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GLOBAL POSITIVE SOLUTIONS OF A GENERALIZED LOGISTIC EQUATION WITH BOUNDED AND UNBOUNDED COEFFICIENTS

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Abstract. In this paper we study the generalized logistic equation

$$\frac{du}{dt} = a(t)u^n - b(t)u^{n+(2k+1)}, \quad n, k \in \mathbb{N},$$

which governs the population growth of a self-limiting specie, with $a(t)$, $b(t)$ being continuous bounded functions. We obtain a unique global, positive and bounded solution which, further, plays the role of a frontier which clarifies the asymptotic behavior or extensibility backwards and further it is an attractor forward of all positive solutions. We prove also that the function

$$\phi(t) = \sqrt[2k+1]{a(t)/b(t)}$$

plays a fundamental role in the study of logistic equations since if it is monotone, then it is an attractor of positive solutions forward in time. Furthermore, we may relax the boundeness assumption on $a(t)$ and $b(t)$ to a boundeness of it. An existence result of a positive periodic solution is also given for the case where $a(t)$ and $b(t)$ are also periodic (actually we derive a necessary and sufficient condition for that). Our technique is a topological one of Knesser's type (connecteness and compactness of the solutions funnel).

1. Introduction

One of the most popular differential equations with various applications in economic and managerial sciences is the logistic equation:

$$\frac{du}{dt} = a(t)u(t) - b(t)u(t)^2, \quad t \in \mathbb{R}. \tag{1.1}$$

While in the case where the coefficients $a(t), b(t)$ are constant, the above-mentioned equation can be solved explicitly, by employing classical techniques, and a stable equilibrium point of population may exists, when $a(t)$ and $b(t)$ are variable the corresponding study becomes much more complicated. As a matter of fact, no explicit solutions can be found in general in this framework (see, among others, [1, 3]) and the equilibrium point may become unstable. However, it is clear that the existence of stable periodic or stable bounded solutions is an essential part of qualitative theory of differential equations. Furthermore, the existence of a solution

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of such type is of fundamental importance biologically, since it concerns the long time survival of species.

Similar problems appear also in the framework of partial differential equations with logistic type nonlinearities (see e.g. [2, 4]) or functional differential equations with discrete or continuous delays (see [9] and [5] for some recent results).

A considerable number of authors have proposed different techniques in order to determine non trivial solutions of the logistic equation and to study their behavior. Among them J. Hale and H. Kocak in [6] discuss the time periodic case and N. Nkashama in his recent paper [7] (published in this journal) works on the study of a bounded solution of (1.1) making ample use of classical techniques.

In this paper we study a generalized logistic equation. Namely, we assume that the change of u in time can be affected by higher order polynomials:

$$\frac{du}{dt} = a(t)u^n - b(t)u^{n+(2k+1)}, \quad n, k \in \mathbb{N}, \tag{1.2}$$

where the carrying capacity $a(t)$ and the self-limiting coefficient $b(t)$ are continuous and bounded functions:

$$0 < a \leq a(t) \leq A, \quad 0 < b \leq b(t) \leq B, \quad t \in \mathbb{R}. \tag{1.3}$$

Based mainly on techniques which involve the so called consequent mapping, presented by the second author in [8], we prove that equation (1.2), which obviously contains (1.1) as a special case, admits a unique global and bounded solution u_b that remains into the interval $I = [\sqrt[2k+1]{a/B}, \sqrt[2k+1]{A/b}]$, for any $t \in \mathbb{R}$ (Theorem 3.1). Furthermore, in Theorem 3.2 we propose a way of relaxing the boundeness conditions (1.3) to

$$0 < m^* \leq \phi(t) \leq M^*, \quad t \in \mathbb{R},$$

for some constants m^* and M^*, obtaining in this way a similar solution.

The above mentioned unique bounded solution plays also the role of a frontier which determines the behavior of all positive solutions of (1.2). Namely, if such a solution u lays bellow u_b for some $t \in \mathbb{R}$, then u approaches the trivial solution backward in time ($\lim_{t\to-\infty} u(t) = 0$), while in the case where u is greater than u_b for some time t, then in general u can not be a global solution. Note here that there is an exception to the later result: If $n = 1$, then global solutions above u_b are possible to exist but they blow up backwards in time: ($\lim_{t\to-\infty} u(t) = +\infty$).

Concerning the behavior of positive solutions u of (1.2) forward in time we prove (Proposition 3.5) that u_b serves as an attractor of them ($\lim_{t\to+\infty} |u_b(t)-u(t)| = 0$) clarifying in this way their asymptotic behavior. In the special case where $\phi(t)$ is monotone then it also serves as an attractor of all positive solutions.

Our approach, apart from being based on new techniques, generalizes the results of Nkashama [7], since the later can be readily obtained as a special case of our study letting the indexes n, k, defined above, to be 1 and 0 respectively.

