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OSCILLATION FOR EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS AND WITH DISTRIBUTED DELAY I: GENERAL RESULTS

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Abstract. We study a scalar delay differential equation with a bounded distributed delay,

$$\dot{x}(t) + \int_{h(t)}^{t} x(s)\, d_s R(t,s) - \int_{g(t)}^{t} x(s)\, d_s T(t,s) = 0,$$

where $R(t,s)$, $T(t,s)$ are nonnegative nondecreasing in s for any t,

$$R(t,h(t)) = T(t,g(t)) = 0, \quad R(t,s) \geq T(t,s).$$

We establish a connection between non-oscillation of this differential equation and the corresponding differential inequalities, and between positiveness of the fundamental function and the existence of a nonnegative solution for a nonlinear integral inequality that constructed explicitly. We also present comparison theorems, and explicit non-oscillation and oscillation results.

In a separate publication (part II), we will consider applications of this theory to differential equations with several concentrated delays, integrodifferential, and mixed equations.

1. Introduction

The study of oscillation properties of non-autonomous delay differential equations with positive and negative coefficients began in the eighties. It was inspired by the study of equations with oscillating coefficients. For example, Chauanxi and Ladas [6] considered the equation

$$\dot{x}(t) + a(t)x(t-\tau) - b(t)x(t-\sigma) = 0, \quad t \geq t_0, \tag{1.1}$$

where $a(t) \geq 0$, $b(t) \geq 0$ are continuous functions, $\tau > \sigma > 0$, and obtained the following result.

Suppose

$$\int_{t-\tau+\sigma}^{t} b(s)ds \leq 1, \quad a(t) \geq b(t-\tau+\sigma), \tag{1.2}$$

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$$\liminf_{t\to\infty}\int_{t-\tau}^{t}[a(s)-b(s-\tau+\sigma)]ds>\frac{1}{e}. \tag{1.3}$$

Then all solutions of (1.1) are oscillatory.

In [16] the inequality (1.3) was improved to

$$\liminf_{t\to\infty}\Big(\int_{t-\tau}^{t}[a(s)-b(s-\tau+\sigma)]\,ds+\frac{1}{e}\int_{t-\tau+\sigma}^{t}b(s-\tau)\,ds\Big)>\frac{1}{e}. \tag{1.4}$$

Recently numerous publications on the oscillation of delay equations with positive and negative coefficients have appeared, for example, [1,4-7,9-11,13-17,19,20]. However all these publications except [4, 7] consider equations with constant delays only. Paper [4] deals with a more general case when the delays are not constant. Some publications study the oscillation of integro-differential equations (see [12] for positive and [18] for oscillatory kernels).

The present paper gives the general insight into the problem. We consider an equation which includes delay differential equations with variable delays, integro-differential equations and mixed differential equations. Thus most of the results of [4] appear as special cases.

We consider the following equation with a distributed delay

$$\dot{y}(t)+\int_{-\infty}^{t}y(s)\,d_sR(t,s)-\int_{-\infty}^{t}y(s)\,d_sT(t,s)=0,\quad t\ge t_0, \tag{1.5}$$

where both $R(t,s)$ and $T(t,s)$ are nondecreasing in s for each t.

Equation (1.5) includes the following special cases:

(1) A delay differential equation

$$\dot{x}(t)+\sum_{k=1}^{n}a_k(t)x(h_k(t))-\sum_{l=1}^{m}b_l(t)x(g_l(t))=0, \tag{1.6}$$

if we assume

$$R(t,s)=\sum_{k=1}^{n}a_k(t)\chi_{[h_k(t),\infty)}(s),\quad T(t,s)=\sum_{l=1}^{m}b_l(t)\chi_{[g_l(t),\infty)}(s), \tag{1.7}$$

where $\chi_{[c,d]}$ is a characteristic function of segment $[c,d]$;

(2) An integro-differential equation

$$\dot{x}(t)+\int_{-\infty}^{t}K_1(t,s)x(s)\,ds-\int_{-\infty}^{t}K_2(t,s)\,ds=0, \tag{1.8}$$

where

$$R(t,s)=\int_{-\infty}^{s}K_1(t,\zeta)\,d\zeta,\quad T(t,s)=\int_{-\infty}^{s}K_2(t,\zeta)\,d\zeta; \tag{1.9}$$

(3) Some types of mixed equations, we will consider two of them:

$$\dot{x}(t)+\sum_{k=1}^{\infty}a_k(t)x(h_k(t))-\int_{-\infty}^{t}K(t,s)x(s)\,ds=0, \tag{1.10}$$

$$\begin{aligned}&\dot{x}(t)+\sum_{k=1}^{n}a_k(t)x(h_k(t))-\sum_{l=1}^{m}b_l(t)x(g_k(t))\\&+\int_{-\infty}^{t}K_1(t,s)x(s)\,ds-\int_{-\infty}^{t}K_2(t,s)x(s)\,ds=0,\end{aligned} \tag{1.11}$$

$R(t,s)$ and $T(t,s)$ are defined similarly.

The basic result of the present paper is the relation between the following properties for (1.5): the existence of a non-oscillatory solution of (1.5), the existence of an eventually positive solution of the corresponding differential inequality and the existence of a nonnegative solution of some nonlinear integral inequality which is explicitly constructed by (1.5). Theorems of this kind are well known and widely applied for delay differential equations with positive coefficients. For (1.5) a result of this type has never been stated before.

This paper is to be continued. In the second part we will consider applications of this theory to equations of types (1.6)–(1.11).

2. Preliminaries

We consider a scalar delay differential equation (1.5) under the following assumptions:

(A1) $R(t,\cdot)$, $T(t,\cdot)$ are left continuous functions of bounded variation and for each s their variations on the segment $[t_0,s]$
$$P_R(t,s) = var_{\tau\in[t_0,s]} R(t,\tau),\ P_T(t,s) = var_{\tau\in[t_0,s]} T(t,\tau) \tag{2.1}$$
are locally integrable functions in t, $R(t,s) = R(t,t+)$, $T(t,s) = T(t,t+)$, $t < s$;

(A2) $R(t,\cdot)$, $T(t,\cdot)$ are nondecreasing functions for each t, $R(t,s) \geq T(t,s)$ for each t,s;

(A3) For each t_1 there exist $s_1 = s(t_1) \leq t_1$, $r_1 = r(t_1) \leq t_1$, such that $R(t,s) = 0$ for $s < s_1$, $t > t_1$, $T(t,s) = 0$ for $s < r_1$, $t > t_1$; in addition, functions $s(t)$, $r(t)$ satisfy
$$\lim_{t\to\infty} s(t) = \infty,\ \lim_{t\to\infty} r(t) = \infty.$$

If (A3) holds then we can introduce the following functions
$$h(t) = \inf_s \{s | R(t,s) \neq 0\},\ g(t) = \inf_s \{s | T(t,s) \neq 0\}, \tag{2.2}$$
such that $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} g(t) = \infty$, and (1.5) can be rewritten as
$$\dot{y}(t) + \int_{h(t)}^t y(s)\, d_s R(t,s) - \int_{g(t)}^t y(s)\, d_s T(t,s) = 0, \quad t \geq t_0. \tag{2.3}$$
If (A2) and (A3) hold, then obviously $h(t) \leq g(t)$.

Together with (2.3) we consider for each $t_0 \geq 0$ an initial-value problem
$$\dot{y}(t) + \int_{h(t)}^t y(s)\, d_s R(t,s) - \int_{g(t)}^t y(s)\, d_s T(t,s) = f(t), \quad t \geq t_0, \tag{2.4}$$
$$x(t) = \varphi(t),\ t < t_0,\ x(t_0) = x_0. \tag{2.5}$$
We also assume that the following hypothesis holds

(A4) $f : [t_0,\infty) \to R$ is a Lebesgue measurable locally essentially bounded function, $\varphi : (-\infty, t_0) \to R$ is a Borel measurable bounded function.

Definition. An absolutely continuous on each interval $[t_0,c]$ function $x : R \to R$ is called *a solution of problem* (2.4), (2.5), if it satisfies (2.4) for almost all $t \in [t_0,\infty)$ and equalities (2.5) for $t \leq t_0$.

