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OSCILLATION AND NONOSCILLATION OF SOLUTIONS TO EVEN ORDER SELF-ADJOINT DIFFERENTIAL EQUATIONS

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Abstract. We establish oscillation and nonoscilation criteria for the linear differential equation

$$(-1)^n \big(t^\alpha y^{(n)}\big)^{(n)} - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y = q(t)y, \quad \alpha \notin \{1,3,\dots,2n-1\},$$

where

$$\gamma_{n,\alpha} = \frac{1}{4^n} \prod_{k=1}^{n} (2k-1-\alpha)^2$$

and q is a real-valued continuous function. It is proved, using these criteria, that the equation

$$(-1)^n \big(t^\alpha y^{(n)}\big)^{(n)} - \big(\frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} + \frac{\gamma}{t^{2n-\alpha}\lg^2 t}\big) y = 0$$

is nonoscillatory if and only if

$$\gamma \le \tilde{\gamma}_{n,\alpha} := \frac{1}{4^n} \prod_{k=1}^{n} (2k-1-\alpha)^2 \sum_{k=1}^{n} \frac{1}{(2k-1-\alpha)^2}.$$

1. Introduction

In this paper we investigate the oscillatory behavior of the two term self-adjoint linear differential equation of the form

$$(-1)^n \big(t^\alpha y^{(n)}\big)^{(n)} = p(t)y, \quad \alpha \notin \{1,3,\dots,2n-1\}, \tag{1.1}$$

where p is a continuous function.

Oscillatory properties of equation (1.1) has been investigated in several recent papers, see [5, 7, 9, 10, 11, 12, 13, 16] and the references given therein. In these papers, equation (1.1) is seen as a perturbation of the one term equation

$$(-1)^n \big(t^\alpha y^{(n)}\big)^{(n)} = 0,$$

or of the Euler-type equation

$$(-1)^n \big(t^\alpha y^{(n)}\big)^{(n)} - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y = 0, \tag{1.2}$$

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where

$$\gamma_{n,\alpha} := (-1)^n \prod_{k=0}^{n-1} (\lambda - k)(\lambda + \alpha - k - n)|_{\lambda = \frac{2n-1-\alpha}{2}} = \frac{1}{4^n} \prod_{k=1}^{n} (2k - 1 - \alpha)^2, \quad (1.3)$$

i.e., (1.1) was considered in the form

$$(-1)^n \big(t^\alpha y^{(n)}\big)^{(n)} - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y = q(t)y, \quad (1.4)$$

in the latter case.

If $n = 1$ and $\alpha = 0$, then (1.1) reduces to the second order equation

$$y'' + p(t)y = 0 \quad (1.5)$$

whose oscillation theory is deeply developed, see [18]. In the classical oscillation criteria, (1.5) is viewed as a perturbation of the equation $y'' = 0$ and oscillatory nature of (1.5) depends on "how much p is positive", the last vague expression being specified in the quantitative way in particular (non)oscillation criteria. If (1.5) is viewed as a perturbation of the Euler equation with the "critical" constant 1/4

$$y'' + \frac{1}{4t^2} y = 0,$$

i.e., (1.5) is written in the form

$$y'' + \frac{1}{4t^2} y + q(t)y = 0, \quad q(t) = p(t) - \frac{1}{4t^2}, \quad (1.6)$$

one gets more refined criteria and (non)oscillation of (1.5) is "measured" by positivity of the difference $p(t) - \frac{1}{4t^2}$. Generally, the second order equation with iterated logarithms

$$y'' + \frac{1}{4t^2}\big(1 + \frac{1}{\lg^2 t} + \dots + \frac{1}{\lg^2 t \lg_2^2 t \dots \lg_{n-1}^2 t} + \frac{1}{\lg^2 t \lg_2^2 t \cdots \lg_n^2 t}\big) y = 0, \quad (1.7)$$

where $\lg_2 t = \lg(\lg t)$, $\lg_n t = \lg(\lg_{n-1} t)$ and lg denotes the natural logarithm, is nonoscillatory and one can view (1.5) as perturbation of (1.7). The more logarithmic terms are involved, the more refined oscillation criteria are obtained.

Here we follow this line in case of higher order equations. Equation (1.1) (and hence also (1.4)) is viewed as a perturbation of the equation

$$(-1)^n \big(t^\alpha y^{(n)}\big)^{(n)} - \big(\frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} + \frac{\gamma}{t^{2n-\alpha} \lg^2 t}\big) y = 0 \quad (1.8)$$

with

$$\gamma = \tilde{\gamma}_{n,\alpha} := \gamma_{n,\alpha} \sum_{k=1}^{n} \frac{1}{(2k-1-\alpha)^2}, \quad (1.9)$$

$\gamma_{n,\alpha}$ being given by (1.3). We establish oscillation and nonoscillation criteria for (1.4) and we show that (1.8) is oscillatory for $\gamma > \tilde{\gamma}_{n,\alpha}$ and nonoscillatory for $\gamma \le \tilde{\gamma}_{n,\alpha}$. This approach can be regarded as a generalization of the results of the papers [7, 13], dealing with the fourth order equations, i.e. with the case $n = 2$.

To study equation (1.4), we use methods based on the factorization of disconjugate operators, variational technique and the relationship between self-adjoint equations and linear Hamiltonian systems, similarly as in the papers mentioned above. We also use some combinatorial identities to determine the exact value of the oscillation constant $\tilde{\gamma}_{n,\alpha}$ of (1.8).

This paper is organized as follows. The next section contains necessary definitions and some auxiliary results concerning self-adjoint equations. In section 3 we present the main results of the paper – oscillation and nonoscillation criteria for (1.4). In Section 4 we discuss some open problems and possibilities of the extension of our results. The last section contains technical computations related to the combinatorial identities used in (non)oscillation criteria of Section 3.

2. Preliminaries

We start with a statement concerning factorization of the formally self-adjoint differential operators

$$L(y) := \sum_{k=0}^{n} (-1)^k \left(r_k(t) y^{(k)}\right)^{(k)} = 0, \quad r_n(t) > 0. \tag{2.1}$$

Note that the differential operator generated by the left-hand side of (1.4) is a special case of the operator L.

Lemma 2.1 ([1]). *Suppose that equation* (2.1) *possesses a system of positive solutions $y_1, \dots, y_{2n}$ such that Wronskians $W(y_1, \dots, y_k) \neq 0$, $k = 1, \dots, 2n$, for large t. Then the operator L admits the factorization for large t*

$$L(y) = \frac{(-1)^n}{a_0(t)} \Big(\frac{1}{a_1(t)} \Big(\dots \frac{r_n(t)}{a_n^2(t)} \Big(\frac{1}{a_{n-1}(t)} \cdots \frac{1}{a_1(t)} \big(\frac{y}{a_0(t)} \big)' \dots \Big)' \dots \Big)' \Big)',$$

where

$$a_0 = y_1, \ a_1 = \big(\frac{y_2}{y_1}\big)', \ a_i = \frac{W(y_1, \dots, y_{i+1}) W(y_1, \dots, y_{i-1})}{W^2(y_1, \dots, y_i)}, \ i = 1, \dots, n-1,$$

and $a_n = (a_0 \cdots a_{n-1})^{-1}$.

An important role is played in our investigation by the following specification of the factorization formula to the differential operator given by the left-hand side of (1.2).

Lemma 2.2. *Let $\alpha \neq \{1, 3, \dots, 2n-1\}$. Then we have for any sufficiently smooth function y*

$$\begin{aligned} l(y) &:= (-1)^n \big(t^\alpha y^{(n)}\big)^{(n)} - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y \\ &= \frac{(-1)^n}{a_0(t)} \Big(\frac{1}{a_1(t)} \Big(\dots \frac{t^\alpha}{a_n^2(t)} \Big(\frac{1}{a_{n-1}(t)} \cdots \frac{1}{a_1(t)} \big(\frac{y}{a_0(t)} \big)' \dots \Big)' \dots \Big)' \Big)', \end{aligned} \tag{2.2}$$

where

$$a_0(t) = t^{\alpha_0}, \ a_k(t) = t^{\alpha_k - \alpha_{k-1} - 1}, \ k = 1, \dots, n-1, \ a_n(t) = t^{(n-1) - \alpha_{n-1}},$$

with $\alpha_0 = \frac{2n-1-\alpha}{2}$ and $\alpha_1 < \dots < \alpha_{n-1}$ the first roots (ordered by their size) of the polynomial

$$P(\lambda) := (-1)^n \prod_{i=0}^{n-1} (\lambda - i)(\lambda - n + \alpha - i) - \gamma_{n,\alpha}. \tag{2.3}$$

Proof. By a direct computation one can verify that the functions $y = t^{\alpha_k}$, $k = 0, \dots, n-1$, where α_k are the roots of the polynomial P, are solutions of (1.2).

