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LIMIT CYCLES FOR A CLASS OF POLYNOMIAL SYSTEMS AND APPLICATIONS

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ABSTRACT. In this paper, we study the existence and uniqueness of limit cycles for a particular polynomial system. Some known results are extended and improved.

1. INTRODUCTION

As well known, the second part of Hilbert's 16th problem deals with polynomial differential equations in the plane. Recall that the second part of Hilbert's 16th problem asks about the following three questions:

(i) Is it true that a planar polynomial vector field has but a finite number of limit cycles?
(ii) Is it true that the number of limit cycles of a planar polynomial vector field is bounded by a constant depending on the degree of the polynomials only?
(iii) Given an upper bound on the number of limit cycles in (ii).

There has been a large body of excellent work on the study of the problems above since 1900 (see, for example, [1-9] and the references cited therein). However, it remains unsolved even for quadratic polynomials. There may be a long distance to solve this problem. So far, most of the previous important works tackle the quadratic polynomial systems, but there aren't many results dealing with cubic and higher degree polynomial system. In this paper, we are concerned with a class of higher degree polynomial differential system which contains as particular cases certain more simple systems considered earlier by other authors. The main purpose of this paper is to study the number and the distribution of limit cycles of higher degree polynomial differential equations. To do so, we consider the following differential system

$$\begin{gathered} dx/dt = -y^\alpha(1+ny)^\alpha + dx^\alpha(1+ny)^\alpha + lx^{\alpha+1}(1+ny)^{\alpha-1} - my^{\alpha+1}(1+ny)^\alpha, \\ dy/dt = x^\alpha[(1+ny)^\alpha + (ax)^\alpha], \end{gathered} \tag{1.1}$$

where $n \ge m \ge 0$, α is a positive odd integer. Without loss of generality, we assume $a \le 0$ (Otherwise, one may make the transformation $x_1 = -x, t_1 = -t$).

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The paper is organized as follows. In the next section, by analyzing an equivalent system of (1.1), we discuss the existence and uniqueness of limit cycles of system (1.1). In section 3, by applying the main results in section 2, we discuss some more simple polynomial differential systems considered earlier by some other authors. Some known results in [2, 4, 5, 9] are extended and improved.

2. The main results and proofs

In this section, we discuss the existence and uniqueness of limit cycles of system (1.1) by using the qualitative theory of ordinary differential equations.

Noting that $y = -1/n$ is a straight line isocline, we set $x = \overline{x}(1+n\overline{y})$, $y = \overline{y}$, $dt = (1+ny)^{1-\alpha}d\overline{t}$, then system (1.1) can be transformed into the following equivalent system (denoting $\overline{x}$, $\overline{y}$, $\overline{t}$ again by x, y, t, respectively):

$$\begin{gathered} dx/dt = -y^{\alpha} - my^{\alpha+1} + (1+ny)^{\alpha}[dx^{\alpha} + (l-n)x^{\alpha+1} - na^{\alpha}x^{2\alpha+1}] \equiv P(x,y), \\ dy/dt = (1+ny)^{\alpha+1}x^{\alpha}(1+a^{\alpha}x^{\alpha}) \equiv Q(x,y). \end{gathered} \tag{2.1}$$

We now formulate our main results for systems (1.1) and (2.1) as follows.

Theorem 2.1. *If $ad[l(\alpha+1)+n\alpha] \le 0$, then system (1.1) has no limit cycle in the whole plane.*

Proof. Take a Dulac's function

$$B(y) = (1+ny)^{\frac{-l(\alpha+1)}{n}}.$$

Along solutions of system (2.1), we derive that

$$\operatorname{div}(BP, BQ) = (1+ny)^{\frac{n\alpha - l(\alpha+1)}{n}} x^{\alpha-1}[ad - (l(\alpha+1)+n\alpha)a^{\alpha}x^{\alpha+1}].$$

Thus, if $ad[l(\alpha+1)+n\alpha] \le 0$, then $\operatorname{div}(BP, BQ)$ is of fixed sign in the region $y + \frac{1}{n} > 0$ or $y + \frac{1}{n} < 0$ and is not identically equal to zero in any subregion since α is a positive odd integer. By Dulac's criterion, we know that system (2.1) has no limit cycle in the whole plane. Since system (1.1) is equivalent to system (2.1), we assert that system (1.1) has no limit cycle in the whole plane. This completes the proof. □

By Theorem 2.1, we can assume $ad[l(\alpha+1)+n\alpha] > 0$. By the assumption of $a \le 0$, we have $d[l(\alpha+1)+n\alpha] < 0$. Hence, in the following, we always assume $l > -n\alpha/(\alpha+1), a < 0$, and $d < 0$.