We conclude this note studying the behavior of u_b in those cases where the function ϕ is monotone (Remark 1) and indicating in the last section how one can establish sufficient and necessary conditions in order to obtain an a priori bounded, positive and periodic solution of the generalized logistic equation.

2. Preliminaries

In this section we present some preliminary material concerning the topological behavior of the solutions' funnel of ordinary differential equations or systems. To this end, let us consider the system

$$\dot{x} = f(t,x), \quad (t,x) \in \Omega \subseteq \mathbb{R} \times \mathbb{R}^n. \tag{2.1}$$

For any subset ω of Ω such that $\Omega - \bar{\omega} \neq \emptyset$, let $P = (\tau, \xi)$ be a point of $\Omega \cap \partial\omega$ and $\mathcal{X}(P)$ the family of solutions of (2.1) through P. It is well-known that the set of all solutions $x \in \mathcal{X}(P)$ emanating from the point P forms a compact and connected (continuum) family. Namely, for any $t \in Dom\mathcal{X}(P) := \cap Dom\,\{x : x \in \mathcal{X}(P)\}$ the cross-section

$$\mathcal{X}(t;P) := \{x(t) : x \in \mathcal{X}(P)\}$$

is a continuum. Palamides in [8] replaces the last cross-section by the set of *consequent points*, which is a subset of $\partial\omega$ determined also by the solution funnel $\mathcal{X}(P)$. If $G(x;P)$ denotes the graph of such a solution and $\mathcal{X}_\omega(P)$ the set of all solutions $x \in \mathcal{X}(P)$ which remain right asymptotic in ω, then P is called a point of *semi-egress* of ω, with respect to the system (2.1), if and only if there exists a solution $x \in \mathcal{X}(P)$, a point t_1 of the domain $Domx$ of x, an $\varepsilon_1 > 0$ and a $\tau > t_1$ such that

$$G(x|[t_1 - \varepsilon_1, t_1); P) \subseteq \omega^o \quad \text{and} \quad G(x|[t_1, \tau]; P) \subseteq \partial\omega.$$

If, in addition, for any solution $x \in \mathcal{X}(P)$ there exists a point $t_2 \in Domx$ and a positive $\varepsilon_2 > 0$ such that

$$G(x|[\tau, t_2]; P) \subseteq \partial\omega \quad \text{and} \quad G(x|(t_2, t_2 + \varepsilon_2]; P) \subseteq \Omega - \overline{\omega},$$

then the point P is called a point of *strict semi-egress* of ω.

The set of all points of semi-egress of ω is denoted by ω^s and those of strict semi-egress by ω^{ss}.

A second point now $Q = (\sigma, \eta) \in \omega^s$, with $\sigma \geq \tau$, will be called a consequent of the initial one $P = (\tau, \xi)$, with respect to the set ω and the system (2.1), if there exists a solution $x \in \mathcal{X}(P,Q) = \mathcal{X}(P) \cap \mathcal{X}(Q)$ and a point $t_1 \in [\tau, \sigma]$ such that $G(x|_{[t_1,\sigma]}) \subseteq \partial\omega$ and $G(x|_{(\tau,t_1)}) \subseteq \omega^o$, for $\tau < t_1$.

The set of all consequent points of P with respect to ω is denoted by $C(\omega;P)$. If we set $S(\omega) = \{Q \in \omega : C(\omega;Q) \neq \emptyset\}$, then the consequent mapping of ω is defined by

$$\mathcal{K}_\omega(P) = C(\omega, P), \ P \in S(\omega).$$

We conclude this section referring to two fundamental results concerning the aforementioned notion which form the appropriate framework for our approach towards the generalized logistic equation. For details and the corresponding proofs we refer the reader to [8].

Proposition 2.1. *If $P \in S(\omega)$ and every $x \in \mathcal{X}(P)$ semi-egresses strictly from ω, then the consequent mapping K_ω is upper semi-continuous at the point P and the image $K_\omega(P)$ is a continuum (i.e. compact and connected subset) of $\partial\omega$. Moreover, the image $K_\omega(A)$ of any continuum A is also a continuum.*

Proposition 2.2. *If $\omega^s = \omega^{ss}$ and $P_0 = (t_0, x_0)$ is a point such that $\mathcal{X}_\omega(P_0) \neq \emptyset$, then either the family $\mathcal{X}(P_0)$ remains asymptotic in ω (i.e. $\mathcal{X}(P_0) = \mathcal{X}_\omega(P_0)$) or every connected component S of $K_\omega(P_0)$ approaches the boundary $\partial\Omega$ of Ω, i.e. $\overline{S} \cap \partial\Omega \neq \emptyset$.*

3. A generalized Logistic Equation

In this section we study a generalization of the classical logistic equation with bounded coefficients. More precisely, we consider the differential equation:

$$\frac{du}{dt} = a(t)u^n - b(t)u^{n+(2k+1)}, \quad n, k \in \mathbb{N}, \tag{3.1}$$

under the following assumptions:

$$0 < a \leq a(t) \leq A, \quad 0 < b \leq b(t) \leq B, \quad t \in \mathbb{R}, \tag{3.2}$$

where a, A, b, B are real and n, k natural numbers. The choice $n = 1$ and $k = 0$ provides the classical logistic equation studied, among other authors, by [1, 3, 6, 7].