Substituting into the formulas in Lemma 2.1 with $y_1 = t^{\alpha_0}$, $y_k = t^{\alpha_{k-1}}$, $k = 2, \dots, n$, we have

$$a_0(t) = t^{\alpha_0} = t^{\frac{2n-1-\alpha}{2}}, \quad a_1(t) = \big(\frac{y_2}{y_1}\big)' = (\alpha_1 - \alpha_0)t^{\alpha_1-\alpha_0-1}.$$

In computing the remaining functions a_k, $k = 2, \dots, n-1$, we use the formula for computation of Wronskians of power functions

$$W(t^{\beta_1}, \dots, t^{\beta_k}) = \prod_{1\le i<j\le k} (\beta_j - \beta_i) t^{\beta_1+\dots+\beta_k-\frac{k(k-1)}{2}}, \tag{2.4}$$

see Lemma 5.4 in the last section. Using (2.4), for $k = 2, \dots, n-1$,

$$\begin{aligned}
&a_k(t)\\
&= \frac{W(t^{\alpha_0}, \dots, t^{\alpha_k})W(t^{\alpha_0}, \dots, t^{\alpha_{k-2}})}{(W(t^{\alpha_0}, \dots, t^{\alpha_{k-1}}))^2}\\
&= \frac{t^{\alpha_0+\dots+\alpha_k-\frac{(k+1)k}{2}} t^{\alpha_0+\dots+\alpha_{k-2}-\frac{(k-1)(k-2)}{2}} \prod_{0\le i<j\le k} (\alpha_j-\alpha_i) \prod_{0\le i<j\le k-2} (\alpha_j-\alpha_i)}{\big(t^{\alpha_0+\dots+\alpha_{k-1}-\frac{(k-1)k}{2}} \prod_{0\le i<j\le k-1}(\alpha_j-\alpha_i)\big)^2}\\
&= \frac{\prod_{i=0}^{k-1}(\alpha_k-\alpha_i)}{\prod_{i=0}^{k-2}(\alpha_k-\alpha_i)} t^{\alpha_k-\alpha_{k-1}-1}\\
&= \frac{(\alpha_k-\alpha_0)(\alpha_k-\alpha_1)\cdots(\alpha_k-\alpha_{k-1})}{(\alpha_{k-1}-\alpha_0)(\alpha_{k-1}-\alpha_1)\cdots(\alpha_{k-1}-\alpha_{k-2})} t^{\alpha_k-\alpha_{k-1}-1}
\end{aligned}$$

and

$$a_n = \frac{t^{(n-1)-\alpha_{n-1}}}{(\alpha_{n-1}-\alpha_0)(\alpha_{n-1}-\alpha_1)\cdots(\alpha_{n-1}-\alpha_{n-2})}.$$

Since the product of all factors $(\alpha_j - \alpha_i)$ appearing in the functions $a_0, \dots, a_n$ equals 1, we can neglect these terms and the factorization of the Euler operator is really as stated. □

Now we recall basic oscillatory properties of self-adjoint differential equations (2.1). These properties can be investigated within the scope of the oscillation theory of linear Hamiltonian systems (further LHS)

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u, \tag{2.5}$$

where A, B, C are $n \times n$ matrices with B, C symmetric. Indeed, if y is a solution of (2.1) and we set

$$x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad u = \begin{pmatrix} (-1)^{n-1}(r_n y^{(n)})^{(n-1)} + \dots + r_1 y' \\ \vdots \\ -(r_n y^{(n)})' + r_{n-1} y^{(n-1)} \\ r_n y^{(n)} \end{pmatrix},$$

then (x, u) solves (2.5) with A, B, C given by

$$B(t) = \operatorname{diag}\{0, \dots, 0, r_n^{-1}(t)\}, \quad C(t) = \operatorname{diag}\{r_0(t), \dots, r_{n-1}(t)\},$$
$$A = A_{i,j} = \begin{cases} 1, & \text{if } j = i+1,\ i = 1, \dots, n-1, \\ 0, & \text{elsewhere.} \end{cases}$$

In this case we say that the solution (x, u) of (2.5) is generated by the solution y of (2.1). Moreover, if $y_1, \dots, y_n$ are solutions of (2.1) and the columns of the matrix solution (X, U) of (2.5) are generated by the solutions $y_1, \dots, y_n$, we say that the solution (X, U) is generated by the solutions $y_1, \dots, y_n$.

Recall that two different points t_1, t_2 are said to be *conjugate* relative to system (2.5) if there exists a nontrivial solution (x, u) of this system such that $x(t_1) = 0 = x(t_2)$. Consequently, by the above mentioned relationship between (2.1) and (2.5), these points are conjugate relative to (2.1) if there exists a nontrivial solution y of this equation such that $y^{(i)}(t_1) = 0 = y^{(i)}(t_2)$, $i = 0, 1, \dots, n-1$. System (2.5) (and hence also equation (2.1)) is said to be *oscillatory* if for every $T \in \mathbb{R}$ there exists a pair of points $t_1, t_2 \in [T, \infty)$ which are conjugate relative to (2.5) (relative to (2.1)), in the opposite case (2.5) (or (2.1)) is said to be *nonoscillatory*.

Using the relation between (2.1), (2.5) and the so-called Roundabout Theorem for linear Hamiltonian systems (see e.g. [17]), one can easily prove the following variational lemma.

Lemma 2.3 ([14]). *Equation* (2.1) *is nonoscillatory if and only if there exists* $T \in \mathbb{R}$ *such that*

$$\mathcal{F}(y; T, \infty) := \int_T^\infty \Big[\sum_{k=0}^n r_k(t)(y^{(k)}(t))^2\Big]\, dt > 0$$

for any nontrivial $y \in W^{n,2}(T, \infty)$ *with compact support in* (T, ∞).

One of the main tools in our investigation is also the following Wirtinger-type inequality.

Lemma 2.4 ([14]). *Let* $y \in W^{1,2}(T, \infty)$ *have compact support in* (T, ∞) *and let* M *be a positive differentiable function such that* $M'(t) \neq 0$ *for* $t \in [T, \infty)$. *Then*

$$\int_T^\infty |M'(t)| y^2\, dt \le 4 \int_T^\infty \frac{M^2(t)}{|M'(t)|} y'^2\, dt.$$

Now we express the quadratic functional associated with (1.2) in a way suitable for the application of the Wirtinger inequality. This statement can be proved using the repeated integration by parts, similarly as in [7, Lemma 4].

Lemma 2.5. *Let* $y \in W_0^{n,2}(T, \infty)$ *have compact support in* (T, ∞). *Then*

$$\begin{aligned}
&\int_T^\infty \Big[t^\alpha \big(y^{(n)}\big)^2 - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y^2\Big]\, dt \\
&= \int_T^\infty \frac{t^\alpha}{a_n} \Big\{ \Big[\frac{1}{a_{n-1}} \Big(\frac{1}{a_{n-2}} \Big(\dots \frac{1}{a_1} \big(\frac{y}{a_0}\big)' \Big)' \dots \Big)' \Big]' \Big\}^2 dt,
\end{aligned}$$

where $a_0, \dots, a_n$ *are given in Lemma 2.2.*

We finish this section with the concept of the principal system of solutions of (2.1). A conjoined basis (X, U) of (2.5) (i.e. a matrix solution of this system with $n \times n$ matrices X, U satisfying $X^T(t)U(t) = U^T(t)X(t)$ and $\operatorname{rank}(X^T, U^T)^T = n$) is said to be the *principal solution* of (2.5) if $X(t)$ is nonsingular for large t and for any other conjoined basis $(\bar X, \bar U)$ such that the (constant) matrix $X^T \bar U - U^T \bar X$ is nonsingular $\lim_{t\to\infty} \bar X^{-1}(t) X(t) = 0$ holds. The last limit equals zero if and only if

$$\lim_{t\to\infty} \Big(\int^t X^{-1}(s) B(s) X^{T-1}(s)\, ds \Big)^{-1} = 0, \tag{2.6}$$

see [17]. A principal solution of (2.5) is determined uniquely up to a right multiple by a constant nonsingular $n \times n$ matrix. If (X, U) is the principal solution, any conjoined basis $(\bar{X}, \bar{U})$ such that the matrix $X^T \bar{U} - U^T \bar{X}$ is nonsingular is said to be a *nonprincipal solution* of (2.5). Solutions $y_1, \dots, y_n$ of (2.1) are said to form the *principal (nonprincipal) system of solutions* if the solution (X, U) of the associated linear Hamiltonian system generated by $y_1, \dots, y_n$ is a principal (nonprincipal) solution. Note that if (2.1) possesses a fundamental system of positive solutions $y_1, \dots, y_{2n}$ satisfying $y_i = o(y_{i+1})$ as $t \to \infty$, $i = 1, \dots, 2n-1$, (the so-called *ordered system* of solutions), then the "small" solutions $y_1, \dots, y_n$ form the principal system of solutions of (2.1). In particular, if $L(y) = (-1)^n (t^\alpha y^{(n)})^{(n)} - \gamma_{n,\alpha} t^{\alpha - 2n} y$ and $\alpha_0, \dots, \alpha_{n-1}$ are the same as in Lemma 2.2, then $y_k = t^{\alpha_k}$, $k = 1, \dots, n-1$, $y_n = t^{(2n-1-\alpha)/2}$ is the ordered principal system of solutions of (1.2).

3. Main results – oscillation and nonoscillation criteria

We start this section with a statement where nonoscillation of (1.4) is compared with nonoscillation of a certain associated second order differential equation.