Definition. For each $s \geq t_0$, a solution $X(t,s)$ of the problem

$$\dot{x}(t)+\int_{h(t)}^{t} x(s)\, d_sR(t,s)-\int_{g(t)}^{t} x(s)\, d_sT(t,s)=0,\ x(t)=0,\ t<s,\ x(s)=1,\quad (2.6)$$

is called *a fundamental function* of (2.3).

We assume $X(t,s)=0$, $t_0 \leq t < s$.

Lemma 2.1 ([2]). *Let (A1), (A3), (A4) hold. Then there exists one and only one solution of problem* (2.4), (2.5) *that can be presented in the form*

$$\begin{aligned} x(t) =& X(t,t_0)x_0 + \int_{t_0}^{t} X(t,s)f(s)ds - \int_{t_0}^{t} X(t,s)\, ds \int_{-\infty}^{s} \varphi(\tau)\, d_\tau R(s,\tau) \\ &+ \int_{t_0}^{t} X(t,s)\, ds \int_{-\infty}^{s} \varphi(\tau)\, d_\tau T(s,\tau), \end{aligned} \quad (2.7)$$

where $\varphi(\tau)=0$, *if* $\tau > t_0$.

Definition. We will say that an equation *has a non-oscillatory solution* if it has an eventually positive or an eventually negative solution. Otherwise all solutions of this equation are oscillatory.

Consider an equation with one distributed delay:

$$\dot{x}(t)+\int_{-\infty}^{t} x(s)\, d_sR(t,s)=0,\ t \geq t_0. \quad (2.8)$$

Lemma 2.2 ([3]). *Let (A1)-(A3) hold for* $R(t,s)$ *(all the conditions on* $T(t,s)$ *in (A1)-(A3) are omitted)*

1) If there exists $t_1 \geq t_0$ *such that the inequality*

$$u(t) \geq \int_{h(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)\, d\tau\right\} d_sR(t,s) \quad (2.9)$$

has a nonnegative locally integrable solution u *for* $t \geq t_1$ *(in* (2.9) *we assume* $u(t)=0$ *for* $t<t_1$*), then* (2.8) *has a non-oscillatory solution.*

2) If

$$\limsup_{t\to\infty} \int_{h(t)}^{t} d\tau \int_{h(\tau)}^{\tau} d_sR(t,s) = \limsup_{t\to\infty} \int_{h(t)}^{t} var_{s\in[h(\tau),\tau]} R(t,s)\, d\tau < \frac{1}{e}, \quad (2.10)$$

where $h(t)$ *is defined by* (2.2)*, then* (2.8) *has a non-oscillatory solution.*

3) If

$$\liminf_{t\to\infty} \int_{h(t)}^{t} d\tau \int_{h(\tau)}^{\tau} d_sR(t,s) = \liminf_{t\to\infty} \int_{h(t)}^{t} var_{s\in[h(\tau),\tau]} R(t,s)\, d\tau > \frac{1}{e}, \quad (2.11)$$

then all the solutions of (2.8) *are oscillatory.*

Let us also consider the equation

$$\dot{x}(t)+\int_{-\infty}^{t} x(s)\, d_sR_1(t,s)=0,\ t \geq t_0. \quad (2.12)$$

Lemma 2.3 ([3]). *Suppose (A1)-(A3) hold for* $R(t,s)$ *and* $R_1(t,s)$*. If a function* $R(t,\cdot)-R_1(t,\cdot)$ *is nondecreasing for each* t *and* (2.8) *has a non-oscillatory solution, then* (2.12) *also has a non-oscillatory solution. If a function* $R_1(t,\cdot)-R(t,\cdot)$ *is nondecreasing for each* t *and all the solutions of* (2.12) *are oscillatory then all solutions of* (2.8) *are oscillatory.*

We will also need the following trivial result.

Lemma 2.4. *Let f,μ be nonnegative functions of bounded variation in $[a,b]$. Suppose in addition that f is non-increasing and $\mu(a)=0$. Then $\int_a^b f(t)\,d\mu(t)\ge 0$.*

Proof. Evidently,

$$\int_a^b f(t)\,d\mu(t)=f(b)\mu(b)-f(a)\mu(a)-\int_a^b \mu(t)\,df(t)=f(b)\mu(b)-\int_a^b \mu(t)\,df(t)\ge 0.$$

3. Non-oscillation criteria

Consider together with (2.3) the delay differential inequality

$$\dot y(t)+\int_{h(t)}^t y(s)\,d_sR(t,s)-\int_{g(t)}^t y(s)\,d_sT(t,s)\le 0,\quad t\ge t_0. \tag{3.1}$$

The next theorem establishes sufficient non-oscillation conditions.

Theorem 3.1. *Suppose (A1)-(A3) hold. Consider the following hypotheses:*
1) There exists $t_1\ge t_0$ such that the inequality

$$u(t)\ge\int_{h(t)}^t \exp\Big\{\int_s^t u(\tau)d\tau\Big\}\,d_sR(t,s)-\int_{g(t)}^t \exp\Big\{\int_s^t u(\tau)d\tau\Big\}\,d_sT(t,s), \tag{3.2}$$

with $t\ge t_1$, has a nonnegative locally integrable solution (we assume $u(t)=0$ for $t<t_1$);
2) There exists $t_2\ge t_0$ such that $X(t,s)>0,\ t\ge s\ge t_2$;
3) Equation (2.3) has a non-oscillatory solution;
4) Inequality (3.1) has an eventually positive solution.
Then the implications $1)\Rightarrow 2)\Rightarrow 3)\Rightarrow 4)$ are valid.

Proof. $1)\Rightarrow 2)$. **Step 1**. Let us prove that the fundamental solution is nonnegative for $t\ge s\ge t_1$. To this end consider an initial-value problem

$$\dot x(t)+\int_{t_1}^t x(s)\,d_sR(t,s)-\int_{t_1}^t x(s)\,d_sT(t,s)=f(t),\quad t\ge t_1,\ x(t)=0,\quad t\le t_1. \tag{3.3}$$

Denote

$$z(t)=\dot x(t)+u(t)x(t),\quad z(t)=0,\ t\le t_1, \tag{3.4}$$

where x is the solution of (3.3) and u is a nonnegative solution of (3.2). Equality (3.4) implies

$$x(t)=\int_{t_1}^t \exp\Big\{-\int_s^t u(\tau)d\tau\Big\}z(s)ds,\quad t\ge t_1. \tag{3.5}$$

After substituting (3.5) into (3.3) we have

$$\begin{aligned} &z(t)-u(t)\int_{t_1}^t \exp\Big\{-\int_s^t u(\tau)\,d\tau\Big\}z(s)\,ds\\ &+\int_{t_1}^t\Big(\int_{t_1}^s \exp\Big\{-\int_\theta^s u(\tau)\,d\tau\Big\}z(\theta)\,d\theta\Big)d_sR(t,s)\\ &-\int_{t_1}^t\Big(\int_{t_1}^s \exp\Big\{-\int_\theta^s u(\tau)\,d\tau\Big\}z(\theta)\,d\theta\Big)d_sT(t,s)=f(t). \end{aligned}$$

In the second and the third integrals (in s) the integrand vanishes for $s < t_1$. After changing the order of integration in the second and the third integrals we have

$$\begin{aligned} z(t) - u(t) \int_{t_1}^{t} \exp\Big\{ - \int_s^t u(\tau)\, d\tau \Big\} z(s)\, ds \\ + \int_{t_1}^{t} z(s)\, ds \int_s^t \exp\Big\{ - \int_s^{\theta} u(\tau)\, d\tau \Big\}\, d_\theta R(t,\theta) \\ - \int_{t_1}^{t} z(s)\, ds \int_s^t \exp\Big\{ - \int_s^{\theta} u(\tau)\, d\tau \Big\}\, d_\theta T(t,\theta) = f(t). \end{aligned}$$