Our main goal here is to prove that (3.1) admits exactly one bounded solution and to study the asymptotic behavior of all positive solutions. To this end, we employ the reparametrization $s = -t$ which leads to the equation:

$$\frac{dv}{ds} = -c(s)v^n + d(s)v^{n+(2k+1)}, \quad n, k \in \mathbb{N}, \tag{3.3}$$

where

$$0 < a \leq -c(s) = a(t) \leq A, \quad 0 < b \leq -d(s) = b(t) \leq B, \quad t \in \mathbb{R}. \tag{3.4}$$

It is worthy to notice that the function

$$\phi(t) = \sqrt[2k+1]{a(t)/b(t)}$$

keeps a fundamental role in our approach, which will be clarified in the sequel of the paper. However, it is necessary to point out here that $\phi(t)$ affects on the monotonicity of all positive u solutions of (3.1) and v of (3.3) respectively:

$$u(t) \text{ is increasing } \Leftrightarrow u(t) < \phi(t)$$
$$v(s) \text{ is increasing } \Leftrightarrow v(s) > \phi(s).$$

Moreover, $\phi(t)$ is a positive and bounded function since for any $t \in \mathbb{R}$:

$$m := \sqrt[2k+1]{a/B} \leq \phi(t) \leq \sqrt[2k+1]{A/b} := M.$$

Under the previous notifications the following theorem holds true.

Theorem 3.1. *The generalized logistic equation (3.1) admits exactly one bounded solution which remains into the interval $I = [m, M]$ for all $t \in \mathbb{R}$.*

Proof. Let ω and ω_0 be the sets given by

$$\omega := \{(s, v) \in \mathbb{R} \times \mathbb{R} : s \geq 0, m \leq v \leq M\},$$
$$\omega_0 := \{(s, v) \in \omega : s = 0\}$$

Then, every solution $v \in \mathcal{X}(P)$, $P \in \omega_0$, of the differential equation (3.3), that reaches the boundary $\partial\omega$ of ω, strictly egresses of it. In other words, using the terminology defined in Section 2, $\omega^s = \omega^{ss}$. As a result, the image $\mathcal{K}_\omega(\omega_0)$ of the consequent mapping $\mathcal{K}_\omega$, with respect to (3.3), has common points with both the lines $v = m$, $v = M$. Therefore, based on the fact that the above-mentioned image has to be a connected set (see Proposition 2.1), we conclude that there exists at least one solution $v = v(s)$ of (3.3) that remains into the interval $I = [m, M]$ for every $s \geq 0$. Then, the corresponding function $u(t) = v(-s)$ is the desired bounded solution of (3.1), for $t \leq 0$. On the other hand, if

$$\omega_1 := \{(t, u) \in \mathbb{R} \times \mathbb{R} : t \geq 0, m \leq u \leq M\},$$

then every point of $\partial\omega_1$ is not an egress one. As a result, the solution $u = u(t)$ remains constantly into I also for all positive values of t.

We proceed now with the proof of the uniqueness of this bounded solution $u(t)$. Let $w(t)$ be another solution of (3.1) with $w(t_1) \neq u(t_1)$ for some $t_1 \in \mathbb{R}$ and let us assume (with no loss of generality) that $w(t_1) < u(t_1)$. Then, due to the uniqueness of solutions upon initial conditions, we would have that $w(t) < u(t)$, for every $t \in \mathbb{R}$. On the other hand, equation (3.1) is equivalent to

$$(\frac{u^{1-n}}{1-n})' = a(t) - b(t)u^{2k+1},$$

in the general case where $n > 1$. As a result,

$$(\frac{u^{1-n}}{1-n} - \frac{w^{1-n}}{1-n})' = b(t)(w^{2k+1} - u^{2k+1})$$

and, therefore, the function $u^{1-n} - w^{1-n}$ is increasing. This fact ensures that

$$\frac{1}{u(t)^{n-1}} - \frac{1}{w(t)^{n-1}} \leq \frac{1}{u(0)^{n-1}} - \frac{1}{w(0)^{n-1}} = c < 0, \quad t \leq 0.$$