Theorem 3.1. *If the second order linear differential equation*

$$(tz')' + \frac{1}{4\tilde{\gamma}_{n,\alpha}} t^{2n-1-\alpha} q(t) z = 0 \tag{3.1}$$

is nonoscillatory, then equation (1.4) *is also nonoscillatory.*

Proof. Let $T \in \mathbb{R}$ and $y \in W^{n,2}(T, \infty)$ be any function having compact support in (T, ∞). Using Lemma 2.5, Wirtinger inequality (Lemma 2.4), which we apply $(n-1)$-times, and also Lemma 5.1 from the last section, we have

$$\begin{aligned}
&\int_T^\infty \Big[t^\alpha \big(y^{(n)}\big)^2 - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y^2 \Big] dt \\
&= \int_T^\infty \frac{t^\alpha}{a_n} \Big\{ \Big[\frac{1}{a_{n-1}} \Big(\frac{1}{a_{n-2}} \Big(\dots \frac{1}{a_1} \big(\frac{y}{a_0} \big)' \Big)' \dots \Big)' \Big]' \Big\}^2 dt \\
&\ge \prod_{k=1}^{n-1} \Big(\frac{2n-1-\alpha}{2} - \alpha_k \Big)^2 \int_T^\infty t \big[\big(\frac{y}{a_0} \big)' \big]^2 dt \\
&= 4\tilde{\gamma}_{n,\alpha} \int_T^\infty t \Big[\Big(\frac{y}{t^{(2n-1-\alpha)/2}} \Big)' \Big]^2 dt.
\end{aligned}$$

If we denote $z = y / t^{(2n-1-\alpha)/2}$ then, since (3.1) is nonoscillatory, it follows from Lemma 2.3, that

$$\int_T^\infty \Big[t(z'(t))^2 - \frac{1}{4\tilde{\gamma}_{n,\alpha}} t^{2n-1-\alpha} q(t) z^2(t) \Big] dt > 0.$$

Summarizing

$$\begin{aligned}
&\int_T^\infty \Big[t^\alpha \big(y^{(n)}\big)^2 - \Big(\frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} + q(t) \Big) y^2 \Big] dt \\
&\ge 4\tilde{\gamma}_{n,\alpha} \int_T^\infty t \Big\{ \Big[\Big(\frac{y}{t^{(2n-1-\alpha)/2}} \Big)' \Big]^2 - \frac{1}{4\tilde{\gamma}_{n,\alpha}} q(t) y^2(t) \Big\} dt \\
&= 4\tilde{\gamma}_{n,\alpha} \int_T^\infty t \Big\{ \Big[\Big(\frac{y}{t^{(2n-1-\alpha)/2}} \Big)' \Big]^2 - \frac{1}{4\tilde{\gamma}_{n,\alpha}} t^{2n-1-\alpha} q(t) \big(\frac{y}{t^{(2n-1-\alpha)/2}} \big)^2 \Big\} dt > 0.
\end{aligned}$$

The nonoscillation of (1.4) follows now from Lemma 2.3. $\square$

In the proof of the next oscillatory counterpart of the previous theorem, in addition to the constant $\tilde\gamma_{n,\alpha}$, three other constants appeared, namely $K_{n,\alpha}$, $\tilde K_{n,\alpha}$ and $L_{n,\alpha}$. To prove the statement of this theorem, we needed the equalities $K_{n,\alpha}=\tilde K_{n,\alpha}$ and $L_{n,\alpha}=\tilde\gamma_{n,\alpha}$. The formulas which defined these constants looked completely different on the first view (compare below given formulas (3.7), (3.8), (3.9)) and the proof of the required equalities leads to interesting combinatorial identites which are presented in the last section.

Theorem 3.2. *Suppose that $q(t)\ge 0$ for large t and*

$$\int^\infty \Big(q(t)-\frac{\tilde\gamma_{n,\alpha}}{t^{2n-\alpha}\lg^2 t}\Big)t^{2n-1-\alpha}\lg t\,dt=\infty. \tag{3.2}$$

Then equation (1.4) *is oscillatory.*

Proof. Let $T\in\mathbb{R}$ be arbitrary, $T<t_0<t_1<t_2<t_3$ (these values will be specified later). Further, let

$$h(t)=t^{\frac{2n-1-\alpha}{2}}\sqrt{\lg t},$$

$f\in C^n[t_0,t_1]$ be any function such that

$$f^{(j)}(t_0)=0,\ f^{(j)}(t_1)=h^{(j)}(t_1),\ j=0,\dots,n-1,$$

and g be the solution of (1.2) satisfying the boundary conditions

$$g^{(j)}(t_2)=h^{(j)}(t_2),\ g^{(j)}(t_3)=0,\ j=0,\dots,n-1. \tag{3.3}$$

We construct a function $0\not\equiv y\in W^{n,2}(T,\infty)$ with compact support in (T,∞), as follows

$$y(t)=\begin{cases}0, & t\le t_0,\\ f(t), & t_0\le t\le t_1,\\ h(t), & t_1\le t\le t_2,\\ g(t), & t_2\le t\le t_3,\\ 0, & t\ge t_3.\end{cases} \tag{3.4}$$

We show that for $t_2,\ t_3$ sufficiently large

$$\mathcal{F}(y;T,\infty):=\int_T^\infty\Big[t^\alpha\big(y^{(n)}(t)\big)^2-\Big(\frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}+q(t)\Big)y^2(t)\Big]dt\le 0$$

and hence (1.4) is oscillatory according to Lemma 2.3. To this end, denote

$$K:=\int_{t_0}^{t_1}\Big[t^\alpha\big(f^{(n)}(t)\big)^2-\Big(\frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}+q(t)\Big)f^2(t)\Big]dt.$$

Concerning the interval $[t_1,t_2]$, let us compute $\big(h^{(n)}\big)^2$. We use the usual convention that the value of a product $\prod_m^n$ equals 1 whenever the lower multiplication limit m

is greater that the upper limit n. Using Lemma 5.2 from the last section we have

$$
\begin{aligned}
&h^{(n)}(t)\\
&= \Big(t^{\frac{2n-1-\alpha}{2}}\sqrt{\lg t}\Big)^{(n)}\\
&= \frac{1}{2^n}\prod_{k=1}^{n}(2k-1-\alpha)t^{\frac{-1-\alpha}{2}}\sqrt{\lg t}\\
&+\sum_{j=1}^{n}\binom{n}{j}\frac{1}{2^{n-j}}\prod_{k=j+1}^{n}(2k-1-\alpha)\frac{(-1)^{j-1}}{t^j}\Big[\frac{a_j}{\sqrt{\lg t}}+\frac{b_j}{\sqrt{\lg^3 t}}+o\big(\lg^{-\frac{3}{2}}t\big)\Big]t^{\frac{2j-1-\alpha}{2}}\\
&= t^{\frac{-1-\alpha}{2}}\Big[\frac{1}{2^n}\prod_{k=1}^{n}(2k-1-\alpha)\sqrt{\lg t}+\frac{A_n}{\sqrt{\lg t}}+\frac{B_n}{\sqrt{\lg^3 t}}+o\big(\lg^{-\frac{3}{2}}t\big)\Big],
\end{aligned}
$$

as $t\to\infty$, where

$$
A_n=\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\frac{a_j}{2^{n-j}}\prod_{k=j+1}^{n}(2k-1-\alpha), \tag{3.5}
$$

$$
B_n=\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\frac{b_j}{2^{n-j}}\prod_{k=j+1}^{n}(2k-1-\alpha). \tag{3.6}
$$

Then we have

$$
\begin{aligned}
\big(h^{(n)}\big)^2 = t^{-1-\alpha}\Big[&\frac{\lg t}{4^n}\prod_{k=1}^{n}(2k-1-\alpha)^2+\frac{A_n}{2^{n-1}}\prod_{k=1}^{n}(2k-1-\alpha)\\
&+\frac{B_n}{2^{n-1}\lg t}\prod_{k=1}^{n}(2k-1-\alpha)+\frac{A_n^2}{\lg t}+2\frac{A_nB_n}{\lg^2 t}+\frac{B_n^2}{\lg^3 t}+O\left(\lg^{-3}t\right)\Big]
\end{aligned}
$$

as $t\to\infty$. If we denote

$$
K_{n,\alpha}:=\frac{A_n}{2^{n-1}}\prod_{k=1}^{n}(2k-1-\alpha), \tag{3.7}
$$

$$
L_{n,\alpha}:=A_n^2+\frac{B_n}{2^{n-1}}\prod_{k=1}^{n}(2k-1-\alpha), \tag{3.8}
$$

we get

$$
\big(h^{(n)}\big)^2=t^{-1-\alpha}\Big[\gamma_{n,\alpha}\lg t+K_{n,\alpha}+\frac{L_{n,\alpha}}{\lg t}+O\left(\lg^{-2}t\right)\Big],
$$

as $t\to\infty$. Consequently,

$$
\int_{t_1}^{t_2}\Big[t^{\alpha}\Big(h^{(n)}(t)\Big)^2-\frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}h^2(t)\Big]dt=K_{n,\alpha}\lg t_2+L_{n,\alpha}\int_{t_1}^{t_2}\frac{dt}{t\lg t}+L_1+o(1),
$$

as $t\to\infty$, where L_1 is a real constant.