Thus the left hand side equals to

$$\begin{aligned} & z(t) - u(t) \int_{t_1}^{h(t)} \exp\Big\{ - \int_s^t u(\tau)\, d\tau \Big\} z(s)\, ds \\ & + \int_{t_1}^{h(t)} z(s)\, ds \int_s^t \exp\Big\{ - \int_s^t u(\tau) d\tau \Big\} \exp\Big\{ \int_\theta^t u(\tau)\, d\tau \Big\}\, d_\theta R(t,\theta) \\ & - \int_{t_1}^{h(t)} z(s)\, ds \int_s^t \exp\Big\{ - \int_s^t u(\tau) d\tau \Big\} \exp\Big\{ \int_\theta^t u(\tau)\, d\tau \Big\}\, d_\theta T(t,\theta) \\ & - u(t) \int_{h(t)}^{t} \exp\Big\{ - \int_s^t u(\tau)\, d\tau \Big\} z(s)\, ds \\ & + \int_{h(t)}^{t} z(s)\, ds \int_{h(t)}^t \exp\Big\{ - \int_s^t u(\tau) d\tau \Big\} \exp\Big\{ \int_\theta^t u(\tau)\, d\tau \Big\}\, d_\theta R(t,\theta) \\ & - \int_{h(t)}^{t} z(s)\, ds \int_{g(t)}^t \exp\Big\{ - \int_s^t u(\tau) d\tau \Big\} \exp\Big\{ \int_\theta^t u(\tau)\, d\tau \Big\}\, d_\theta T(t,\theta) \\ & - \int_{h(t)}^{t} z(s)\, ds \int_{h(t)}^s \exp\Big\{ - \int_s^{\theta} u(\tau)\, d\tau \Big\}\, d_\theta [R(t,\theta) - T(t,\theta)] \\ & - \int_{h(t)}^{t} z(s)\, ds \int_{h(t)}^{g(t)} \exp\Big\{ - \int_s^{\theta} u(\tau)\, d\tau \Big\}\, d_\theta T(t,\theta) \end{aligned}$$

which is equal to

$$\begin{aligned} & z(t) - u(t) \int_{t_1}^{h(t)} \exp\Big\{ - \int_s^t u(\tau)\, d\tau \Big\} z(s)\, ds \\ & + \int_{t_1}^{h(t)} z(s)\, ds \int_s^t \exp\Big\{ - \int_{h(t)}^t u(\tau) d\tau \Big\} \exp\Big\{ \int_\theta^t u(\tau)\, d\tau \Big\}\, d_\theta R(t,\theta) \\ & - \int_{t_1}^{h(t)} z(s)\, ds \int_s^t \exp\Big\{ - \int_{g(t)}^t u(\tau) d\tau \Big\} \exp\Big\{ \int_\theta^t u(\tau)\, d\tau \Big\}\, d_\theta T(t,\theta) \\ & - u(t) \int_{h(t)}^{t} \exp\Big\{ - \int_s^t u(\tau)\, d\tau \Big\} z(s)\, ds \\ & + \int_{h(t)}^{t} z(s)\, ds \int_{h(t)}^t \exp\Big\{ - \int_s^t u(\tau) d\tau \Big\} \exp\Big\{ \int_\theta^t u(\tau)\, d\tau \Big\}\, d_\theta R(t,\theta) \\ & - \int_{h(t)}^{t} z(s)\, ds \int_{g(t)}^t \exp\Big\{ - \int_s^t u(\tau) d\tau \Big\} \exp\Big\{ \int_\theta^t u(\tau)\, d\tau \Big\}\, d_\theta T(t,\theta) \end{aligned}$$

$$-\int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{s} \exp\Big\{-\int_{s}^{\theta} u(\tau)\,d\tau\Big\}\, d_\theta[R(t,\theta)-T(t,\theta)]$$
$$-\int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{g(t)} \exp\Big\{-\int_{s}^{\theta} u(\tau)\,d\tau\Big\}\, d_\theta T(t,\theta)$$

This in turn is equal to

$$\begin{aligned}
&z(t) - \int_{t_1}^{t} \exp\Big\{-\int_{s}^{t} u(\tau)d\tau\Big\} z(s)\,ds \\
&\times \Big[u(t) - \int_{h(t)}^{t} \exp\Big\{\int_{\theta}^{t} u(\tau)\,d\tau\Big\}\, d_\theta R(t,\theta) + \int_{g(t)}^{t} \exp\Big\{\int_{\theta}^{t} u(\tau)\,d\tau\Big\}\, d_\theta T(t,\theta)\Big] \\
&- u(t)\int_{t_1}^{h(t)} \exp\Big\{-\int_{s}^{t} u(\tau)\,d\tau\Big\} z(s)\,ds \\
&- \int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{s} \exp\Big\{-\int_{s}^{\theta} u(\tau)\,d\tau\Big\}\, d_\theta[R(t,\theta)-T(t,\theta)] \\
&- \int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{g(t)} \exp\Big\{-\int_{s}^{\theta} u(\tau)\,d\tau\Big\}\, d_\theta T(t,\theta),
\end{aligned}$$

since $R(t,s)=0$, $s<h(t)$, $T(t,s)=0$, $s<g(t)$. Consequently we obtain an operator equation

$$z - Hz = f, \tag{3.6}$$

which is equivalent to (3.3), where

$$\begin{aligned}
(Hz)(t) = &\int_{t_1}^{t} \exp\Big\{-\int_{s}^{t} u(\tau)\,d\tau\Big\} z(s)\,ds \Big[u(t) - \int_{h(t)}^{t} \exp\Big\{\int_{\theta}^{t} u(\tau)\,d\tau\Big\}\, d_\theta R(t,\theta) \\
&+ \int_{g(t)}^{t} \exp\Big\{\int_{\theta}^{t} u(\tau)\,d\tau\Big\}\, d_\theta T(t,\theta)\Big] \\
&+ \int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{g(t)} \exp\Big\{-\int_{s}^{\theta} u(\tau)\,d\tau\Big\}\, d_\theta T(t,\theta) \\
&+ u(t)\int_{t_1}^{h(t)} \exp\Big\{-\int_{s}^{t} u(\tau)\,d\tau\Big\} z(s)\,ds \\
&+ \int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{s} \exp\Big\{-\int_{s}^{\theta} u(\tau)\,d\tau\Big\}\, d_\theta[R(t,\theta)-T(t,\theta)].
\end{aligned}$$

Let $z(t)\ge 0$. Then by (3.2) the first term is positive, the last term is nonnegative due to Lemma 2.4 ($R(t,\theta)-T(t,\theta)$ is nonnegative and $\exp\big\{-\int_s^\theta u(\tau)\,d\tau\big\}$ is non-increasing in θ), i.e. operator H is positive.

Besides, in each final interval $[t_2,b]$ H is a sum of integral Volterra operators, which are compact in the space of integrable functions. Hence [8], p.519 its spectral radius $r(H)=0<1$ and consequently if in (3.6) right hand side f is nonnegative, then

$$z(t) = f(t) + (Hf)(t) + (H^2 f)(t) + (H^3 f)(t) + \cdots \ge 0.$$

We recall that the solution of (3.3) has form (3.5), with z being a solution of (3.6). Thus if in (3.3) $f(t)\ge 0$, then $x(t)\ge 0$. On the other hand, the solution of (3.3) has representation

$$x(t) = \int_{t_1}^{t} X(t,s) f(s)\, ds.$$

As was demonstrated above, $f(t) \ge 0$ implies $x(t) \ge 0$. Hence the kernel of the integral operator is nonnegative, i.e. $X(t,s) \ge 0$ for $t \ge s > t_1$.

Step 2 Let us prove that in fact the strict inequality $X(t,s) > 0$ holds. Denote

$$x(t) = X(t,t_1) - \exp\Big\{ -\int_{t_1}^{t} u(s) ds \Big\}, \quad x(t) = 0, \quad t \le t_1.$$

and substitute it into the left hand side of (3.3):

$$\begin{aligned}
&X_t'(t,t_1) + u(t)\exp\{ -\int_{t_1}^{t} u(s)\, ds\} + \int_{t_1}^{t} X(s,t_1)\, d_s R(t,s) - \int_{t_1}^{t} X(s,t_1)\, d_s T(t,s) \\
&- \int_{t_1}^{t} \exp\Big\{ -\int_{t_1}^{s} u(\tau)\, d\tau \Big\} d_s R(t,s) + \int_{t_1}^{t} \exp\Big\{ -\int_{t_1}^{s} u(\tau)\, d\tau \Big\} d_s T(t,s) \\
&= 0 + \exp\Big\{ -\int_{t_1}^{t} u(s)\, ds \Big\} \Big[u(t) - \int_{t_1}^{t} \exp\Big\{ \int_{s}^{t} u(s)\, ds \Big\}\, d_s(R(t,s) - T(t,s)) \Big] \ge 0.
\end{aligned}$$

Therefore, $x(t)$ is a solution of (3.3) with a nonnegative right hand side. Hence as shown above $x(t) \ge 0$. Consequently

$$X(t,t_1) \ge \exp\Big\{ -\int_{t_1}^{t} u(s) ds \Big\} > 0.$$

For $s > t_1$ inequality $X(t,s) > 0$ can be proved similarly.