Trivial calculations turns the previous result to

$$(w(t) - u(t))\frac{w(t)^{n-2} + w(t)^{n-3}u(t) + \cdots + w(t)u(t)^{n-3} + u(t)^{n-2}}{(u(t)w(t))^{n-1}} \leq c < 0,$$

where the fraction emerged is a positive and bounded mapping. Hence, there exists a real $\delta > 0$ such that

$$u(t) - w(t) \geq \delta, \quad \forall t \leq 0.$$

Based on this we obtain

$$\begin{aligned}
&(\frac{w^{1-n}}{1-n} - \frac{u^{1-n}}{1-n})' = b(t)(u - w)(u^{2k} + u^{2k-1}w + \cdots + uw^{2k-1} + w^{2k}) \\
&\geq b\delta(2k+1)m^{2k} =: \varepsilon_0 > 0,
\end{aligned}$$

and by integration on the interval [t,0]:

$$w(t)^{1-n} - u(t)^{1-n} > (w(0)^{1-n} - u(0)^{1-n}) + (1-n)\varepsilon_0 t.$$

Taking now the limits when $t \to -\infty$ we obtain that

$$\lim_{t\to-\infty}(\frac{1}{w(t)^{n-1}} - \frac{1}{u(t)^{n-1}}) = +\infty,$$

which cannot be true, since $w(t)$ has been assumed bounded.

In the case where $n = 1$, we proceed in a similar way with only some differences in the proof of the uniqueness of bounded solution. More precisely, (3.1) is now equivalent to

$$(\ln u)' = a(t) - b(t)u(t)^{2k+1}$$

and assuming that $w(t)$ is a second solution of it with

$$0 \leq m \leq w(t) < u(t) \leq M, \quad t \in \mathbb{R},$$

we may check that $(\ln w - \ln u)' > 0$, hence, $w(t)/u(t)$ is increasing. As a result, $\frac{w(t)}{u(t)} \leq \frac{w(0)}{u(0)} = c < 1$, for $t \leq 0$, and

$$u(t) - w(t) \geq (1-c)m.$$

Therefore,

$$\begin{aligned}(\ln w - \ln u)' &= b(t)(u^{2k+1} - w^{2k+1}) \\ &= b(t)(u-w)(u^{2k} + u^{2k-1}w + \dots + uw^{2k-1} + w^{2k}) \\ &\geq b(1-c)m(2k+1)m^{2k} = \varepsilon_1 > 0.\end{aligned}$$

Integrating on the interval [t,0] we obtain

$$\ln(\frac{w(0)/u(0)}{w(t)/u(t)}) > -\varepsilon_1 t \Leftrightarrow \frac{w(t)}{u(t)} < \frac{w(0)}{u(0)} e^{\varepsilon_1 t}, \quad t \leq 0.$$

Then,

$$\lim_{t\to-\infty} (\frac{w(t)}{u(t)}) = 0$$

which is a contradiction to the fact that $\frac{w(t)}{u(t)} \geq \frac{m}{M} > 0$ and this concludes the proof. □

It is worthy to notice here that the boundeness conditions (3.2) can be relaxed according to the next result.

Theorem 3.2. *Let m^*, M^* be positive constants such that*

$$0 < m^* \leq \phi(t) = \sqrt[2k+1]{a(t)/b(t)} \leq M^*, \ t \in \mathbb{R}. \tag{3.5}$$

Then, the differential equation (3.1) admits exactly one bounded solution that remains into the interval $I^* := [m^*, M^*]$ for all $t \in \mathbb{R}$.

Proof. If ω stands for the set

$$\omega := \left\{(s,v) \in \mathbb{R}^2 : s \geq 0 \ \text{ and } \ m^* \leq v \leq M^*\right\},$$

then we may readily pattern the “existence part” of the above proof, under the obvious modifications. Furthermore, if we consider the Banach space

$$C_\infty := \left\{x : \mathbb{R} \to \mathbb{R} : x \text{ is continuous and } \lim_{t\to\pm\infty} x(t) \text{ exist}\right\}$$

equipped with the norm $||x|| := \sup\{|x(t)|, \ t \in \mathbb{R}\}$, then the family of all bounded in I^* solutions of (3.1) forms a compact set

$$\mathcal{X}_0 = \mathcal{X}_\omega(I^*) \subset C_\infty.$$

Indeed, this is a direct consequence of Proposition 2.1 and the uniqueness of solutions upon initial values, since we may find solutions

$$u_M(t) = \max \mathcal{X}_0 \quad \text{and} \quad u_m(t) = \min \mathcal{X}_0, \ t \in \mathbb{R},$$

such that

$$\lim_{t\to-\infty} u_M(t) = M^*, \quad \lim_{t\to-\infty} u_m(t) = m^* \quad \text{and} \quad u_m(t) \leq u_M(t), \ t \in \mathbb{R}.$$

Following the lines of the proof in Theorem 3.1, we conclude that $\mathcal{X}_0$ is a single-point set. □

The unique bounded solution u_b of (3.1), obtained in Theorem 3.1, provides a solid criterion for the behavior backward in time of all positive solutions of the equation in study. This fact is clarified in next two propositions.

Proposition 3.3. *If a positive solution u of (3.1) lies bellow u_b for some time t_1, then u approaches the trivial solution backward in time:* $\lim_{t\to-\infty} u(t) = 0$.