Now we turn our attention to the interval $[t_2,t_3]$. Using the assumption that $q(t)\ge 0$ for large t we have

$$
\begin{aligned}
&\int_{t_2}^{t_3}\Big[t^{\alpha}\Big(g^{(n)}(t)\Big)^2-\Big(\frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}+q(t)\Big)g^2(t)\Big]dt\\
&\le\int_{t_2}^{t_3}\Big[t^{\alpha}\Big(g^{(n)}(t)\Big)^2-\frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}g^2(t)\Big]dt.
\end{aligned}
$$

Further, denote

$$x=\begin{pmatrix} g\\ g'\\ \vdots\\ g^{(n-1)}\end{pmatrix},\quad u=\begin{pmatrix}(-1)^{n-1}(t^\alpha g^{(n)})^{(n-1)}\\ \vdots\\ -(t^\alpha g^{(n)})'\\ t^\alpha g^{(n)}\end{pmatrix},\quad \tilde h=\begin{pmatrix} h\\ h'\\ \vdots\\ h^{(n-1)}\end{pmatrix}.$$

Since g is a solution of (1.2), we can use a relationship between this equation and associated LHS (2.5). In this case

$$B(t)=\operatorname{diag}\{0,\dots,0,t^{-\alpha}\},\quad C(t)=\operatorname{diag}\{-\frac{\gamma_{n,\alpha}}{t^{2n-\alpha}},0,\dots,0\},$$
$$A=A_{i,j}=\begin{cases}1, & \text{if } j=i+1,\ i=1,\dots,n-1,\\ 0, & \text{elsewhere.}\end{cases}$$

This relationship, together with conditions (3.3) imply that

$$\begin{aligned}
&\int_{t_2}^{t_3}\Big[t^\alpha\big(g^{(n)}\big)^2(t)-\frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}g^2(t)\Big]dt\\
&=\int_{t_2}^{t_3}[u^T(t)B(t)u(t)+x^T(t)C(t)x(t)]dt\\
&=\int_{t_2}^{t_3}[u^T(t)(x'(t)-Ax(t))+x^T(t)C(t)x(t)]dt\\
&=u^T(t)x(t)\big|_{t_2}^{t_3}+\int_{t_2}^{t_3}x^T(t)[-u'(t)-A^Tu(t)+C(t)x(t)]dt\\
&=-u^T(t_2)x(t_2).
\end{aligned}$$

Let (X,U) be the principal solution of the LHS associated with (1.2).Then $(\bar X,\bar U)$ defined by

$$\bar X(t)=X(t)\int_t^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds,$$
$$\bar U(t)=U(t)\int_t^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds-X^{T-1}(t)$$

is also a conjoined basis of this LHS, and according to (3.3) we get

$$\begin{aligned}
x(t)&=X(t)\int_t^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds\Big(\int_{t_2}^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds\Big)^{-1}\\
&\quad\times X^{-1}(t_2)\tilde h(t_2),\\
u(t)&=\Big(U(t)\int_t^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds-X^{T-1}(t)\Big)\\
&\quad\times\Big(\int_{t_2}^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds\Big)^{-1}X^{-1}(t_2)\tilde h(t_2)
\end{aligned}$$

and hence

$$\begin{aligned}
-u^T(t_2)x(t_2)&=\tilde h^T(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds\right)^{-1}X^{-1}(t_2)\tilde h(t_2)\\
&\quad-\tilde h^T(t_2)U(t_2)X^{-1}(t_2)\tilde h(t_2).
\end{aligned}$$

Using the fact that the principal solution of LHS associated with (1.2) is generated by $y_1(t) = t^{\alpha_1},\ y_2(t) = t^{\alpha_2}, \dots, y_{n-1}(t) = t^{\alpha_{n-1}},\ y_n(t) = t^{\alpha_0} = t^{\frac{2n-1-\alpha}{2}}$, where α_k, $k = 0, \dots, n-1$, are the roots of (2.3), we have

$$X(t) = \begin{pmatrix} t^{\alpha_1} & \dots & t^{\alpha_{n-1}} & t^{\frac{2n-1-\alpha}{2}} \\ \alpha_1 t^{\alpha_1 - 1} & \dots & \alpha_{n-1} t^{\alpha_{n-1}-1} & \frac{2n-1-\alpha}{2} t^{\frac{2n-3-\alpha}{2}} \\ \vdots & & & \vdots \\ \prod_{k=0}^{n-2} (\alpha_1 - k) t^{\alpha_1 - n + 1} & \dots & \prod_{k=0}^{n-2} (\alpha_{n-1} - k) t^{\alpha_{n-1} - n+1} & \frac{1}{2^{n-1}} \prod_{k=1}^{n} (2k-1-\alpha) t^{\frac{1-\alpha}{2}} \end{pmatrix}$$

$$U(t) = \begin{pmatrix} \ell_{n,\alpha}^{[1,1]} t^{\alpha_1 + \alpha - 2n+1} & \dots & \ell_{n,\alpha}^{[1,n]} t^{\frac{\alpha - 2n+1}{2}} \\ \vdots & & \vdots \\ \ell_{n,\alpha}^{[n,1]} t^{\alpha_1 + \alpha - n} & \dots & \ell_{n,\alpha}^{[n,n]} t^{\frac{\alpha-1}{2}} \end{pmatrix},$$

where

$$\begin{aligned} \ell_{n,\alpha}^{[1,1]} &= (-1)^{n-1} \prod_{j=0}^{n-1} (\alpha_1 - j) \prod_{j=0}^{n-2} (\alpha_1 + \alpha - n - j), \\ \ell_{n,\alpha}^{[1,n]} &= \frac{2n-1-\alpha}{2} \prod_{j=0}^{n-1} \Big(\frac{2n-1-2j-\alpha}{2} \Big)^2, \\ \ell_{n,\alpha}^{[n,1]} &= \prod_{j=0}^{n-1} (\alpha_1 - j), \\ \ell_{n,\alpha}^{[n,n]} &= \prod_{j=0}^{n-1} \Big(\frac{2n-1-2j-\alpha}{2} \Big), \end{aligned}$$

and

$$\tilde h(t) = \begin{pmatrix} t^{\frac{2n-1-\alpha}{2}} \sqrt{\lg t} \\ t^{\frac{2n-3-\alpha}{2}} \Big(\frac{2n-1-\alpha}{2} \sqrt{\lg t} + O(\lg^{-\frac12}(t)) \Big) \\ \vdots \\ t^{\frac{1-\alpha}{2}} \Big(\frac{1}{2^{n-1}} \prod_{k=1}^n (2k-1-\alpha) \sqrt{\lg t} + O(\lg^{-\frac12} t) \Big) \end{pmatrix}, \quad \text{as } t \to \infty$$

Next we compute the asymptotic formula for $(U^T \tilde h)^T X^{-1} \tilde h \big|_{t=t_2}$. Using Lemma 5.4 we have

$$\begin{gathered} (X^{-1} \tilde h)_i \sim t^{\alpha_i - \frac{2n-1-\alpha}{2}} \frac{1}{\sqrt{\lg t}}, \quad i = 1, \dots, n-1, \\ (X^{-1} \tilde h)_n = \sqrt{\lg t} \left(1 + O(\lg^{-1} t)\right), \quad \text{as } t \to \infty. \end{gathered}$$

Here $f_1 \sim f_2$ for a pair of functions f_1, f_2 means that $\lim_{t \to \infty} \frac{f_1(t)}{f_2(t)} = L$ exists and $0 < L < \infty$. By a direct computation, for $i = 1, \dots, n-1$,

$$(U^T \tilde h)_i = l_{n,\alpha}^{[i]} t^{\alpha_i - \frac{2n-1-\alpha}{2}} \sqrt{\lg t},$$

the constants $l_{n,\alpha}^{[i]}$ can be computed explicitly, but their values are not important. As for $(U^T \tilde h)_n$ denote by u_n the last column of U. Then, again by a direct computation,

$$(U^T \tilde h)_n = u_n^T \tilde h = \tilde K_{n,\alpha} \sqrt{\lg t} (1 + O(\lg^{-1} t)), \quad \text{as } t \to \infty,$$

where

$$\tilde{K}_{n,\alpha} = \frac{2}{4^n} \prod_{k=1}^{n} (2k-1-\alpha)^2 \sum_{k=1}^{n} \frac{1}{(2k-1-\alpha)}. \tag{3.9}$$

Consequently,

$$\tilde{h}^T(t_2)U(t_2)X^{-1}(t_2)\tilde{h}(t_2) = \tilde{K}_{n,\alpha} \lg t_2 + L_2 + o(1), \quad \text{as } t_2 \to \infty,$$

where L_2 is a real constant.

Summarizing all the above computations

$$\begin{aligned}\mathcal{F}(y;T,\infty) \le K + K_{n,\alpha} \lg t_2 + L_{n,\alpha} \int_{t_1}^{t_2} \frac{dt}{t \lg t} + L_1 + o(1) - \int_{t_1}^{t_2} q(t)h^2(t)dt \\ + \tilde{h}^T(t_2)X^{T-1}(t_2) \left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)ds \right)^{-1} X^{-1}(t_2)\tilde{h}(t_2) \\ - \tilde{K}_{n,\alpha} \lg t_2 - L_2 - o(1), \quad \text{as } t_2 \to \infty.\end{aligned}$$

It follows from Lemma 5.8 that $K_{n,\alpha} = \tilde{K}_{n,\alpha}$ and $L_{n,\alpha} = \tilde{\gamma}_{n,\alpha}$ according to Lemma 5.9. Using (3.2) let $t_2 > t_1$ be such that

$$\begin{aligned}L_{n,\alpha} \int_{t_1}^{t_2} \frac{dt}{t \lg t} - \int_{t_1}^{t_2} q(t)h^2(t)dt = - \int_{t_1}^{t_2} \left(q(t) - \frac{\tilde{\gamma}_{n,\alpha}}{t^{2n-\alpha} \lg^2 t} \right) t^{2n-1-\alpha} \lg t dt \\ \le -(K + L_1 - L_2 + 2).\end{aligned}$$

Since (X, U) is the principal solution, it is possible to choose $t_3 > t_2$ such that

$$\tilde{h}^T(t_2)X^{T-1}(t_2) \left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)ds \right)^{-1} X^{-1}(t_2)\tilde{h}(t_2) \le 1.$$

Finally, if t_2 is so large that the sum of all the terms $o(1)$ is less then 1, then for these t_2, t_3 we have

$$\mathcal{F}(y;T,\infty) \le K - (K + L_1 - L_2 + 2) + L_1 + 1 + 1 - L_2 = 0,$$

which means that (1.4) is oscillatory. □

Corollary 3.3. *The equation* (1.8) *is nonoscillatory if and only if* $\gamma \le \tilde{\gamma}_{n,\alpha}$.