Implication 2) ⇒ 3): A function $x(t) = X(t,t_1)$ is a positive solution of (2.3) for $t \ge t_1$.

Implication 3) ⇒ 4) is evident, which completes the proof. □

Necessary conditions for non-oscillation require some more constraints on R and T.

(A5) For any t, $R(t,s) - T(t, s - h(t) + g(t))$ is non-decreasing in s and

$$\limsup_{t\to\infty} T(t,t+)[g(t) - h(t)] \le l < 1.$$

Theorem 3.2. *Under assumptions (A1), (A2), (A3), and (A5), the hypotheses 1)-4) of Theorem 3.1 are equivalent.*

Proof. Let us prove that 4) ⇒ 1). Let $y(t)$ be a positive solution of inequality (3.1) for $t \ge t_1$.

Step 1. First we will prove that $\dot{y}(t) \le 0$. Hypotheses (A3),(A5) imply the existence of a point t_2 such that $h(t) \ge t_1,\ g(t) \ge t_1$ and for $t \ge t_2$,

$$T(t,t+)[g(t) - h(t)] \le l < 1\,. \tag{3.7}$$

Inequality (3.1) for $t \geq t_2$ can be rewritten as

$$\begin{aligned}
&\dot{y}(t) + \int_{h(t)}^{t} y(s)\, d_sR(t,s) - \int_{g(t)}^{t} y(s)\, d_sT(t,s) \\
&= \dot{y}(t) + \int_{h(t)}^{t-g(t)+h(t)} y(s)\, d_sR(t,s) \\
&\quad - \int_{h(t)}^{t-g(t)+h(t)} y(s-h(t)+g(t))\, d_sT(t,s-h(t)+g(t)) \\
&\quad + \int_{t-g(t)+h(t)}^{t} y(s)\, d_sR(t,s) \qquad (3.8)\\
&= \dot{y}(t) + \int_{t-g(t)+h(t)}^{t} y(s)\, d_sR(t,s) \\
&\quad + \int_{h(t)}^{t-g(t)+h(t)} (y(s) - y(s-h(t)+g(t)))\, d_sT(t,s-h(t)+g(t)) \\
&\quad + \int_{h(t)}^{t-g(t)+h(t)} y(s)\, d_s(R(t,s) - T(t,s-h(t)+g(t))) \leq 0.
\end{aligned}$$

Here the first integral term is nonnegative, the third term is also nonnegative by (A5) and

$$\begin{aligned}
&\int_{h(t)}^{t-g(t)+h(t)} [y(s) - y(s-h(t)+g(t))]\ d_sT(t,s-h(t)+g(t)) \\
&= \int_{g(t)}^{t} [y(s+h(t)-g(t)) - y(s)]\ d_sT(t,s) \\
&= \int_{g(t)}^{t} d_sT(t,s) \int_{s}^{s+h(t)-g(t)} \dot{y}(\tau)\, d\tau\,.
\end{aligned}$$

Let us denote by $L_\infty[t_2, c]$ the space of all essentially bounded on $[t_2, c]$ functions with the norm $\|x\| = \operatorname{ess\,sup}_{t_2 \leq t \leq c} |x(t)|$ and evaluate the norm of the following operator in this space

$$(Hx)(t) = \int_{g(t)}^{t} d_sT(t,s) \int_{s+h(t)-g(t)}^{s} x(\tau)\, d\tau.$$

Then in the $L_\infty[t_2, c]$-norm

$$\begin{aligned}
\|Hx_1 - Hx_2\| &= \operatorname*{ess\,sup}_{t_2 \leq t \leq c} \Big| \int_{g(t)}^{t} d_sT(t,s) \int_{s+h(t)-g(t)}^{s} [x_1(\tau) - x_2(\tau)]\, d\tau \Big| \\
&\leq \operatorname*{ess\,sup}_{s \in [t_2, c)} \int_{s+h(t)-g(t)}^{s} |x_1(\tau) - x_2(\tau)|\ d\tau \cdot var_{s \in [g(t),t]} T(t,s) \\
&\leq (T(t,t+) - T(t,g(t)) \|x_1 - x_2\| (g(t) - h(t)) \\
&= T(t,t+) \|x_1 - x_2\| (g(t) - h(t)).
\end{aligned}$$

(here $T(t, g(t)) = 0$ by the definition of $g(t)$).

Since $T(t,t+)(g(t) - h(t)) \leq l < 1$, then H is a contracting mapping in $L_\infty[t_2, c]$. Besides, $Hz \geq 0$, if $z \geq 0$, $Hz \leq 0$, if $z \leq 0$. For any y, (3.8) can be rewritten in the form

$$\dot{y} = H\dot{y} + q, \tag{3.9}$$

where $q(t) \le 0$ for $t \ge t_2$. Banach contraction theorem implies (3.9) has the unique solution and for this solution we have $\dot{y} = \lim z_n$, where

$$z_n = Hz_{n-1} + q, \quad z_0 = q.$$

Inequality $q(t) \le 0$ yields $z_n(t) \le 0$, hence $\dot{y}(t) \le 0$, $t_2 \le t \le c$. Since $c \ge t_2$ is an arbitrary number we have $\dot{y}(t) \le 0$, $t \ge t_2$.

Step 2. Now let us prove that inequality (3.2) has an eventually nonnegative solution. Denote

$$u(t) = -\frac{d}{dt} \ln \frac{y(t)}{y(t_2)}, \quad t \ge t_2,$$

where inequality $\dot{y}(t) \le 0$ implies that $u(t) \ge 0$. Then

$$y(t) = y(t_2) \exp\Big\{ -\int_{t_2}^{t} u(s)ds \Big\}, \quad t \ge t_2. \tag{3.10}$$

We substitute (3.10) into (3.1) and obtain by carrying the exponent out of the brackets:

$$\begin{aligned} -\exp\Big\{ -\int_{t_2}^{s} u(\tau)\,d\tau \Big\} y(t_2) \Big[u(t) - \int_{h(t)}^{t} \exp\Big\{ \int_{s}^{t} u(\tau)\,d\tau \Big\}\, d_s R(t,s) \\ + \int_{g(t)}^{t} \exp\Big\{ \int_{s}^{t} u(\tau)\,d\tau \Big\}\, d_s T(t,s) \Big] \le 0, \quad t \ge t_2, \end{aligned}$$

which implies (3.2) and completes the proof of the theorem. □

Remark. Condition (A5) is rather restrictive, for the delay equation (1.6) it describes the case of two delays ($n = m = 1$), with $a(t) \ge b(t)$, $h(t) \le g(t)$ and $\limsup_{t\to\infty} b(t)[g(t) - h(t)] \le l < 1$. This result coincides with [4, Thm. 1].

To obtain other necessary oscillation conditions we consider the following form of equation (2.3):

$$\dot{y}(t) + \sum_{k=1}^{n} \int_{h_k(t)}^{t} y(s)\,d_s R_k(t,s) - \sum_{l=1}^{m} \int_{g_l(t)}^{t} y(s)\,d_s T_l(t,s) = 0, \quad t \ge t_0, \tag{3.11}$$

where R_k, T_l, h_k, g_l satisfy the following conditions:

- (A1*) $R_k(t,\cdot)$, $T_l(t,\cdot)$ are left continuous functions of bounded variation and for each s their variations on the segment $[t_0, s]$ $P_{R_k}(t,s)$, $P_{T_l}(t,s)$ are locally integrable functions in t, $R_k(t,s) = R_k(t,t+)$, $T_l(t,s) = T_l(t,t+)$, $t < s$;
- (A2*) $R_k(t,\cdot)$, $T_l(t,\cdot)$ are nondecreasing functions for each t, and $\sum_k R_k(t,s) \ge \sum_l T_l(t,s)$ for each t, s;
- (A3*) For each k, l $\lim_{t\to\infty} h_k(t) = \infty$, $\lim_{t\to\infty} g_l(t) = \infty$.