Proof. Since u_b is the unique solution of (3.1) that remains into the interval $I = [m, M]$, for every $t \in \mathbb{R}$, u must egresses out of I for some $t_2 \le t_1$. Taking into account that u is then bellow $\phi(t) = \sqrt[2k+1]{a(t)/b(t)}$ and, therefore, according to the relevant remarks given before Theorem 3.1, increasing, there exists a positive ε and a $t_3 \le t_2$ such that $u(t) \le m - \varepsilon$ for every $t \le t_3$. As a result,

$$(\frac{u(t)^{1-n}}{1-n})' \ge a(t) - b(t)(m-\varepsilon)^{2k+1} > a - B(m-\varepsilon)^{2k+1} := \varepsilon_0 > 0.$$

By integration onto $[t, t_3]$ we obtain:

$$u(t)^{1-n} > (n-1)\varepsilon_0(t_3 - t) + u(t_3)^{1-n},$$

which, in turn, gives that $\lim_{t\to-\infty} u(t) = 0$. The same conclusion is also reached in the case where $n = 1$, with only some modifications in our calculations since the function $\ln u$ is then replacing $\frac{u^{1-n}}{1-n}$. □

The previous proposition shows that there is a common behavior backward in time, for all possible values of the indexes n, k, of the positive solutions of the generalized logistic equation that remain bellow the unique global solution u_b. This is not the case for those solutions of (3.1) which are above u_b.

Proposition 3.4. *Let u be a solution of the generalized logistic equation (3.1) that lays above the unique bounded solution u_b for some time t_1. Then u blows up backward in time and, more precisely,*

(i) *There exists a $t_0 \le t_1$ such that $\lim_{t\to t_0} u(t) = +\infty$, when $n > 1$.*
(ii) $\lim_{t\to-\infty} u(t) = +\infty$, *when $n = 1$.*

Proof. We study first the case where the index n is greater than 1. Let $u(t)$ be a positive solution with $u(t_1) > u_b(t_1)$, for some time t_1. Then, $u(t) > u_b(t)$, for every $t \in Dom(u) \cap Dom(u_b)$, and $u(t_2) > M$ for some $t_2 \le t_1$ due to the uniqueness of u_b. Taking into account that u is then decreasing, we obtain $u(t) \ge M + \delta$, for all $t \le t_2$, for a positive δ, and

$$\frac{u'}{u^n} = a(t) - b(t)u^{2k+1} < A - b(\frac{A}{b} + \delta^{2k+1}) = -b\delta^{2k+1} < 0.$$

Integrating on the interval $[t, t_2]$ we get

$$u(t_2)^{1-n} - u(t)^{1-n} > b\delta^{2k+1}(n-1)(t_2 - t).$$

If we assume that the demand of (i) does not fulfilled at any time, then the function $u(t)$ would be defined for every $t_n < t_2$: $u(t_n) = M_n > 0$ and

$$\frac{1}{u(t_2)^{n-1}} > \frac{1}{u(t_2)^{n-1}} - \frac{1}{M_n^{n-1}} > b\delta^{2k+1}(n-1)(t_2 - t_n),$$

which leads to a contradiction if we take $t_n \to -\infty$. As a result, a point t_0 satisfying $\lim_{t\to t_0} u(t) = +\infty$ must exists.

In the case where $n = 1$, equation (3.1) takes the form

$$(\ln u)' = a(t) - b(t)u^{2k+1}$$

and the existence of a positive solution $u > u_b$, as in previous, leads to

$$\frac{u'}{u} \leq -\delta_1 < 0 \Rightarrow \int_t^{t_2} (\ln u)' dt \leq -\int_t^{t_2} \delta_1 dt \Rightarrow$$

$$\ln(\frac{u(t)}{u(t_2)}) \geq \delta_1(t_2 - t) \Rightarrow u(t) \geq e^{\delta_1(t_2-t)} u(t_2), \quad t \leq t_2.$$

As a result, $\lim_{t\to-\infty} u(t) = +\infty$. □

The next result describes the behavior of solutions of (3.1) forward in time. The unique bounded solution u_b is, again, the key since it attracts all such positive solutions.

Proposition 3.5. *The unique bounded solution $u_b(t)$ is an attractor of all positive solutions $w(t)$ of (3.1) forward in time in the sense that*

$$\lim_{t\to+\infty} |u_b(t) - w(t)| = 0.$$

Proof. Let us consider the case $n > 1$ and let $w(t)$ be an arbitrarily chosen positive solution of (3.1). If we assume that $w(t) < u_b(t)$, $t \in \mathbb{R}$, then relation

$$(\frac{u_b^{1-n}}{1-n} - \frac{w^{1-n}}{1-n})' = b(t)(w^{2k+1} - u^{2k+1})$$

ensures that the function $\frac{1}{u_b^{n-1}} - \frac{1}{w^{n-1}}$ will be increasing. Taking also into account that it is bounded, we conclude that its limit, when $t \to +\infty$, exists: $\lim_{t\to+\infty}(\frac{1}{u_b^{n-1}(t)} - \frac{1}{w^{n-1}(t)}) = c \leq 0$.