Proof. If $\gamma \le \tilde{\gamma}_{n,\alpha}$, then the second order equation

$$(tz')' + \frac{1}{4\tilde{\gamma}_{n,\alpha}} t^{2n-1-\alpha} \frac{\gamma}{t^{2n-\alpha} \lg^2 t} z = 0$$

is nonoscillatory, which follows from the fact, that the equation

$$(tz')' + \frac{\mu}{t \lg^2 t} z = 0$$

is nonoscillatory for $\mu \le \frac{1}{4}$. Hence, (1.8) is nonoscillatory according to Theorem 3.1. Conversely, if $\gamma > \tilde{\gamma}_{n,\alpha}$, then for $q(t) = \frac{\gamma}{t^{2n-\alpha} \lg^2 t}$ condition (3.2) holds and we have oscillation of (1.8) using Theorem 3.2. □

4. Remarks and open problems

(i) The oscillation criterion given in Theorem 3.2 is proved under the assumption

$$q(t) \geq 0 \quad \text{for large } t. \tag{4.1}$$

This restriction has been successfully removed in oscillation criteria presented in some recent papers [3, 4, 5, 13, 16]. In particular, it was proved for equation (1.1) with $n = 2$ that the function g/h (the function h, g apper in (3.4)) is monotonically decreasing and this fact enabled to remove the assumption (4.1) via the second mean value theorem of integral calculus, see [4]. The computations proving monotonicity of g/h are rather complex even for $n = 2$ and we have not been able to prove this monotonicity in the general case yet. However, we believe that the function g/h is monotonic also in the general case treated in our paper and we conjecture that Theorem 3.2 remains valid without assumption (4.1).

(ii) In [6] we have discussed the problem of the value of the best constants in oscillation and nonoscillation criteria for equations of the form (1.1) and (1.4). In particular, it is known (see [6, 9]) that equation (1.4) is oscillatory if

$$M := \lim_{t\to\infty} \lg t \int_t^\infty q(s)s^{2n-1-\alpha}\, ds > \omega_{n,\alpha} \tag{4.2}$$

and it is nonoscillatory if the above limit is less than $\omega_{n,\alpha}/4$, where

$$\omega_{n,\alpha} = \frac{(-1)^n \prod_{i=0}^{n-1}(\lambda - i)(\lambda - n + \alpha - i) - \gamma_{n,\alpha}}{\left(\lambda - \frac{2n-1-\alpha}{2}\right)^2}\Big|_{\lambda = \frac{2n-1-\alpha}{2}}. \tag{4.3}$$

An open problem remained what is the oscillatory nature of (1.4) if the limit in (4.2) is between $\omega_{n,\alpha}/4$ and $\omega_{n,\alpha}$. Here we answer this question by showing that the "right" oscillation constant is $\omega_{n,\alpha}/4$, i.e. (1.4) is oscillatory if the limit in (4.2) is greater than this constant.

Observe that $\omega_{n,\alpha}/4 = \tilde\gamma_{n,\alpha}$, this identity is proved in Lemma 5.5 of the last section. If $q(t) = \frac{\lambda}{t^{2n-\alpha}\lg^2 t}$, the next statement is in the full agreement with Corollary 3.3.

Theorem 4.1. *Suppose that* (4.1) *holds. Equation* (1.4) *is oscillatory if the limit* M *in* (4.2) *is greater than* $\tilde\gamma_{n,\alpha}$ *and it is nonoscillatory if it is less than this constant.*

Proof. If $M > \omega_{n,\alpha}$ in (4.2), equation (1.4) is oscillatory by [9, Theorem 4.1]. Hence we suppose that $\tilde\gamma_{n,\alpha} < M \leq \omega_{n,\alpha} = 4\tilde\gamma_{n,\alpha}$. In this case we use Theorem 3.2. Since $M > \tilde\gamma_{n,\alpha}$, there exist $\varepsilon > 0$ and $T \in \mathbb{R}$ such that

$$\int_t^\infty q(s)s^{2n-1-\alpha}\, ds > \frac{\tilde\gamma_{n,\alpha} + \varepsilon}{\lg t} \quad \text{for } t \geq T,$$

and hence, multiplying the last inequality by $\frac{1}{t}$ and integrating it from T to b we get

$$\int_T^b \frac{1}{t}\int_t^\infty q(s)s^{2n-1-\alpha}\, ds > (\tilde\gamma_{n,\alpha} + \varepsilon) \lg\left(\frac{\lg b}{\lg T}\right)$$

for $b > T$. Integration by parts yields

$$\begin{aligned}
&\int_T^b \Big(q(s) - \frac{\tilde\gamma_{n,\alpha}}{t^{2n-\alpha}\lg^2 t}\Big)t^{2n-1-\alpha}\lg t\,dt\\
&= \int_T^b q(t)t^{2n-1-\alpha}\lg t\,dt - \tilde\gamma_{n,\alpha}\lg\big(\frac{\lg b}{\lg T}\big)\\
&= -\lg t\int_t^\infty q(s)s^{2n-1-\alpha}\,ds\Big|_T^b + \int_T^b \frac{1}{t}\Big(\int_t^\infty q(s)s^{2n-1-\alpha}\,ds\Big)dt - \tilde\gamma_{n,\alpha}\lg\big(\frac{\lg b}{\lg T}\big)\\
&> -\lg t\int_t^\infty q(s)s^{2n-1-\alpha}\,ds\Big|_T^b + (\tilde\gamma_{n,\alpha} + \varepsilon - \tilde\gamma_{n,\alpha})\lg\big(\frac{\lg b}{\lg T}\big) \to \infty
\end{aligned}$$

as $b \to \infty$ since the first term in the last line of the previous computation is bounded as $t \to \infty$. Hence, by Theorem 3.2 equation (1.4) is oscillatory. □

Note that assumption (4.1) in the oscillatory part of Theorem 4.1 can be removed if the conjecture formulated in the previous remark turns out to be true.

(iii) Let L be a formally self-adjoint differential operator given by (2.1) and consider the equation

$$L(y) = \lambda w(t)y, \tag{4.4}$$

where w is a positive continuous function. This equation is said to be *conditionally oscillatory* if there exists a constant λ_0, the so called *oscillation constant*, such that (4.4) is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. If we put now

$$L(y) := (-1)^n\big(t^\alpha y^{(n)}\big)^{(n)} - \left(\frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} + \frac{\tilde\gamma_{n,\alpha}}{t^{2n-\alpha}\lg^2 t}\right)y \tag{4.5}$$

a natural question is for what fuction w equation (4.4) with L given by (4.5) is conditionally oscillatory equation. Theorem 3.1 and oscillatory behavior of the second order equation (3.1) lead to the conjecture (whose proof is a subject of the present investigation) that this term is

$$w(t) = \frac{1}{t^{2n-\alpha}\lg^2 t\lg^2(\lg t)}$$

and that the oscillation constant is $\lambda_0 = \tilde\gamma_{n,\alpha}$.

(iv) The previous remark, again together with Theorem 3.1, lead to the following conjecture.

Conjecture 4.2. *Let*

$$K := \lim_{t\to\infty}\lg(\lg t)\int_t^\infty \Big(q(s) - \frac{\tilde\gamma_{n,\alpha}}{s^{2n-\alpha}\lg^2 s}\Big)s^{2n-\alpha-1}\lg s\,ds.$$

There exists a constant $\hat\gamma$ (presumably $\hat\gamma = \tilde\gamma_{n,\alpha}$) such that (1.4) is oscillatory provided $K > \hat\gamma$.

Note that the nonoscillatory complement of the previous conjecture is true by Theorem 3.1. Indeed, the second order equation (3.1), when written in the form

$$(tu')' + \frac{1}{4t\lg^2 t}u + \Big(\frac{t^{2n-1-\alpha}q(t)}{4\tilde\gamma_{n,\alpha}} - \frac{1}{4t\lg^2 t}\Big)u = 0 \tag{4.6}$$

is nonoscillatory, provided

$$\frac{t^{2n-1-\alpha}q(t)}{4\tilde\gamma_{n,\alpha}} - \frac{1}{4t\lg^2 t} \ge 0$$

for large t and

$$\lim_{t\to\infty} \lg(\lg t)\int_t^\infty \Big(\frac{s^{2n-1-\alpha}q(s)}{4\tilde\gamma_{n,\alpha}}-\frac{1}{4s\lg^2 s}\Big)\lg s\,ds<\frac14 \tag{4.7}$$

and the last conditions just the condtion $K<\tilde\gamma_{n,\alpha}$. Condition (4.7) follows from the Hille nonoscillation criterion which states that the second order differential equation

$$(r(t)x')'+c(t)x=0$$

with $c(t)\ge 0$, $\int^\infty c(t)\,dt<\infty$ and $\int^\infty r^{-1}(t)\,dt=\infty$ is nonoscillatory provided

$$\lim_{t\to\infty}\Big(\int^t r^{-1}(s)\,ds\Big)\Big(\int_t^\infty c(s)\,ds\Big)<\frac14.$$

The transformation $u=\sqrt{\lg t}\,v$ transforms (4.6) into the equation

$$(t\lg t v')'+\Big(\frac{t^{2n-1-\alpha}q(s)}{4\tilde\gamma_{n,\alpha}}-\frac{1}{4t\lg^2 t}\Big)\lg t\,v=0$$

and Hille's criterion applied to this equation gives (4.7).