Denote

$$\begin{gathered} R(t,s) = \sum_{k=1}^{n} R_k(t,s), \quad T(t,s) = \sum_{l=1}^{m} T_l(t,s), \\ h(t) = \max_k h_k(t), \quad g(t) = \min_l g_l(t). \end{gathered} \tag{3.12}$$

Let us also introduce the following additional constraints:

(add1) $m = n$, $R_k(t,s) \ge T_k(t,s)$ for each t, s, $k = 1, \dots, n$, for any t, k, function $R_k(t,s) - T_k(t, s - h_k(t) + g_k(t))$ is nondecreasing in s and

$$\limsup_{t\to\infty} \sum_{k=1}^{n} T_k(t,t+)[g_k(t) - h_k(t)] < 1.$$

(add2) $h(t) \le g(t)$, $R(t,s) > T(t,s)$, $R(t,s) - T(t, s - h(t) + g(t))$ is nondecreasing in s for any t and

$$\begin{aligned} \limsup_{t\to\infty} \Big[T(t,t+)[g(t) - h(t)] &+ \sum_{k=1}^{n} R_k(t,t+)\,(h(t) - h_k(t)) \\ &+ \sum_{l=1}^{m} T_l(t,t+)\,(g_l(t) - g(t)) \Big] < 1. \end{aligned} \tag{3.13}$$

Remark. If (add2) holds, then (3.11) can be reduced to an equation for which (add1) is satisfied with $n + m + 1$ terms:

$$\begin{aligned} \dot y(t) + \sum_{k=1}^{n} \int_{h_k(t)}^{t} y(s)\, d_s R_k(t,s) &- \sum_{k=1}^{n} \int_{h(t)}^{t} y(s)\, d_s R_k(t,s) \\ + \int_{h(t)}^{t} y(s)\, d_s R(t,s) &- \int_{g(t)}^{t} y(s)\, d_s T(t,s) \\ + \sum_{l=1}^{m} \int_{g(t)}^{t} y(s)\, d_s T_l(t,s) &- \sum_{l=1}^{m} \int_{g_l(t)}^{t} y(s)\, d_s T_l(t,s) = 0, \quad t \ge t_0, \end{aligned} \tag{3.14}$$

Together with (3.11) consider the delay differential inequality

$$\dot y(t) + \sum_{k=1}^{n} \int_{h_k(t)}^{t} y(s)\, d_s R_k(t,s) - \sum_{l=1}^{m} \int_{g_l(t)}^{t} y(s)\, d_s T_l(t,s) \le 0, \quad t \ge t_0. \tag{3.15}$$

The following theorem establishes non-oscillation criteria for (3.11).

Theorem 3.3. *Suppose R_k, T_l, h_k, g_k satisfy (A1*)-(A3*) and at least one of conditions (add1), (add2) hold. Then the following hypotheses are equivalent:*
1) There exists $t_1 \ge t_0$ such that the inequality

$$\begin{aligned} u(t) \ge \sum_{k=1}^{n} \int_{h_k(t)}^{t} &\exp\Big\{ \int_s^t u(\tau) d\tau \Big\}\, d_s R_k(t,s) \\ - \sum_{l=1}^{m} \int_{g_l(t)}^{t} &\exp\Big\{ \int_s^t u(\tau) d\tau \Big\}\, d_s T_l(t,s), \quad t \ge t_1, \end{aligned} \tag{3.16}$$

has a nonnegative locally integrable solution (we assume $u(t) = 0$ for $t < t_1$);
2) There exists $t_2 \ge t_0$ such that $X(t,s) > 0$, $t \ge s \ge t_2$;
3) Equation (3.11) has a non-oscillatory solution;
4) Inequality (3.15) has an eventually positive solution.

Proof. Since (3.11) is a special case of (2.3), it is enough to prove the implication 4) $\Rightarrow$ 1). Step 2 of the proof repeats the proof in the case $n = m = 1$, so we will present here only Step 1.

Suppose (add1) holds. Let us prove that 4) $\Rightarrow$ 1). Let $y(t)$ be a positive solution of inequality (3.15) for $t \ge t_1$.

First we will prove that $\dot{y}(t) \le 0$. Hypotheses $(A3^\star)$, (add1) imply the existence of a point t_2 such that $h(t) \ge t_1, g(t) \ge t_1$ and for $t \ge t_2$,

$$\sum_{k=1}^{n} T_k(t,t+)[g_k(t)-h_k(t)] \le l < 1\,. \tag{3.17}$$

Inequality (3.15) for $t \ge t_2$ can be rewritten as

$$\begin{aligned}
&\dot{y}(t) + \sum_{k=1}^{n} \Big[\int_{h_k(t)}^{t} y(s)\, d_s R_k(t,s) - \int_{g_k(t)}^{t} y(s)\, d_s T_k(t,s) \Big] \\
&= \dot{y}(t) + \sum_{k=1}^{n} \Big[\int_{h_k(t)}^{t-g_k(t)+h_k(t)} y(s)\, d_s R(t,s) \\
&\quad - \int_{h_k(t)}^{t-g_k(t)+h_k(t)} y(s-h_k(t)+g_k(t))\, d_s T_k(t,s-h_k(t)+g_k(t)) \\
&\quad + \int_{t-g_k(t)+h_k(t)}^{t} y(s)\, d_s R_k(t,s) \Big] \\
&= \dot{y}(t) + \sum_{k=1}^{n} \Big[\int_{t-g(t)+h(t)}^{t} y(s)\, d_s R_k(t,s) \\
&\quad + \int_{h_k(t)}^{t-g_k(t)+h_k(t)} (y(s)-y(s-h(t)+g(t)))\ d_s T_k(t,s-h_k(t)+g_k(t)) \\
&\quad + \int_{h_k(t)}^{t-g_k(t)+h_k(t)} y(s)\, d_s (R_k(t,s)-T_k(t,s-h_k(t)+g_k(t))) \Big] \le 0.
\end{aligned}$$

Here the first integral term in the brackets is nonnegative, the third term is also nonnegative by (add1) for each k and

$$\begin{aligned}
&\int_{h_k(t)}^{t-g_k(t)+h_k(t)} [y(s)-y(s-h_k(t)+g_k(t))]\, d_s T_k(t,s-h_k(t)+g_k(t)) \\
&= \int_{g_k(t)}^{t} [y(s+h_k(t)-g_k(t))-y(s)]\, d_s T_k(t,s) \\
&= \int_{g_k(t)}^{t} d_s T_k(t,s) \int_{s}^{s+h_k(t)-g_k(t)} \dot{y}(\tau)\, d\tau.
\end{aligned}$$

Let us evaluate the norm of the operators

$$(H_k x)(t) = \int_{g_k(t)}^{t} d_s T_k(t,s) \int_{s+h_k(t)-g_k(t)}^{s} x(\tau)\, d\tau, \quad (Hx)(t) = \sum_{k=1}^{n} (H_k x)(t)$$

in $L_\infty[t_2,c]$. Then, similar to the proof of Theorem 3.2, in $L_\infty[t_2,c]$-norm,

$$\|H_k x_1 - H_k x_2\| \le T_k(t,t+)\|x_1-x_2\|(g_k(t)-h_k(t)),$$

thus $\|H_k\| \le T_k(t,t+)(g_k(t)-h_k(t))$ and

$$\|H\| \le \sum_{k=1}^{n} \|H_k\| \le \sum_{k=1}^{n} T_k(t,t+)(g_k(t)-h_k(t)) < 1.$$

Consequently, H is a contracting mapping in $L_\infty[t_2, c]$. Besides, $Hz \geq 0$, if $z \geq 0$, $Hz \leq 0$, if $z \leq 0$. The rest of the proof coincides with the proof of Theorem 3.2. The case (add2) is reduced to (add1). □

Theorems 1-3 yield the following comparison result between the oscillation properties of the equation

$$\dot{y}(t) + \sum_{k=1}^{n} \int_{-\infty}^{t} y(s)\, d_s L_k(t,s) - \sum_{l=1}^{m} \int_{-\infty}^{t} y(s)\, d_s D_l(t,s) = 0, \quad t \geq t_0, \tag{3.18}$$

and the oscillation properties of (3.11).