If c is negative, then there will exist a point t_1 such that

$$\begin{aligned}
&\frac{1}{u_b^{n-1}(t)} - \frac{1}{w^{n-1}(t)} \\
&= (w(t) - u_b(t))\frac{w^{n-2}(t) + w^{n-3}(t)u_b(t) + \cdots + w(t)u_b^{n-3}(t) + u_b^{n-2}(t)}{(w(t)u_b(t))^{n-1}} \\
&< c^* < 0,
\end{aligned}$$

for any $t \geq t_1$. However, the latter fraction is a positive bounded mapping, thus $w(t) - u_b(t)$ would be less than a negative constant when $t \geq t_1$. As a result, the same would be true for $(\frac{u_b^{1-n}}{1-n} - \frac{w^{1-n}}{1-n})'$ a fact that, by integration on the interval $[t_1, t]$ will give

$$\frac{1}{u_b^{n-1}(t)} - \frac{1}{w^{n-1}(t)} > \hat{M}(t - t_1) + k,$$

where $\hat{M}$ is a positive constant. This directly gives that $\lim_{t\to+\infty}(\frac{1}{u_b^{n-1}(t)}) = +\infty$ which obviously contradicts the fact that the solution u_b remains into the interval $[m, M]$. Therefore, $\lim_{t\to+\infty}(\frac{1}{u_b^{n-1}(t)} - \frac{1}{w^{n-1}(t)}) = 0$ which, in turns, gives rise to the desired $\lim_{t\to+\infty} |u_b(t) - w(t)| = 0$.

Analogously we work in the case where the solution w lays above u_b. In the special case where $n = 1$ we proceed similarly just adopting the formalism presented in Theorem 3.1. □

We conclude this section with a detailed study of the asymptotic behavior of positive solutions of the generalized Logistic Equation (3.1) in the special case where $\phi(t) = \sqrt[2k+1]{a(t)/b(t)}$ is monotone. First, we outline in next Remark the behavior of the unique bounded solution u_b.

Remark. (i) If $\phi(t)$ is always decreasing, then the following choices are possible for u_b:

(A) $u_b(t)$ lays always over $\phi(t)$ being constantly decreasing. In this case there is no possibility of u_b and ϕ to intersect since if such an incident occurs at a time t_0, then we would have
$$u_b'(t_0) = 0, u_b(t_0 + h) < \phi(t_0 + h), \quad \text{for } h > 0.$$
As a result,
$$\phi'(t_0) = \lim_{h\to 0+} \frac{\phi(t_0 + h) - \phi(t_0)}{h} \geq \lim_{h\to 0+} \frac{u_b(t_0 + h) - u_b(t_0)}{h} = u_b'(t_0) = 0$$
which contradicts the fact that ϕ is decreasing.

(B) u_b begins bellow ϕ. Then either $u_b(t) < \phi(t)$, for all $t \in \mathbb{R}$, and u_b is increasing or u_b intersects ϕ at a unique point t_0 and then follows the behavior described in case (A). Moreover it is obvious that $\max\{u_b(t) : t \in \mathbb{R}\} = u_b(t_0)$.

(ii) If $\phi(t)$ is always increasing in $\mathbb{R}$, then $u_b(t)$ has an analogous behavior with three possible choices:

(C) Remains constantly increasing bellow ϕ,

(D) Remains above ϕ for every $t \in \mathbb{R}$ and approaching it being constantly decreasing.

(E) Begins above ϕ decreasing until they intersect and falling in case (C) thereafter. Then clearly $\min\{u_b(t) : t \in \mathbb{R}\} = u_b(t_0)$.

(iii) In the special case where $\phi(t)$ is constant, it coincides with the unique bounded solution u_b.

In view of the previous thoughts, we are now in a position to prove the next basic result which illustrates the asymptotic behavior of all positive solutions of the generalized Logistic Equation (3.1) under the assumptions (3.2) in the case where the function $\phi(t) = \sqrt[2k+1]{a(t)/b(t)}$ is monotone.

Proposition 3.6. *If $\phi(t)$ is monotone, then it is an attractor of all positive solutions $w(t)$ of (3.1) forward in time in the sense that*
$$\lim_{t\to+\infty} |\phi(t) - w(t)| = 0.$$

Proof. We study the case where $\phi(t)$ is decreasing. The second option ($\phi(t)$ increasing) can be developed analogously. Since $\phi(t)$ is bounded, it will converge (when $t \to +\infty$) to its infimum: $\lim_{t\to+\infty} \phi(t) = m_1 > 0$. On the other hand, if the unique bounded solution u_b of (3.1) lays above $\phi(t)$ (case A of Remark 1), then it will also be decreasing and bounded converging to a positive m_2. If we assume that $m_2 > m_1$, then there will exists a $t_1 > 0$ such that $m_1 \leq \phi(t) < m_2 - \frac{\delta}{2}$, for any $t \geq t_1$, where $\delta := m_2 - m_1$, since $m_2 - \frac{\delta}{2} > m_1$. Therefore,
$$u_b(t) - \phi(t) > \frac{\delta}{2}, \quad t \geq t_1.$$