(v) Throughout the paper we consider the case $\alpha\notin\{1,3,\dots,2n-1\}$ only. The reason is that for $\alpha\in\{1,3,\dots,2n-1\}$ the Euler equation

$$(-1)^n\big(t^\alpha y^{(n)}\big)^{(n)}-\frac{\lambda}{t^{2n-\alpha}}y=0 \tag{4.8}$$

is no longer conditionally oscillatory and one has to consider the equation

$$(-1)^n\big(t^\alpha y^{(n)}\big)^{(n)}-\frac{\lambda}{t^{2n-\alpha}\lg^2 t}y=0. \tag{4.9}$$

According to [6] and [10], equation (4.9) is oscillatory for the values $\lambda>\nu_{n,m}:=[m!(n-m-1)!]^2/4$ and $m:=(2n-1-\alpha)/2$, and nonoscillatory in the opposite case. In the proof of Theorem 3.2 we have defined the function g as the solution of (1.2) (satisfying certain boundary conditions) and we have used the fact that we know solutions (even if with generally unknown exponents) of (1.2). Concerning equation (4.9), we do not know solutions explicitly even for $n=2$ and $\lambda=\nu_{2,m}$, so we cannot apply directly the method used in the proof of Theorem 3.2. Nevertheless, we conjecture that the equation

$$(-1)^n\big(t^\alpha y^{(n)}\big)^{(n)}-\frac{\nu_{n,m}}{t^{2n-\alpha}\lg^2 t}y=q(t)y,\quad \alpha\in\{1,3,\dots,2n-1\}, \tag{4.10}$$

is oscillatory provided $q(t)\ge 0$ for large t and

$$\lim_{t\to\infty}\lg(\lg t)\int_t^\infty q(s)s^{2n-1-\alpha}\lg s\,ds>\nu_{n,m}. \tag{4.11}$$

Note that by [10, Theorem 3.1] equation (4.10) is nonoscillatory provided the second order equation

$$(tu')'+\frac{t^{2m}}{4\nu_{n,m}}\Big(q(t)+\frac{\nu_{n,m}}{t^{2n-\alpha}\lg^2 t}\Big)u=0 \tag{4.12}$$

is nonoscillatory. The application of the nonoscillation criterion (4.7) with $\tilde\gamma_{n,\alpha}$ replaced by $\nu_{n,m}$ to this equation gives nonoscillation of (4.12) if the limit in (4.11) is less than $\nu_{n,m}$.

5. Technical results

In this section we present some technical lemmata needed in the proofs of our main results.

Lemma 5.1. *Let α_k, $k = 1, \dots, n-1$, be the first $n-1$ roots (ordered by size) of the polynomial* (2.3) *and $\alpha_0 = \frac{2n-1-\alpha}{2}$. Then*

$$4\tilde{\gamma}_{n,\alpha} = \prod_{k=1}^{n-1} \Big(\frac{2n-1-\alpha}{2} - \alpha_k\Big)^2,$$

where $\tilde{\gamma}_{n,\alpha}$ is given by (1.9).

Proof. Denote $\beta_k := \frac{2n-1-\alpha}{2} - \alpha_k$, $k = 1, \dots, n-1$. Then, since the roots of (2.3) are α_k, $2n-1-\alpha-\alpha_k$, $k = 1, \dots, n-1$, $\alpha_0 = (2n-1-\alpha)/2$ (α_0 is the double root), we can write them in the form $\alpha_k = \frac{2n-1-\alpha}{2} - \beta_k$, $2n-1-\alpha-\alpha_k = \frac{2n-1-\alpha}{2} + \beta_k$. The substitution $\mu = \frac{2n-1-\alpha}{2} - \lambda$ converts the polynomial

$$P(\lambda) := (-1)^n \prod_{i=0}^{n-1} (\lambda - i)(\lambda - n + \alpha - i) - \gamma_{n,\alpha}.$$

into the polynomial

$$\begin{aligned}\tilde{P}(\mu) &= (-1)^n \prod_{i=1}^{n} \Big(\frac{2i-1-\alpha}{2} - \mu\Big)\Big(\frac{-2i+1+\alpha}{2} - \mu\Big) - \gamma_{n,\alpha} \\ &= (-1)^n \prod_{i=1}^{n} \Big[\mu^2 - \Big(\frac{2i-1-\alpha}{2}\Big)^2\Big] - \gamma_{n,\alpha}.\end{aligned}$$

The coefficient by μ^2 in $(-1)^n \tilde{P}(\mu)$ is

$$\begin{aligned}&\frac{1}{4^{n-1}} \big[(2n-3-\alpha)^2(2n-5-\alpha)^2 \cdots (1-\alpha)^2 \\ &+ (2n-1-\alpha)^2(2n-5-\alpha)^2 \dots (1-\alpha)^2 + \cdots \\ &+ (2n-1-\alpha)^2(2n-3-\alpha)^2 \cdots (3-\alpha)^2\big] \\ &= \frac{1}{4^{n-1}} \prod_{k=1}^{n} (2k-1-\alpha)^2 \sum_{k=1}^{n} \frac{1}{(2k-1-\alpha)^2} = 4\tilde{\gamma}_{n,\alpha}.\end{aligned}$$

On the other hand, according to the above substitution, since the roots of $\tilde{P}(\mu)$ are $\pm\beta_k$, $k = 1, \dots, n-1$, $\beta_0 = 0$ (double root), it is possible to express $(-1)^n \tilde{P}(\mu)$ in the form

$$(-1)^n \tilde{P}(\mu) = \mu^2(\mu^2 - \beta_1^2)(\mu^2 - \beta_2^2) \cdots (\mu^2 - \beta_{n-1}^2)$$

and the coefficient by μ^2 in $\tilde{P}(\mu)$ is $\prod_{k=1}^{n-1} \beta_k^2$. Comparing the both expressions by β^2 we have the result of this lemma. $\square$

Lemma 5.2. *For arbitrary $j \in \mathbb{N}$*

$$\big(\sqrt{\lg t}\big)^{(j)} = \frac{(-1)^{j-1}}{t^j} \Big(\frac{a_j}{\sqrt{\lg t}} + \frac{b_j}{\sqrt{\lg^3 t}} + o\big(\lg^{-\frac{3}{2}} t\big)\Big),$$

where a_j, b_j are given by recursion

$$a_1 = \frac{1}{2},\ a_{k+1} = k a_k; \quad b_1 = 0,\ b_{k+1} = k b_k + \frac{a_k}{2}. \tag{5.1}$$

Proof. If $j=1$, then $\big(\sqrt{\lg t}\big)'=\frac{1}{2t\sqrt{\lg t}}$, and hence $a_1=\frac12$, $b_1=0$. By induction

$$\begin{aligned}
\big(\sqrt{\lg t}\big)^{(k+1)} &= \Big[\frac{(-1)^{k-1}}{t^k}\Big(\frac{a_k}{\sqrt{\lg t}}+\frac{b_k}{\sqrt{\lg^3 t}}+o\big(\lg^{-\frac32} t\big)\Big)\Big]' \\
&= \frac{(-1)^k k}{t^{k+1}}\Big(\frac{a_k}{\sqrt{\lg t}}+\frac{b_k}{\sqrt{\lg^3 t}}+o\big(\lg^{-\frac32} t\big)\Big) \\
&\quad + \frac{(-1)^k}{t^{k+1}}\Big(\frac{a_k}{2\sqrt{\lg^3 t}}+\frac{3b_k}{2\sqrt{\lg^5 t}}+o\big(\lg^{-\frac32} t\big)\Big) \\
&= \frac{(-1)^k}{t^{k+1}}\Big(\frac{ka_k}{\sqrt{\lg t}}+\frac{kb_k+a_k/2}{\sqrt{\lg^3 t}}+o\big(\lg^{-\frac32} t\big)\Big).
\end{aligned}$$

□

Remark 5.3. Using (5.1) we have $a_{n+1}=na_n$, which implies

$$a_n=a_1\prod_{j=1}^{n-1} j=\frac12(n-1)!$$

and $b_{n+1}=nb_n+\frac12 a_n=nb_n+\frac14(n-1)!$. Solving the last difference equation using variation of parameters method, we obtain

$$b_n=\frac{(n-1)!}{4}\sum_{j=1}^{n-1}\frac1j,\ n\ge 2.$$

The next lemma presents basic rules for computation of Wronskians.