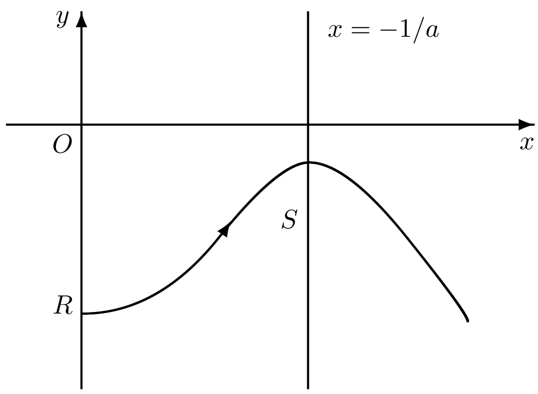
Lemma 2.2. *If system (2.1) has limit cycles, then they must intersect the line $x = -\alpha d/[l(\alpha+1)+n\alpha]$.*

Proof. Take a Dulac's function $B(y) = (1+ny)^{\alpha}$. Then it follows that

$$\operatorname{div}(BP, BQ) = x^{\alpha-1}(1+ny)^{2\alpha}[\alpha d + (l(\alpha+1)+n\alpha)x].$$

Hence, if system (2.1) has limit cycles, then they must intersect the line $x = -\alpha d/[l(\alpha+1)+n\alpha]$. The proof is complete. □

Lemma 2.3. *If system (2.1) has limit cycles around the origin, then they must not intersect the line $x = -1/a$.*



Proof. Let Γ be a limit cycle around the origin which intersects the y-axis at R and the line $x = -1/a$ on S. Since $Q(x, y) = (1 + ny)^{\alpha+1} x^{\alpha}(1 + a^{\alpha} x^{\alpha}) < 0$ on the right hand of the line $x = -1/a$, Γ will go in a right downward direction after it passes through the point S. Moreover, it is impossible for it to go around the origin. This contradicts the assumption that Γ is a limit cycle around the origin. The proof is complete. $\square$

Theorem 2.4. *If $ad \geq [l(\alpha + 1) + n\alpha]/\alpha$, then system* (2.1) *has no limit cycle around the origin.*

Proof. By Lemma 2.2, if system (2.1) has a limit cycle around the origin, it must intersect the line $x = -\alpha d/[l(\alpha + 1) + n\alpha]$. By Lemma 2.3, it must not intersect the line $x = -1/a$. Hence the line $x = -\alpha d/[l(\alpha+1)+n\alpha]$ must lie in the left hand of the line $x = -1/a$, i.e. $-\alpha d/[l(\alpha + 1) + n\alpha] < -1/a$. Noticing the assumptions of $l > -n\alpha/(\alpha + 1), a < 0, d < 0$, the inequality above can be written in the form $ad < [l(\alpha+1)+n\alpha]/\alpha$. Therefore, if $ad \geq [l(\alpha+1)+n\alpha]/\alpha$, then system (2.1) has no limit cycle around the origin.The proof is complete. $\square$

From what has been discussed above, we can assume $0 < ad < [l(\alpha+1)+n\alpha]/\alpha$, and we know that if system (2.1) has a limit cycle around the origin, it must lie in the region $D = \{(x, y) : x < -1/a,\ y > -1/n\}$.

Let

$$\hat{x} = x, \quad e^{-\hat{y}} = 1 + ny, \quad d\hat{t} = -(1 + ny)^{\alpha} dt. \tag{2.2}$$

On dropping the hats for ease of notation, system (2.1) becomes

$$\begin{aligned} dx/dt &= -\varphi(y) - F(x), \\ dy/dt &= g(x), \end{aligned} \tag{2.3}$$

where

$$\varphi(y) = (e^y - 1)^{\alpha}[(n - m) + me^{-y}]/n^{\alpha+1},$$

$$F(x) = dx^{\alpha} + (l - n)x^{\alpha+1} - na^{\alpha}x^{2\alpha+1},$$

$$g(x) = nx^{\alpha}(1 + a^{\alpha}x^{\alpha}).$$

Denote

$$f(x) = F'(x) = \alpha d x^{\alpha-1} + (\alpha+1)(l-n)x^{\alpha} - n(2\alpha+1)a^{\alpha}x^{2\alpha},$$

$$G(x) = \int_0^x g(x)dx = \frac{n}{\alpha+1}x^{\alpha+1} + \frac{na^{\alpha}}{2\alpha+1}x^{2\alpha+1}.$$

Lemma 2.5. *If $d < 0, |d| \ll 1$, then the singular point O of system* (1.1) *is a stable focus; if $d = 0$, then the singular point O is an unstable focus.*

Proof. Let $z = G(x)$. We denote by $x_1(z) > 0$ the inverse function of $G(x)$ in the interval $(0, -1/a)$, and by $x_2(z)$ the inverse function of $G(x)$ in the half plane $x < 0$. Since $G(x) \sim \frac{n}{\alpha+1}x^{\alpha+1}$ as $x \to 0$, we have $x_1(z) \sim (\frac{\alpha+1}{n})^{1/(\alpha+1)}z^{1/(\alpha+1)}$ and $x_2(z) \sim -x_1(z)$ as $z \to 0$. Denote $F_i(z) = F(x_i(z))$, $i = 1, 2$. It is easy to verify that $\varphi(y)$ and $F_i(z)$ satisfy all the conditions of Corollary 4 in [3] for $d \le 0$ and $a < 0$.

