(s,y)}\Big)\, ds. \tag{3.13}$$

The inequality (3.13) may be treated as a one-dimensional Bihari inequality [1] for any fixed $y, y \in \mathbb{R}_+$, which implies that

$$u(x,y) \le F(x,y)n(x,y),$$

where $F(x,y)$ is defined by (3.3). Now, by following the last argument as in the proof of Theorem 2.5, we obtain desired inequality in (3.12). □

Theorem 3.4 *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $f(x,y)$, L, M, ϕ, and ϕ^{-1} be as defined in Theorem 2.5, and let $g \in S$. Assume that a function $\overline{n}(x,y)$ is non-increasing in x and $\overline{n}(x,y) \geq 1$, which is defined by*

$$\overline{n}(x,y) = a(x,y) + f(x,y)\phi\Big(\int_x^\infty \int_y^\infty F(s,t,u(s,t))\, dtds\Big)$$

for $x, y \in \mathbb{R}_+$. If $u(x,y) \leq \overline{n}(x,y) + \int_x^\beta b(s,y)b\big(u(s,y)\big)\, ds$ for $\beta, x, y \in \mathbb{R}_+$ and $x \leq \beta$, then

$$\begin{aligned} u(x,y) \leq & \overline{F}(x,y)\Big\{a(x,y) + f(x,y)\phi\Big[\overline{e}(x,y) \\ & \times \exp\Big(\int_x^\infty \int_y^\infty M\Big(s,t,\overline{F}(s,t)a(s,t)\Big)\phi^{-1}\big(\overline{F}(s,t)f(s,t)\big)\, dtds\Big)\Big]\Big\} \end{aligned}$$

for $x, y \in \mathbb{R}_+$, where $\overline{F}$ is defined in (3.9), $\overline{e}(x,y)$ is defined in (2.27), Ω is defined in (3.5), Ω^{-1} is the inverse function of Ω and $\Omega(1) + \int_x^\beta b(s,y)\, ds$ is in the domain of Ω^{-1} for $x, y \in \mathbb{R}_+$.

The proof of this theorem follows by an argument similar to that in Theorem 3.3 with suitable changes. We omit the details.

4 Some Applications

In this section we present some immediate applications of Theorem 2.3 to study certain properties of solutions of the following terminal-value problem for the hyperbolic partial differential equation

$$u_{xy}(x,y) = h(x,y,u(x,y)) + r(x,y), \tag{4.1}$$
$$u(x,\infty) = \sigma_\infty(x), u(0,y) = \tau(y), u(0,\infty) = k, \tag{4.2}$$

where $h : \mathbb{R}_+^2 \times R \to R$, $r : \mathbb{R}_+^2 \to R$, $\sigma_\infty, \tau(y) : \mathbb{R}_+ \to R$ are continuous functions and k is a real constant.

The following example deals with the estimate on the solution of the partial differential equation (4.1) with the conditions (4.2).

Example Assume that functions are defined and continuous on their respective domains of definitions and such that

$$|h(x,y,u)| \leq d(x,y)W\big(|\, u\,|\big) \tag{4.3}$$

and

$$\begin{aligned} &\Big|\sigma_\infty(x) + \tau(y) - k - \int_0^x \int_y^\infty r(s,t)\, dtds\Big| \\ &\qquad \leq a(x,y) + b(x,y)\int_\alpha^x c(s,y)g\big(|u(s,y)|\big)\, ds, \end{aligned} \tag{4.4}$$

where $a(x,y)$, $b(x,y)$, $c(x,y)$ and $W(u)$ are as defined in Theorem 2.3. If $u(x,y)$ is a solution of (4.1) with the conditions (4.2), then it can be written as [1, p. 80]

$$u(x,y) = \sigma_\infty(x) + \tau(y) - k - \int_0^x \int_y^\infty \big(h(s,t,u(s,t)) + r(s,t)\big)\,dtds \qquad (4.5)$$

for $x, y \in \mathbb{R}_+$. From (4.3), (4.4), (4.5) we get

$$\begin{aligned} |u(x,y)| \le & a(x,y) + b(x,y)\int_\alpha^x c(s,y)g\big(|u(s,y)|\big)\,ds \\ & + \int_0^x \int_y^\infty d(s,t)W\big(|u(s,t)|\big)\,dtds. \end{aligned} \qquad (4.6)$$

Now, a suitable application of Theorem 2.3 with $f(x,y) = 1$ and $H(u) = u$ to (4.6) yields the required estimate, Therefore,

$$\begin{aligned} |u(x,y)| \le p(x,y)\Big\{a(x,y) + G^{-1}\Big[G\Big(\int_0^\infty \int_0^\infty d(s,t)W\big(p(s,t)a(s,t)\big)\,dtds\Big) \\ + \int_0^x \int_y^\infty d(s,t)W\big(p(s,t)\big)\,dtds\Big]\Big\} \end{aligned}$$

for $x, y \in \mathbb{R}_+$, where $p(x,y), G$, and G^{-1} are define in Theorem 2.3.

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References

[1] D. Bainov and P. Simeonov, *Integral Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, 1992.

[2] P. R. Beesack, *Gronwall inequalities*, Carleton University Mathematical Lecture Notes, No. 11, 1987.

[3] S. S. Dragomir, *On Gronwall type lemmas and applications*, “Monografii Matematics” Univ. Timişoara , No. 29, 1987.

[4] S. S. Dragomir and N. M. Ionescu, *On nonlinear integral inequalities in two independent variables*, Studia Univ. Babeş-Bolyai, Math., Vol. 34 (1989), 11-17.

[5] A. Mate and P. Neval, *Sublinear perturbations of the differential equation* $y^{(n)} = 0$ *and of the analogous difference equation*, J. Differential Equations, Vol. 52 (1984), 234-257.

[6] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1991.

[7] B. G. Pachpatte, *On some fundamental integral inequalities and their discrete analogues*, JIPAM. J. Inequal. Pure Appl. Math., Vol. 2 (2001), Issue 2, Article 15.

[8] B. G. Pachpatte, *On some finite difference inequalities in two independent variables*, J. Math. Anal. Appl., Vol. 254 (2001), 587-598.

Sever S. Dragomir
School of Communications and Informatics, Victoria University of Technology
P O Box 14428, MCMC Melbourne, Victoria 8001, Australia
email: sever.dragomir@vu.edu.au

Young-Ho Kim
Department of Applied Mathematics, Changwon National University,
Changwon 641-773, Korea
email: yhkim@sarim.changwon.ac.kr