Lemma 5.4. *Let $W(f_1,\dots,f_n)$ denote the Wronskian of the functions in brackets. Then the following statements hold.*

(i) *We have (with a function r) $W(rf_1,\dots,rf_n)=r^nW(f_1,\dots,f_n)$. In particular, if $f_i(t)\neq 0$ for some $i\in\{1,\dots,n\}$, then*

$$W(f_1,\dots,f_n)=(-1)^{i-1}f_i^nW\Big(\big(\frac{f_1}{f_i}\big)',\dots,\big(\frac{f_{i-1}}{f_i}\big)',\big(\frac{f_{i+1}}{f_i}\big)',\dots,\big(\frac{f_n}{f_i}\big)'\Big).$$

(ii) *Let $f_1=t^{\beta_1},\dots,f_n=t^{\beta_n}$, then*

$$W(f_1,\dots,f_n)=t^{\sum_{i=1}^n\beta_i-\frac{n(n-1)}{2}}\prod_{1\le j<i\le n}(\beta_i-\beta_j)$$

(iii) *Let X be the Wronskian matrix of the functions $f_1,\dots,f_n$, and let $\tilde h=(h,h',\dots,h^{(n-1)})^T$. Then*

$$(X^{-1}\tilde h)_i=\frac{W(f_1,\dots,f_{i-1},h,f_{i+1},\dots,f_n)}{W(f_1,\dots,f_n)}.$$

(iv) *If f_i, $i=1,\dots,n$, are the same as in (ii) with $\beta_i\neq\beta_j$, $i\neq j$, X is their Wronskian matrix and $h(t)=t^{\beta_n}\sqrt{\lg t}$, then*

$$\begin{gathered}
(X^{-1}\tilde h)_i\sim t^{\beta_i-\beta_n}\frac{1}{\sqrt{\lg t}},\quad i=1,\dots,n-1,\\
(X^{-1}\tilde h)_n=\sqrt{\lg t}(1+O(\lg^{-1} t)),
\end{gathered}$$

here $f_1\sim f_2$ for a pair of functions f_1,f_2 means that $\lim_{t\to\infty}\frac{f_1(t)}{f_2(t)}=L$ exists and $0<L<\infty$.

Proof. The statements (i) and (iii) are proved in [1, Chap. III]. The statement (ii) can be found e.g. in [8] and the claim (iv) can be proved by a direct computation using the rules (i)–(iii). □

Lemma 5.5. *Let $\omega_{n,\alpha}$ and $\tilde{\gamma}_{n,\alpha}$ be given by* (4.3) *and* (1.9)*, respectively. Then* $\omega_{n,\alpha}/4 = \tilde{\gamma}_{n,\alpha}$

Proof. Using the substitution $\mu = \lambda - \frac{2n-1-\alpha}{2}$ and then $i = n-1-j$ in the product formula for $\omega_{n,\alpha}$, we have

$$\begin{aligned}
\omega_{n,\alpha} &= \frac{(-1)^n \prod_{i=0}^{n-1}(\lambda - i)(\lambda - n + \alpha - i) - \gamma_{n,\alpha}}{\left(\lambda - \frac{2n-1-\alpha}{2}\right)^2}\Big|_{\lambda = \frac{2n-1-\alpha}{2}} \\
&= \frac{(-1)^n \prod_{j=0}^{n-1}\left(\mu + \frac{2n-1-\alpha-2j}{2}\right)\left(\mu - \frac{2n-1-\alpha-2j}{2}\right) - \gamma_{n,\alpha}}{\mu^2}\Big|_{\mu=0} \\
&= \frac{(-1)^n \prod_{j=0}^{n-1}\left(\mu^2 - \frac{(2n-1-\alpha-2j)^2}{4}\right) - \gamma_{n,\alpha}}{\mu^2}\Big|_{\mu=0} \\
&= 4\prod_{i=0}^{n-1}\Big(\frac{2n-1-\alpha-2i}{2}\Big)^2\Big\{\frac{1}{(2n-1-\alpha)^2} + \dots + \frac{1}{(1-\alpha)^2}\Big\} = 4\tilde{\gamma}_{n,\alpha}.
\end{aligned}$$

□

The proofs of the next two combinatorial results can be found in [15]

Lemma 5.6. *Let $x,\ z \in \mathbb{R}$ be arbitrary, $n \in \mathbb{N}$. Then*

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j+x}{j}^{-1} = \frac{x}{n+x}, \tag{5.2}$$

$$\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\binom{j+x}{j}^{-1}\Big(\sum_{i=1}^{j}\frac{1}{i+x}\Big) = \frac{n}{(x+n)^2}, \tag{5.3}$$

$$\begin{aligned}
&\sum_{j=1}^{n}\binom{n}{j}\binom{z}{j}\binom{j+x}{j}^{-1}\Big(\sum_{i=1}^{j}\frac{1}{i+x}\Big) \\
&= \binom{x+z+n}{n}\binom{x+n}{n}^{-1}\Big[\sum_{i=1}^{n}\frac{1}{i+x} - \sum_{i=1}^{n}\frac{1}{i+x+z}\Big].
\end{aligned} \tag{5.4}$$

Lemma 5.7. *Let $n \in \mathbb{N}$ and $\{F_n\}$, $\{f_n\}$ be sequences such that $F_n = \sum_{j=1}^{n}\binom{n}{j}f_j$. Then*

$$f_n = (-1)^n\sum_{j=1}^{n}(-1)^j\binom{n}{j}F_j.$$

Lemma 5.8. *Let $n \in \mathbb{N}$. Then*

$$\frac{A_n}{2^{n-1}}\prod_{k=1}^{n}(2k-1-\alpha) = \frac{2}{4^n}\prod_{k=1}^{n}(2k-1-\alpha)^2\sum_{k=1}^{n}\frac{1}{(2k-1-\alpha)},$$

where A_n is given by (3.5)*.*

Proof. Using (3.5) and Remark 5.3 we get

$$
\begin{aligned}
A_n &= \sum_{j=1}^{n}\Big[(-1)^{j-1}\binom{n}{j}\frac{(j-1)!}{2^{n-j+1}}\prod_{k=j+1}^{n}(2k-1-\alpha)\Big]\\
&= \frac{1}{2^n}\prod_{j=1}^{n}(2j-1-\alpha)\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\frac{2^{j-1}(j-1)!}{\prod_{k=1}^{j}(2k-1-\alpha)}\\
&= \frac{1}{2^n}\prod_{j=1}^{n}(2j-1-\alpha)\sum_{j=0}^{n-1}(-1)^{j}\binom{n}{j+1}\frac{2^{j}j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)}.
\end{aligned}
$$

We have to show that

$$
A_n = \frac{1}{2^n}\prod_{k=1}^{n}(2k-1-\alpha)\sum_{k=1}^{n}\frac{1}{2k-1-\alpha}, \tag{5.5}
$$

i.e.

$$
\sum_{j=0}^{n-1}(-1)^{j}\binom{n}{j+1}\frac{2^{j}j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)} = \sum_{k=1}^{n}\frac{1}{2k-1-\alpha}.
$$

Denote C_n the expression on the left-hand side and $\tilde C_n$ the expression on the right-hand side of the last formula. If $n=2$, then $C_2 = \frac{2(2-\alpha)}{(1-\alpha)(3-\alpha)} = \tilde C_2$. By induction, it remains to show that $\Delta C_n = \Delta \tilde C_n$, where

$$
\begin{aligned}
\Delta C_n &= C_{n+1} - C_n\\
&= \sum_{j=0}^{n}(-1)^{j}\Big[\binom{n+1}{j+1}-\binom{n}{j+1}\Big]\frac{2^{j}j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)}\\
&= \sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\frac{2^{j}j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)},\\
\Delta\tilde C_n &= \tilde C_{n+1} - \tilde C_n = \frac{1}{2n+1-\alpha}.
\end{aligned}
$$

Substituting $x = (1-\alpha)/2$ into (5.2) we obtain

$$
\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\binom{j+\frac{1-\alpha}{2}}{j}^{-1} = \frac{1-\alpha}{2n+1-\alpha}
$$

and since

$$
\binom{j+\frac{1-\alpha}{2}}{j} = \frac{\prod_{k=2}^{j+1}(2k-1-\alpha)}{2^{j}j!}, \tag{5.6}
$$

we have $\Delta C_n = \Delta\tilde C_n$. $\square$

Lemma 5.9. *Let A_n, B_n be given by (3.5), (3.6) respectively. Then for arbitrary $n\in\mathbb{N}$*

$$
\tilde\gamma_{n,\alpha} = A_n^2 + \frac{B_n}{2^{n-1}}\prod_{k=1}^{n}(2k-1-\alpha).
$$

Proof. Similarly as in the proof of Lemma 5.8, using Remark 5.3 we can express B_n in the following form

$$\begin{aligned}
B_n &= \sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\frac{b_j}{2^{n-j}}\frac{\prod_{k=1}^{n}(2k-1-\alpha)}{\prod_{k=1}^{j}(2k-1-\alpha)} \\
&= \frac{\prod_{k=1}^{n}(2k-1-\alpha)}{2^n}\sum_{j=2}^{n}(-1)^{j-1}\binom{n}{j}\frac{2^j b_j}{\prod_{k=1}^{j}(2k-1-\alpha)} \\
&= \frac{\prod_{k=1}^{n}(2k-1-\alpha)}{2^n}\sum_{j=2}^{n}(-1)^{j-1}\binom{n}{j}\frac{2^j (j-1)!}{4\prod_{k=1}^{j}(2k-1-\alpha)}\Big(\sum_{i=1}^{j-1}\frac{1}{i}\Big) \\
&= \frac{\prod_{k=1}^{n}(2k-1-\alpha)}{2^n}\sum_{j=1}^{n-1}(-1)^{j}\binom{n}{j+1}\frac{2^{j-1} j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)}\Big(\sum_{i=1}^{j}\frac{1}{i}\Big).
\end{aligned}$$