Since

$$F_1(z) - F_2(z) \sim 2d\left(\frac{\alpha+1}{n}\right)^{\alpha/(\alpha+1)} z^{\frac{\alpha}{\alpha+1}} - 2na^{\alpha}\left(\frac{\alpha+1}{n}\right)^{(2\alpha+1)/(\alpha+1)} z^{\frac{2\alpha+1}{\alpha+1}}$$

as $z \to 0$, it is easy to see that for $0 < z \ll 1$, if $d < 0$, then $F_1(z) - F_2(z) < 0$ and if $d = 0$, then $F_1(z) - F_2(z) > 0$. Hence, if $d < 0$, then the origin O is an unstable focus of system (2.3); if $d = 0$, then the origin O is a stable focus. Noticing the transformation (2.2), it follows that if $d < 0$, then the origin O is a stable focus of system (1.1); if $d = 0$, then the origin O is an unstable focus. The proof is complete. □

Theorem 2.6. *If $0 < ad < [l(\alpha+1) + n\alpha]/\alpha$ and $|d| \ll 1$, then system* (1.1) *has limit cycles around the origin.*

Proof. By Lemma 2.5 and the Hopf's bifurcation Theorem in [7], the conclusion of Theorem 2.6 can be obtained directly. This completes the proof. □

Theorem 2.7. *If $0 < ad < [l(\alpha+1) + n\alpha]/\alpha$, then system* (1.1) *has at most one limit cycle around the origin, and if it exists, it must be unstable.*

Proof. Consider system (2.3). It is easy to determine $\varphi'(y) > 0$, $\varphi(0) = 0$ and $xg(x) > 0$ for $x < -1/a$ and $x \neq 0$, and

$$\Big(\frac{f(x)}{g(x)}\Big)' = -\frac{\alpha}{nx^2(1+a^{\alpha}x^{\alpha})^2}\{[n\alpha + l(\alpha+1)]a^{\alpha}x^{\alpha+1} + (\alpha+1)da^{\alpha}x^{\alpha} + d\}.$$

Let

$$h(x) = [n\alpha + l(\alpha+1)]a^{\alpha}x^{\alpha+1} + (\alpha+1)da^{\alpha}x^{\alpha} + d.$$

Then it follows

$$\begin{aligned} h'(x) &= (\alpha+1)[n\alpha + l(\alpha+1)]a^{\alpha}x^{\alpha} + \alpha(\alpha+1)da^{\alpha}x^{\alpha-1} \\ &= (1+\alpha)\{[l(\alpha+1) + n\alpha]x + d\alpha\}a^{\alpha}x^{\alpha-1}. \end{aligned}$$

It is easy to verify that $h(x)$ has a unique point, namely $x_0 = -\alpha d/[n\alpha + l(\alpha+1)]$ at which $h(x)$ attains its maximum, and $h(x_0) = d\{1 - [\frac{\alpha a d}{l(\alpha+1)+n\alpha}]^{\alpha}\}$. Noticing that $0 < ad < [l(\alpha+1) + n\alpha]/\alpha$, we know that $h(x_0) < 0$, hence $h(x) < 0$. Therefore, it follows that $(\frac{f(x)}{g(x)})' > 0$.

By Lemma 2.5, if $d < 0$, then the origin O is a stable focus of system (1.1). Applying the Zhang Zhifen's Theorem in [8], we have that system (1.1) has at most one limit cycle around the origin, and if it exists, it must be unstable. This completes the proof. □

3. The applications of the main results

In [4], the author considered the system

$$\begin{aligned} dx/dt &= -y + dx + x^2 + dxy - (m+1)y^2 - my^3, \\ dy/dt &= x(1 + ax + y), \quad (0 \le m \le 1), \end{aligned} \tag{3.1}$$

which is a special case of (1.1) when $n = 1, \alpha = 1$, $l = 1$ in system (1.1). It is easy to derive the following corollaries by Theorems 2.1, 2.4, 2.6, 2.7.