Theorem 3.4. *1. If $(A1^\star) - (A3^\star)$ and anyone of the conditions (add1),(add2) holds for* (3.18) *(where R_k, T_l are changed by L_k, D_l), $L_k(t,s) \geq R_k(t,s)$, $D_l(t,s) \leq T_l(t,s)$ and* (3.18) *has a non-oscillatory solution, then (3.11) has a non-oscillatory solution.*
2. If $(A1^\star) - (A3^\star)$ and any one of the conditions (add1),(add2) holds for (3.11), *$L_k(t,s) \leq R_k(t,s)$, $D_l(t,s) \geq T_l(t,s)$ and all solutions of* (3.18) *are oscillatory, then all solutions of (3.11) are oscillatory.*

Proof. 1). If (3.18) has a non-oscillatory solution, then there exists a solution $u(t) \geq 0$ of inequality (3.16), where the functions R_k, T_l are replaced by L_k, D_l. Hence u is also a solution of (3.16) with parameters R_k, T_l. Theorem 3.3 implies now (3.11) has a non-oscillatory solution.
2) follows from 1) which completes the proof. □

Let us study the asymptotic behavior of non-oscillatory solutions of (3.11).

Theorem 3.5. *Suppose ($A1^\star$)-($A3^\star$) and anyone of the following conditions holds:*
1) (add1) is satisfied and for some k

$$\int_{t_0}^{\infty} [R_k(t,t+) - T_k(t,t+)]\, dt = \infty; \tag{3.19}$$

2) (add2) is satisfied and

$$\int_{t_0}^{\infty} [R(t,t+) - T(t,t+)]\, dt = \infty. \tag{3.20}$$

Then any non-oscillatory solution y of (3.11) *satisfies* $\lim_{t\to\infty} y(t) = 0$.

Proof. Assume $y(t) > 0$, $t \geq t_1$, is a positive solution of (3.11). The proof of Theorem 3.2 implies the existence of a point $t_2 \geq t_1$ such that $\dot{y}(t) \leq 0$ for $t \geq t_2$. Then $u(t) = -\frac{\dot{y}(t)}{y(t)}, \quad t \geq t_2$, is a nonnegative solution of (3.16). Suppose 1) holds. Then this inequality yields

$$\begin{aligned}
u(t) \geq \sum_{k=1}^{n} \Big[&\int_{h_k(t)}^{t-g_k(t)+h_k(t)} \exp\Big\{ \int_s^t u(\tau)d\tau \Big\}\, d_s R_k(t,s) \\
&- \int_{h_k(t)}^{t-g_k(t)+h_k(t)} \exp\Big\{ \int_{s+g(t)-h(t)}^{t} u(\tau)d\tau \Big\}\, d_s T_k(t, s+g(t)-h(t)) \\
&+ \int_{t-g_k(t)+h_k(t)}^{t} \exp\Big\{ \int_s^t u(\tau)d\tau \Big\}\, d_s R_k(t,s) \Big] \\
\geq \sum_{k=1}^{n} \Big[&\int_{h_k(t)}^{t-g_k(t)+h_k(t)} \exp\Big\{ \int_s^t u(\tau)d\tau \Big\}\, d_s (R_k(t,s) - T_k(t, s+g(t)-h(t)))
\end{aligned}$$

$$
\begin{aligned}
&+\int_{h_k(t)}^{t-g_k(t)+h_k(t)} \Big(\exp\Big\{\int_s^t u(\tau)d\tau\Big\} - \exp\Big\{\int_{s+g(t)-h(t)}^t u(\tau)d\tau\Big\}\Big)\\
&\times\, d_s T_k(t, s+g(t)-h(t))\Big]\\
&+\int_{t-g_k(t)+h_k(t)}^t \exp\left\{\int_s^t u(\tau)d\tau\right\} d_s R_k(t,s)\\
\geq &\int_{h_k(t)}^{t+h_k(t)-g_k(t)} d_s[R_k(t,s) - T_k(t, s-h_k(t)+g_k(t))]\\
&+\int_{t-g_k(t)+h_k(t)}^t d_s R_k(t,s)\\
= &\int_{h_k(t)}^t d_s R_k(t,s) - \int_{h_k(t)}^{t+h_k(t)-g_k(t)} d_s T_k(t, s-h_k(t)+g_k(t))\\
= &\, R_k(t,t+) - T_k(t,t+).
\end{aligned}
$$

Thus if (3.19) holds, then $\int_{t_0}^{\infty} u(s)ds = \infty$. The solution y of (3.11) has the form (3.10). Then $\lim_{t\to\infty} y(t) = 0$. Case 2) is treated similarly. □

4. Explicit Oscillation and Non-Oscillation Results

To obtain some explicit oscillation results we will need to reformulate Lemma 2.2. Consider the equation

$$\dot{x}(t) + \sum_{k=1}^n \int_{h_k(t)}^t x(s) m_k(t,s)\, d_s R_k(t,s) = 0, \tag{4.1}$$

which is a special case of (2.8). Denote $H_1(t) = \min_k h_k(t), H_2(t) = \max_k h_k(t)$. Then Lemma 2.2 can be reformulated in the following way.

Lemma 4.1. *Suppose (A1)-(A3) hold for $R_k(t,s)$ (all the constraints on $T_k(t,s)$ in (A1)-(A3) are omitted), m_k are locally absolutely continuous in s for each t and locally essentially bounded in t for each s, $m_k \geq 0$.*
1) If there exists $t_1 \geq t_0$ such that the inequality

$$u(t) \geq \sum_{k=1}^n \int_{h_k(t)}^t \exp\Big\{\int_s^t u(\tau)\, d\tau\Big\} m_k(t,s) d_s R_k(t,s) \tag{4.2}$$

has a nonnegative locally integrable solution u for $t \geq t_1$ (in (4.2) we assume $u(t) = 0$ for $t < t_2$), then (4.1) has a non-oscillatory solution.
2) If

$$\limsup_{t\to\infty} \sum_{k=1}^n \int_{H_1(t)}^t d\tau \int_{h(\tau)}^{\tau} m_k(t,s) d_s R(t,s) < \frac{1}{e}, \tag{4.3}$$

where $h(t)$ is defined by (2.2), then equation (4.1) has a non-oscillatory solution.
3) If

$$\liminf_{t\to\infty} \sum_{k=1}^n \int_{H_2(t)}^t d\tau \int_{h(\tau)}^{\tau} m_k(t,s) d_s R_k(t,s) > \frac{1}{e}, \tag{4.4}$$

then all the solutions of (4.1) are oscillatory.

Consider the following two equations which are also partial cases of (2.8):

$$\begin{aligned} &\dot{x}(t)+\int_{h(t)}^{t} x(s)\, d_s[R(t,s)-T(t,s-h(t)+g(t))] \\ &+\int_{g(t)}^{t} x(s)\Big(\exp\Big\{\int_{s+h(t)-g(t)}^{s}[R(\tau,\tau+)-T(\tau,\tau+)]d\tau\Big\}-1\Big)\, d_sT(t,s)=0 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\dot{x}(t)+\int_{g(t)}^{t} x(s)\Big(\exp\Big\{\int_{s-g(t)+h(t)}^{s}[R(\tau,\tau+)-T(\tau,\tau+)]d\tau\Big\}-1\Big) \\ &\times d_sR(t,s-g(t)+h(t))+\int_{g(t)}^{t} x(s)\, d_s[R(t,s-g(t)+h(t))-T(t,s)]=0. \end{aligned} \tag{4.6}$$

The oscillation properties of these equations will be compared to the properties of (2.3).