Using this formula and (5.5), we obtain

$$\begin{aligned}
A_n^2 + \frac{B_n}{2^{n-1}}\prod_{k=1}^{n}(2k-1-\alpha) = \frac{\prod_{j=1}^{n}(2j-1-\alpha)^2}{4^n}\Big[\Big(\sum_{k=1}^{n}\frac{1}{2k-1-\alpha}\Big)^2 \\
+ \sum_{j=1}^{n-1}(-1)^{j}\binom{n}{j+1}\frac{2^{j} j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)}\Big(\sum_{i=1}^{j}\frac{1}{i}\Big)\Big].
\end{aligned}$$

¿From the definition of $\tilde{\gamma}_{n,\alpha}$, it suffices to show that $D_n = \tilde{D}_n$, where we have denoted

$$\begin{aligned}
D_n &= \sum_{j=1}^{n-1}(-1)^{j}\binom{n}{j+1}\frac{2^{j} j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)}\Big(\sum_{i=1}^{j}\frac{1}{i}\Big), \\
\tilde{D}_n &= \sum_{k=1}^{n}\frac{1}{(2k-1-\alpha)^2} - \Big(\sum_{k=1}^{n}\frac{1}{2k-1-\alpha}\Big)^2.
\end{aligned}$$

One can see that $D_2 = -\frac{2}{(3-\alpha)(1-\alpha)} = \tilde{D}_2$ and we verify equality $\Delta D_n = \Delta\tilde{D}_n$. We have

$$\begin{aligned}
\Delta D_n &= \sum_{j=1}^{n}(-1)^{j}\binom{n+1}{j+1}\frac{2^{j} j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)}\Big(\sum_{i=1}^{j}\frac{1}{i}\Big) \\
&\quad - \sum_{j=1}^{n-1}(-1)^{j}\binom{n}{j+1}\frac{2^{j} j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)}\Big(\sum_{i=1}^{j}\frac{1}{i}\Big) \\
&= \sum_{j=1}^{n-1}(-1)^{j}\binom{n}{j}\frac{2^{j} j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)}\Big(\sum_{i=1}^{j}\frac{1}{i}\Big) \\
&\quad + (-1)^n\frac{2^n n!}{\prod_{k=1}^{n+1}(2k-1-\alpha)}\Big(\sum_{i=1}^{n}\frac{1}{i}\Big) \\
&= \sum_{j=1}^{n}(-1)^{j}\binom{n}{j}\frac{2^{j} j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)}\Big(\sum_{i=1}^{j}\frac{1}{i}\Big)
\end{aligned}$$

and

$$\begin{aligned}\Delta\tilde{D}_n &= \sum_{k=1}^{n+1}\frac{1}{(2k-1-\alpha)^2}-\sum_{k=1}^{n}\frac{1}{(2k-1-\alpha)^2}\\ &\quad-\Big[\Big(\sum_{k=1}^{n+1}\frac{1}{2k-1-\alpha}\Big)^2-\Big(\sum_{k=1}^{n}\frac{1}{2k-1-\alpha}\Big)^2\Big]\\ &=\frac{1}{(2n+1-\alpha)^2}-\frac{1}{2n+1-\alpha}\Big[2\sum_{k=1}^{n}\frac{1}{2k-1-\alpha}+\frac{1}{2n+1-\alpha}\Big]\\ &=-\frac{2}{2n+1-\alpha}\sum_{k=1}^{n}\frac{1}{2k-1-\alpha}.\end{aligned}$$

Using (5.3) for $x=(1-\alpha)/2$ and (5.6) we get the following identity

$$\sum_{j=1}^{n}(-1)^j\binom{n}{j}\frac{2^j j!}{\prod_{k=1}^{j+1}(2k-1-\alpha)}\Big(\sum_{i=1}^{j}\frac{1}{i+\frac{1-\alpha}{2}}\Big)=-\frac{n}{(1-\alpha)\left(\frac{1-\alpha}{2}+n\right)^2}. \tag{5.7}$$

Next we substitute $x=\frac{1-\alpha}{2}$, $z=\frac{\alpha-1}{2}$ into (5.4). Using (5.6) and the fact that

$$\binom{\frac{\alpha-1}{2}}{j}\binom{\frac{1-\alpha}{2}+j}{j}^{-1}=(-1)^j\frac{1-\alpha}{2j+1-\alpha},$$

we obtain

$$\begin{aligned}&\sum_{j=1}^{n}(-1)^j\binom{n}{j}\frac{1}{2j+1-\alpha}\Big(\sum_{i=1}^{j}\frac{1}{1+\frac{1-\alpha}{2}}\Big)\\ &=\frac{2^n n!}{\prod_{k=1}^{n+1}2k-1-\alpha}\Big[\sum_{i=1}^{n}\frac{1}{i+\frac{1-\alpha}{2}}-\sum_{i=1}^{n}\frac{1}{i}\Big].\end{aligned}$$

Next we use Lemma 5.7 in case $f_j:=(-1)^j\frac{1}{2j+1-\alpha}\big(\sum_{i=1}^{j}\frac{1}{1+\frac{1-\alpha}{2}}\big)$ and F_n indicates the term on the right-hand side of the above equality. Then

$$\frac{1}{2n+1-\alpha}\Big(\sum_{i=1}^{n}\frac{1}{1+\frac{1-\alpha}{2}}\Big)=\sum_{j=1}^{n}(-1)^j\binom{n}{j}\frac{2^j j!}{\prod_{k=1}^{j+1}2k-1-\alpha}\Big[\sum_{i=1}^{j}\frac{1}{i+\frac{1-\alpha}{2}}-\sum_{i=1}^{j}\frac{1}{i}\Big]$$

and hence, using (5.7)

$$\frac{1}{2n+1-\alpha}\Big(\sum_{i=1}^{n}\frac{1}{i+\frac{1-\alpha}{2}}\Big)=-\frac{n}{(1-\alpha)\left(\frac{1-\alpha}{2}+n\right)^2}-\Delta D_n,$$

which implies

$$\begin{aligned}\Delta D_n &= -\frac{2}{2n+1-\alpha}\sum_{i=1}^{n}\frac{1}{2i+1-\alpha}-\frac{n}{(1-\alpha)\left(\frac{1-\alpha}{2}+n\right)^2}\\ &=-\frac{2}{2n+1-\alpha}\sum_{i=1}^{n}\frac{1}{2i-1-\alpha}=\Delta\tilde{D}_n.\end{aligned}$$

$\square$

References

[1] W. A. Coppel, Disconjugacy, Lectures Notes in Math., No. 220, Springer Verlag, Berlin-Heidelberg 1971.

[2] O. Došlý, *Oscillation criteria and the discreteness of the spectrum of self-adjoint, even order, differential operators*, Proc. Roy. Soc. Edinburgh **119A** (1991), 219–232.

[3] O. Došlý, *Oscillation criteria for self-adjoint linear differential equations*, Math. Nachr. **166** (1994), 141-153.

[4] O. Došlý, *Nehari-type oscillation criteria for self-adjoint linear differential equations*, J. Math. Anal. Appl. **182** (1994), 69-89.

[5] O. Došlý, *Oscillation and spectral properties of a class of singular self-adjoint differential operators*, Math. Nachr. **188** (1997), 49-68.

[6] O. Došlý, *Constants in oscillation theory of higher order Sturm-Liouville differential equations*, Electron. J. Differ. Equ. **2002** (2002), No. 34, pp.1-12.

[7] O. Došlý, *Oscillatory properties of fourth order Sturm-Liouville differential equations,* Acta Univ. Palacki. Olomouc., Fac. Rer. Nat. Mathematica **41** (2002), 49-59.

[8] O. Došlý, J. Osička, *Kneser-type oscillation criteria for self-adjoint, two terms, differential equations*, Georgian J. Math. **2** (1995), 241-258.

[9] O. Došlý, J. Osička, *Oscillation and nonoscillation of higher order self-adjoint differential equations*, Czech. Math. J. **52 (127)** (2002), 833-849.

[10] O. Došlý, J. Osička, *Oscillatory properties of higher order Sturm-Liouville differential equations*, Studies Univ. Žilina, Math,. Ser. **15** (2002), 25-40.

[11] F. Fiedler, *Oscillation criteria for a class of* $2n$*-order ordinary differential operators*, J. Differential Equations, **42** (1982), 155-188.

[12] F. Fiedler, *Oscillation criteria for a class of* $2n$*-order ordinary differential equations*, Math. Nachr. **131** (1987), 205-218.

[13] S. Fišnarová, *Oscillatory properties of fourth order self-adjoint differential equations,* to appear in Arch. Math.

[14] I. M. Glazman, Direct Methods of Qualitative Analysis of Singular Differential Operators, Davey, Jerusalem 1965.

[15] J. Kaucký, Kombinatorial Identities, Veda, Bratislava 1975 (in Czech).

[16] E. Müller-Pfeiffer, *An oscillation theorem for self-adjoint differential equations*, Math. Nachr. **108** (1982), 79-92.

[17] W. T. Reid, Sturmian Theory for Ordinary Differential Equations, Springer Verlag, New York-Heidelberg-Berlin 1980.

[18] C. A. Swanson, Comparison and Oscillation Theory of Linear Differential Equation, Acad. Press, New York, 1968.

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