Corollary 3.1. *For system* (3.1)*, if $ad \le 0$, then there is no limit cycle in the whole plane; if $ad \ge 3$, then there is no limit cycle around the origin.*

Corollary 3.2. *For system* (3.1)*, we have the following conclusions:*

- (i) *If $0 < ad < 3$, then there is at most one limit cycle around the origin, and if it exists, it must be unstable;*
- (ii) *If $0 < ad < 3$ and $|d| \ll 1$, then there is a unique unstable limit cycle around the origin.*

In [9], the author discussed the quadratic system

$$\begin{aligned} dx/dt &= -y + dx + x^2 + dxy - y^2 \\ dy/dt &= x(1 + ax + y), \end{aligned} \tag{3.2}$$

which is a special case of (3.1) when $m = 0$ in system (3.1) or a special case of (1.1) when $n = 1, \alpha = 1$, $l = 1$ and $m = 0$ in system (1.1). From what has been discussed above, it is easy to see that above Corollaries 3.1, 3.2 still hold for system (3.2).

Remark 3.3. Corollaries 3.1, 3.2 not only contain all results in [4] and [9], but also show sufficient conditions for the existence of limit cycles which was not involved in [9].

In [2], the authors studied the system

$$\begin{aligned} dx/dt &= dx^\alpha(1 - ny)^\alpha + lx^{\alpha+1}(1 - ny)^{\alpha-1}, \\ dy/dt &= x^\alpha[(1 - ny)^\alpha + (ax)^\alpha]. \end{aligned} \tag{3.3}$$

On substituting

$$\hat{x} = x, \quad \hat{y} = -y, \quad \hat{t} = -t \tag{3.4}$$

into (3.3) and dropping the hats for ease of notation, system (3.3) reduces to the equivalent system

$$\begin{aligned} dx/dt &= -y^\alpha(1 + ny)^\alpha + dx^\alpha(1 + ny)^\alpha - lx^{\alpha+1}(1 + ny)^{\alpha-1}, \\ dy/dt &= x^\alpha[(1 + ny)^\alpha + (ax)^\alpha], \end{aligned} \tag{3.5}$$

which is a special case of (1.1) when $m = 0$ in system (1.1). Noticing the negative sign before l in system (3.5) and the transformation (3.4), by Theorems 2.1, 2.4, 2.6, 2.7, we have the following results.

Corollary 3.4. *If $ad[l(\alpha+1)-n\alpha]\geq 0$, then system* (3.5) *has no limit cycle in the whole plane; If $ad\geq [n\alpha-l(\alpha+1)]/\alpha$, then system* (3.5) *has no limit cycle around the origin.*

Corollary 3.5. *For system* (3.5)*, we have the following conclusions:*

(i) *If $0<ad<[n\alpha-l(\alpha+1)]/\alpha$, then system* (3.5) *has at most one limit cycle around the origin, and if it exists, it must be stable;*

(ii) *If $0<ad<[n\alpha-l(\alpha+1)]/\alpha$ and $|d|\ll 1$, then system* (3.5) *has a unique stable limit cycle around the origin.*

In [5], the author discussed the following system:

$$\begin{aligned} dx/dt &= -y^{\alpha}(1-y)^{\alpha} - dx^{\alpha}(1-y)^{\alpha} + lx^{\alpha+1}(1-y)^{\alpha-1},\\ dy/dt &= x^{\alpha}[(1-y)^{\alpha}+(ax)^{\alpha}], \end{aligned} \tag{3.6}$$

which is a special case of (3.5) when $n=1$ in system (3.5). by Corollary 3.4 and Corollary 3.5, we have the following results directly.

Corollary 3.6. *If $ad[l(\alpha+1)-\alpha]\geq 0$, then system* (3.6) *has no limit cycle in the whole plane; If $ad\geq [\alpha-l(\alpha+1)]/\alpha$, then system* (3.6) *has no limit cycle around the origin.*

Corollary 3.7. *For system* (3.6)*, we have the following conclusions:*

(i) *If $0<ad<[\alpha-l(\alpha+1)]/\alpha$, then system* (3.6) *has at most one limit cycle around the origin, and if it exists, it must be stable;*

(ii) *If $0<ad<[\alpha-l(\alpha+1)]/\alpha$ and $|d|\ll 1$, then system* (3.6) *has a unique stable limit cycle around the origin.*

In [1], the author considered the following quadratic system with a degenerate critical point:

$$\begin{aligned} dx/dt &= -y-dx+lx^2+dxy+y^2,\\ dy/dt &= x(1+ax-y), \end{aligned} \tag{3.7}$$

which is a special case of (3.6) when $\alpha=1$ in system (3.6) or a special case of (1.1) when $n=1,\alpha=1$, and $m=0$ in system (1.1). From what has been discussed above, it is easy to see that above Corollaries 3.6, 3.7 in which when $\alpha=1$ still hold for system (3.7).

Remark 3.8. Corollaries 3.4, 3.5 and corollaries 3.6, 3.7 contain the main results in [2] and [5] respectively.

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