Theorem 4.2. *Suppose (A1)-(A3), (A5) hold for* (2.3). *If all solutions of either* (4.5) *or* (4.6) *are oscillatory, then all solutions of* (2.3) *are also oscillatory.*

Proof. 1) Suppose all solutions of (4.5) are oscillatory and (2.3) has a non-oscillatory solution. By Theorem 3.2 there exists a nonnegative solution $u(t)$ of the inequality (3.2), i.e.

$$\begin{aligned} &u(t) \\ &\geq \int_{h(t)}^{t}\exp\Big\{\int_{s}^{t}u(\tau)d\tau\Big\}\, d_sR(t,s)-\int_{g(t)}^{t}\exp\Big\{\int_{s}^{t}u(\tau)d\tau\Big\}\, d_sT(t,s) \\ &=\int_{h(t)}^{t}\exp\Big\{\int_{s}^{t}u(\tau)d\tau\Big\}\, d_sR(t,s)-\int_{h(t)}^{t}\exp\Big\{\int_{s}^{t}u(\tau)d\tau\Big\}\, d_sT(t,s-h(t)+g(t)) \\ &+\int_{h(t)}^{t}\exp\Big\{\int_{s}^{t}u(\tau)d\tau\Big\}\, d_sT(t,s-h(t)+g(t))-\int_{g(t)}^{t}\exp\Big\{\int_{s}^{t}u(\tau)d\tau\Big\}\, d_sT(t,s) \\ &=\int_{h(t)}^{t}\exp\Big\{\int_{s}^{t}u(\tau)d\tau\Big\}\, d_s[R(t,s)-T(t,s-h(t)+g(t))] \\ &\quad+\int_{g(t)}^{t+g(t)-h(t)}\exp\Big\{\int_{s+h(t)-g(t)}^{t}u(\tau)d\tau\Big\}\, d_sT(t,s) \\ &\quad-\int_{g(t)}^{t}\exp\Big\{\int_{s}^{t}u(\tau)d\tau\Big\}\, d_sT(t,s) \\ &\geq \int_{h(t)}^{t}\exp\Big\{\int_{s}^{t}u(\tau)d\tau\Big\}\, d_s[R(t,s)-T(t,s-h(t)+g(t))] \\ &\quad+\int_{g(t)}^{t}\exp\Big\{\int_{s}^{t}u(\tau)d\tau\Big\}\Big(\exp\Big\{\int_{s+h(t)-g(t)}^{s}u(\tau)d\tau\Big\}-1\Big)\, d_sT(t,s). \end{aligned}$$

The second term in the right hand side is obviously nonnegative, thus

$$\begin{aligned} u(t) &\geq \int_{h(t)}^{t}\exp\Big\{\int_{s}^{t}u(\tau)d\tau\Big\}\, d_s[R(t,s)-T(t,s-h(t)+g(t))] \\ &\geq \int_{h(t)}^{t} d_s[R(t,s)-T(t,s-h(t)+g(t))] \end{aligned}$$

$$= R(t,t+) - T(t,t+),$$

which yields

$$\begin{aligned} u(t) \ge & \int_{h(t)}^{t} \exp\Big\{ \int_s^t u(\tau)d\tau \Big\}\, d_s[R(t,s) - T(t,s-h(t)+g(t))] \\ & + \int_{g(t)}^{t} \exp\Big\{ \int_s^t u(\tau)d\tau \Big\} \\ & \times \Big(\exp\Big\{ \int_{s+h(t)-g(t)}^{s} [R(\tau,\tau) - T(\tau,\tau)]d\tau \Big\} - 1 \Big)\, d_sT(t,s). \end{aligned}$$

By Lemma 4.1 equation (4.5) has a non-oscillatory solution, which leads to a contradiction.
2) Similarly, let (2.3) have a non-oscillatory solution. Then there exists a nonnegative function $u(t)$, such that

$$\begin{aligned} & u(t) \\ & \ge \int_{g(t)}^{t+g(t)-h(t)} \exp\Big\{ \int_{s-g(t)+h(t)}^{t} u(\tau)d\tau \Big\}\, d_sR(t,s-g(t)+h(t)) \\ & \quad - \int_{g(t)}^{t} \exp\Big\{ \int_s^t u(\tau)d\tau \Big\}\, d_sT(t,s) \\ & = \int_{t}^{t+g(t)-h(t)} \exp\Big\{ \int_{s-g(t)+h(t)}^{t} u(\tau)d\tau \Big\}\, d_sR(t,s-g(t)+h(t)) \\ & \quad + \int_{g(t)}^{t} \exp\Big\{ \int_{s-g(t)+h(t)}^{t} u(\tau)d\tau \Big\}\, d_sR(t,s-g(t)+h(t)) \\ & \quad - \int_{g(t)}^{t} \exp\Big\{ \int_s^t u(\tau)d\tau \Big\}\, d_sT(t,s) \\ & = \int_{t-g(t)+h(t)}^{t} \exp\Big\{ \int_s^t u(\tau)d\tau \Big\}\, d_sR(t,s) \\ & \quad + \int_{g(t)}^{t} \exp\Big\{ \int_s^t u(\tau)d\tau \Big\} \Big(\exp\Big\{ \int_{s-g(t)+h(t)}^{s} u(\tau)d\tau \Big\} - 1 \Big)\, d_sR(t,s-g(t)+h(t)) \\ & \quad + \int_{g(t)}^{t} \exp\Big\{ \int_s^t u(\tau)d\tau \Big\}\, d_s[R(t,s-g(t)+h(t)) - T(t,s)] \\ & \ge \int_{g(t)}^{t} \exp\Big\{ \int_s^t [R(\tau,\tau+) - T(\tau,\tau+)]d\tau \Big\} \\ & \times \Big(\exp\Big\{ \int_{s-g(t)+h(t)}^{s} u(\tau)d\tau \Big\} - 1 \Big)\, d_sR(t,s-g(t)+h(t)) \\ & + \int_{g(t)}^{t} \exp\Big\{ \int_s^t u(\tau)d\tau \Big\}\, d_s[R(t,s-g(t)+h(t)) - T(t,s)]. \end{aligned}$$

Therefore, by Lemma 4.1 equation (4.6) has a non-oscillatory solution.

Corollary 4.3. *Suppose (A1)-(A3), (A5) hold for (2.3). If all solutions of either*

$$\begin{aligned}\dot{x}(t) &+ \int_{h(t)}^{t} x(s)\, d_s[R(t,s) - T(t, s-h(t)+g(t))] \\ &+ \int_{g(t)}^{t} x(s)\Big(\int_{s+h(t)-g(t)}^{s} [R(\tau,\tau+) - T(\tau,\tau+)]\Big)\, d_s T(t,s) = 0\end{aligned} \tag{4.7}$$

or

$$\begin{aligned}\dot{x}(t) + \int_{g(t)}^{t} x(s)\Big(\int_{s-g(t)+h(t)}^{s} [R(\tau,\tau+) - T(\tau,\tau+)]d\tau\Big)\, d_s R(t, s-g(t)+h(t)) \\ + \int_{g(t)}^{t} x(s)\, d_s[R(t, s-g(t)+h(t)) - T(t,s)] = 0\end{aligned} \tag{4.8}$$

are oscillatory, then all solutions of (2.3) are also oscillatory.

Proof. The statement of the corollary is an immediate consequence of the inequality $e^x - 1 \geq x$, $x \geq 0$, and comparison Lemma 2.3 (preliminary Lemma 2.3 has to be reformulated to include a more general case, similar to the statement of Lemma 4.1 when compared to Lemma 2.2).