Taking into account that the generalized Logistic Equation (3.1) is equivalent to

$$\begin{aligned}(\frac{u(t)^{1-n}}{1-n})' &= b(t)(\phi(t)^{2k+1} - u(t)^{2k+1})\\ &= b(t)[\phi(t) - u(t)][\phi(t)^{2k} + \phi(t)^{2k-1}u(t) + \cdots + \phi(t)u(t)^{2k-1} + u(t)^{2k}],\end{aligned}$$

we obtain $(\frac{u_b(t)^{1-n}}{1-n})' < -b\frac{\delta}{2}m_1^{2k}$ which, by integration on $[t_1, t]$ gives

$$\frac{1}{u_b(t)^{n-1}} > \frac{1}{u_b(t_1)^{n-1}} + (n-1)b\frac{\delta}{2}m_1^{2k}(t - t_1).$$

Taking the limits when $t \to +\infty$, we obtain $\lim_{t\to+\infty} \frac{1}{u_b(t)^{n-1}} = +\infty$, which contradicts the fact that $u_b(t)$ remains into the interval $[m, M]$ for every $t \in \mathbb{R}$ (Theorem 3.1). Therefore, the limits m_1, m_2 have to coincide and $\lim_{t\to+\infty} |\phi(t) - u_b(t)| = 0$.

If we assume now that u_b is bellow ϕ for some time (case B of Remark 1), then either it intersects ϕ at a unique point and then follows the behavior described above, approaching ϕ when $t \to +\infty$, or it remains bellow ϕ for all $t \in \mathbb{R}$. If this is the case, then u_b is constantly increasing and bounded, so that $\lim_{t\to+\infty} u_b(t) = m_2 > 0$. If m_2 does not coincides with m_1, then $m_2 = m_1 - \delta$, $\delta > 0$, and

$$u_b(t) \le m_2 \le \phi(t) - \delta, \quad t \in \mathbb{R}.$$

Following now similar thoughts as before, we are leading to the contradiction $\lim_{t\to+\infty} u_b(t) = 0$. As a result, in this case too, m_2 and m_1 have to be equal and $\lim_{t\to+\infty} |\phi(t) - u_b(t)| = 0$.

Taking in mind that $u_b(t)$ is an attractor of all positive solutions of (2.1) (Proposition 3.5) we reach the desired result of the proposition. □

4. The Periodic Problem

In this section we consider again the differential equation

$$\frac{du}{dt} = a(t)u^n - b(t)u^{n+(2k+1)}, \quad n, k \in \mathbb{N}, \tag{4.1}$$

under the assumption

$$0 < m^* \le \phi(t) = \sqrt[2k+1]{a(t)/b(t)} \le M^*, \quad t \in \mathbb{R}, \tag{4.2}$$

where m^* and M^* are positive constants. Our main goal is to study the conditions under which (4.1) admits periodic solutions. Next result sets up the basis for this study.

Theorem 4.1. *For any $T > 0$ and $\tau \in \mathbb{R}$ the generalized logistic equation (4.1) admits a positive solution $x = x(t)$, $t \in [\tau, \tau + T]$, which satisfies the periodic condition*

$$x(\tau) = x(\tau + T).$$

Furthermore (4.1) has exactly one (classical) global T-periodic solution $u = u(t)$, $t \in \mathbb{R}$, provided that both functions $a(t)$ and $b(t)$ are also T-periodic.

Proof. We consider the set

$$\Omega := \{(t, u) \in \mathbb{R}^2 : m^* \le u \le M^*\}$$

and, for a fixed time $t = \tau$, its subset

$$\Omega[\tau, \tau + T] := \{(t, u) \in \Omega : \tau \le t \le \tau + T\}.$$

Let also

$$\Omega(\tau) := \{(t,u) \in \Omega : t = \tau\} \quad \text{and} \quad \Omega(\tau+T) := \{(t,u) \in \Omega : t = \tau + T\}$$

be the cross-sections of Ω at the time $t = \tau$ and $t = \tau + T$ respectively, where we notice that

$$\Omega[\tau] = \Omega[\tau+T] = I^* = [m^*, M^*].$$

Taking into account the sign of the nonlinearity of the differential equation (4.1), it is clear that any solution $x \in \mathcal{X}(I^*)$ egresses strictly from $\Omega[\tau, \tau+T]$, through the face $\Omega(\tau+T)$. Thus, the consequent mapping

$$\mathcal{K} : I^* \to I^*$$

is well defined and continuous and $\mathcal{K}$ admits a fixed point P_0. In other words, there exists a solution $x \in \mathcal{X}(I^*)$ of (4.1) remaining in $\Omega[\tau, \tau+T]$ such that

$$x(\tau) = x(\tau+T),\ \tau \in \mathbb{R}.$$

Now we extend periodically the obtaining solution $x = x(t)$, $t \in [\tau, \tau+T]$. More precisely, for any integer n we set

$$u(t) := x(t - nT) = x(s), \quad t \in [\tau + nT, \tau + (n+1)T]\,.$$

Then, clearly $u = u(t)$, $t \in \mathbb{R}$, is a periodic function. Furthermore (notice that $s = t - nT \in [\tau, \tau+T]$) by the periodicity of $a(t)$ and $b(t)$, we obtain

$$\begin{aligned} u'(t) &= x'(t-nT) = x'(s) \\ &= a(s)x^n(s) - b(s)x^{n+(2k+1)}(s) \\ &= a(t-nT)u^n(t) - b(t-nT)u^{n+(2k+1)}(t) \\ &= a(t)u^n(t) - b(t)u^{n+(2k+1)}(t). \end{aligned}$$

As a result, $u = u(t)$ is also a solution of equation (4.1) remaining in Ω for all $t \in \mathbb{R}$. The uniqueness of the obtained periodic solution follows by Theorem 3.1. □

The assumption of T-periodicity of $a(t)$ and $b(t)$ is essential at the above Theorem. Indeed, this is clarified in the next Example given by the referee.

4.1. **Example.** Let us consider the equation

$$\frac{du}{dt} = u - b(t)u^4, \tag{4.3}$$

where $b(t)$ is converging to a positive number when $t \to +\infty$. When $u \neq 0$, we have from (4.3)

$$u^{-4}\frac{du}{dt} = u^{-3} - b(t).$$

Let $x = u^{-3}$, then $\frac{dx}{dt} = -3u^{-4}\frac{du}{dt}$. Hence, equation (4.3) becomes

$$\frac{dx}{dt} = -3x + 3b(t). \tag{4.4}$$

The solution of equation (4.4) with initial value $x(0) = x^0$ is

$$x(t) = e^{-3t}[x_0 + \int_0^t b(s)e^{3s}ds],$$

which implies

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x_0 e^{-3t} + \lim_{t\to+\infty} \frac{\int_0^t b(s)e^{3s}ds}{e^{3t}} = \lim_{t\to+\infty} \frac{b(t)}{3}.$$

This shows that (4.4), and consequently (4.3), has no nonconstant periodic solution. In fact, zero is always a constant periodic solution of (4.3).

We conclude this paper by proving, further, that the periodicity of the coefficients $a(t)$ and $b(t)$ is an almost necessary and sufficient condition for the existence of a global periodic solution of the generalized logistic equation (4.1). Namely:

Theorem 4.2. *The generalized logistic equation (4.1) admits a positive T-periodic solution, for any $T > 0$, if and only if both functions $a(t)$ and $b(t)$ are T-periodic, provided that the function $u_p(t) := \sqrt[2k+1]{\frac{a(t+T)-a(t)}{b(t+T)-b(t)}}$ is not a periodic solution. In the case where u_p is a solution of (4.1) and the function $\phi = \phi(t)$ is nonconstant and periodic, then u_p is the unique (particular) periodic solution of the logistic equation.*

Proof. The sufficiency of the T-periodicity of both $a(t)$ and $b(t)$ has been proven in Theorem 4.1.

If we assume now that a T-periodic solution u exists then, taking into account the uniqueness of the bounded solution u_b of (4.1), we obtain that $u = u_b$ and $u'(t+T) = u'(t)$. Consequently, since $u(t)$ satisfies the equation (4.1), we easily get

$$a(t+T) - a(t) = [b(t+T) - b(t)]u^{2k+1}(t).$$

As a result, if $u = u_p(t)$ is not a solution of (4.1) or it is not a periodic function, then necessarily both the coefficients $a(t)$ and $b(t)$ must be T-periodic functions. Hence the first part of the theorem is established.

Suppose now that $a(t)$ or $b(t)$ is not T-periodic and further that the map $u = u_p(t)$ is a solution of (4.1). In order to finish the proof, it is enough to show that $u_p(t)$ is a T-periodic function. However, the T-periodicity of $u_p(t)$ is equivalent to:

$$\frac{a(t) - a(t-T)}{b(t) - b(t-T)} = \frac{a(t+T) - a(t)}{b(t+T) - b(t)}$$

or

$$(a(t) - a(t-T))(b(t+T) - b(t)) = (a(t+T) - a(t))(b(t) - b(t-T)).$$

The last equality holds if the function $\phi(t)$ (or simply the map $a(t)/b(t)$) is T-periodic, since it is equivalent to

$$\begin{aligned} &\phi^{2k+1}(t) - \phi^{2k+1}(t-T)\frac{b(t-T)}{b(t)} + \phi^{2k+1}(t-T)\frac{b(t-T)}{b(t+T)} \\ &= \phi^{2k+1}(t+T) - \phi^{2k+1}(t+T)\frac{b(t-T)}{b(t)} + \phi^{2k+1}(t)\frac{b(t-T)}{b(t+T)} \end{aligned}$$

□

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