Corollary 4.4. *Suppose (A1)-(A3), (A5) hold for (2.3) and at least one of the following four inequalities hold:*

$$\begin{aligned}&\liminf_{t\to\infty} \Big\{ \int_{h(t)}^{t} [R(t,\tau) - T(t, \tau-h(t)+g(t))]d\tau \\ &+ \int_{g(t)}^{t} d\tau \int_{h(\tau)}^{\tau} \Big(\exp\Big\{ \int_{s+h(t)-g(t)}^{s} [R(u,u+) - T(u,u+)]du \Big\} - 1 \Big)\, d_s T(t,s) \Big\} > \frac{1}{e}\end{aligned}$$

$$\begin{aligned}&\liminf_{t\to\infty} \Big\{ \int_{h(t)}^{t} [R(t,\tau+) - T(t, \tau-h(t)+g(t))]\, d\tau \\ &\quad + \int_{g(t)}^{t} d\tau \int_{h(\tau)}^{\tau} \Big(\int_{s+h(t)-g(t)}^{s} [R(u,u+) - T(u,u+)]du \Big)\, ds T(t,s) \Big\} > \frac{1}{e}\end{aligned}$$

$$\begin{aligned}&\liminf_{t\to\infty} \Big\{ \int_{g(t)}^{t} d\tau \int_{g(\tau)}^{\tau} \Big(\exp\Big\{ \int_{s-g(t)+h(t)}^{s} [R(u,u+) - T(u,u+)]du \Big\} - 1 \Big) \\ &\quad \times d_s R(t, s-g(t)+h(t)) + \int_{g(t)}^{t} [R(t, \tau-g(t)+h(t)) - T(t,\tau)]\, d\tau \Big\} > \frac{1}{e}\end{aligned}$$

$$\begin{aligned}&\liminf_{t\to\infty} \Big\{ \int_{g(t)}^{t} d\tau \int_{g(\tau)}^{\tau} \Big(\int_{s-g(t)+h(t)}^{s} [R(u,u+) - T(u,u+)] \\ &\quad \times du \Big)\, d_s R(t, s-g(t)+h(t)) + \int_{g(t)}^{t} [R(t, \tau-g(t)+h(t)) - T(t,\tau)] \Big\} > \frac{1}{e}\end{aligned}$$

Then all solutions of (1.5) are oscillatory.

The proof of this corollary follows from Lemma 4.1. Similar results can be obtained for (3.11).

Theorem 4.5. *Suppose R_k, T_l, h_k, g_l satisfy (A1*)-(A3*) and condition (add1). If all solutions of either*

$$\dot{x}(t) + \sum_{k=1}^{n} \int_{h_k(t)}^{t} x(s)\, d_s[R_k(t,s) - T_k(t, s - h_k(t) + g_k(t))] + \sum_{k=1}^{n} \int_{g_k(t)}^{t} x(s) \times \Big(\exp\Big\{ \int_{s+h_k(t)-g_k(t)}^{s} [R_k(\tau,\tau+) - T_k(\tau,\tau+)] d\tau \Big\} - 1 \Big) d_s T_k(t,s) = 0 \quad (4.9)$$

or

$$\dot{x}(t) + \sum_{k=1}^{n} \int_{g_k(t)}^{t} x(s) \Big(\exp\Big\{ \int_{s-g_k(t)+h_k(t)}^{s} [R_k(\tau,\tau+) - T_k(\tau,\tau+)] d\tau \Big\} - 1 \Big) \times d_s R_k(t, s - g_k(t) + h_k(t)) + \sum_{k=1}^{n} \int_{g_k(t)}^{t} x(s)\, d_s[R_k(t, s-g_k(t)+h_k(t)) - T_k(t,s)] = 0 \quad (4.10)$$

are oscillatory, then all solutions of (3.11) are also oscillatory.

Let us proceed to non-oscillation conditions.

Theorem 4.6. *Suppose (A1)-(A3), (A5) hold for (2.3) and there exists λ, $0 < \lambda < 1$, such that*

$$\limsup_{t\to\infty} \int_{h(t)}^{g(t)} [R(s,s+) - \lambda T(s,s+)]\, ds < \frac{1}{e} \ln \frac{1}{\lambda}, \quad (4.11)$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} [R(s,s+) - \lambda T(s,s+)] ds < \frac{1}{e}. \quad (4.12)$$

Then (2.3) has a non-oscillatory solution.

Proof. By (4.12) there exists $t_1 \ge t_0$ such that for $t \ge t_1$ the function

$$u(t) = e[R(t,t+) - \lambda T(t,t+)] \quad (4.13)$$

is a solution of the inequality

$$u(t) \ge \exp\Big\{ \int_{h(t)}^{t} u(\tau) d\tau \Big\} [R(t,t+) - \lambda T(t,t+)],$$

which can be rewritten in the form

$$\begin{aligned} u(t) &\ge \exp\Big\{ \int_{h(t)}^{t} u(\tau) d\tau \Big\} \Big[\int_{h(t)}^{t} d_s R(t,s) - \lambda \int_{g(t)}^{t} d_s T(t,s) \Big] \\ &= \exp\Big\{ \int_{h(t)}^{t} u(\tau) d\tau \Big\} \int_{h(t)}^{t} d_s[R(t,s) - \lambda T(t, s - h(t) + g(t))] \\ &= \int_{h(t)}^{t} \exp\Big\{ \int_{h(t)}^{t} u(\tau) d\tau \Big\} d_s[R(t,s) - \lambda T(t, s-h(t)+g(t))]. \end{aligned}$$

The function $R(t,s) - \lambda T(t, s-h(t)+g(t)) = (R(t,s) - T(t, s-h(t)+g(t))) + (1 - \lambda T(t, s-h(t)+g(t)))$ is nondecreasing as a sum of two nondecreasing functions, consequently, the integral becomes smaller if the function under the integral is changed by a smaller one:

$$\begin{aligned} u(t) &\geq \int_{h(t)}^{t} \exp\Big\{\int_s^t u(\tau)d\tau\Big\} d_s[R(t,s) - \lambda T(t, s-h(t)+g(t))] \\ &= \int_{h(t)}^{t} \exp\Big\{\int_s^t u(\tau)d\tau\Big\} d_sR(t,s) - \int_{g(t)}^{t} \exp\Big\{\int_s^t u(\tau)d\tau\Big\} d_sT(t,s) \\ &\quad + \Big[\int_{g(t)}^{t} \exp\Big\{\int_s^t u(\tau)d\tau\Big\} d_sT(t,s) \\ &\quad - \lambda\int_{h(t)}^{t} \exp\Big\{\int_s^t u(\tau)d\tau\Big\} d_sT(t, s-h(t)+g(t))\Big] \end{aligned} \tag{4.14}$$

Let us demonstrate that the expression in the brackets is nonnegative. Inequality (4.11) implies for u defined by (4.13), t being large enough and any $s \geq g(t)$:

$$\int_{s+h(t)-g(t)}^{s} u(\tau)d\tau \leq \ln\frac{1}{\lambda}, \tag{4.15}$$

which yields

$$\begin{aligned} &\int_{g(t)}^{t} \exp\Big\{\int_s^t u(\tau)d\tau\Big\} d_sT(t,s) - \lambda\int_{h(t)}^{t} \exp\Big\{\int_s^t u(\tau)d\tau\Big\} d_sT(t, s-h(t)+g(t)) \\ &= \int_{g(t)}^{t} \Big(\exp\Big\{\int_s^t u(\tau)d\tau\Big\} - \lambda\exp\Big\{\int_{s+h(t)-g(t)}^{t} u(\tau)d\tau\Big\}\Big) d_sT(t,s) \\ &= \int_{g(t)}^{t} \exp\Big\{\int_s^t u(\tau)d\tau\Big\}\Big(1 - \lambda\exp\Big\{\int_{s+h(t)-g(t)}^{s} u(\tau)d\tau\Big\}\Big) d_sT(t,s) \geq 0. \end{aligned}$$

By inequalities (4.14) and (4.15) we have

$$u(t) \geq \int_{h(t)}^{t} \exp\Big\{\int_s^t u(\tau)d\tau\Big\} d_sR(t,s) - \int_{g(t)}^{t} \exp\Big\{\int_s^t u(\tau)d\tau\Big\} d_sT(t,s),$$

Hence u is a nonnegative solution of (3.2). By Theorem 3.1 equation (2.3) has a non-oscillatory solution. □

Corollary 4.7. *Suppose (A1)-(A3), (A5) hold for (2.3) and*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \Big[R(s, s+) - \frac{1}{e}T(s, s+)\Big] ds < \frac{1}{e}. \tag{4.16}$$

Then (2.3) has a non-oscillatory solution.

This corollary is obtained by setting $\lambda = 1/e$ in Theorem 4.6. Similar to Theorem 4.6 the following result is obtained.

Theorem 4.8. *Suppose $n = m$, conditions (A1*)-(A3*), (add1) and the following inequality*

$$\limsup_{t\to\infty} \sum_{k=1}^{n} \int_{h_k(t)}^{t} [R_k(s, s+) - \frac{1}{e}T_k(s, s+)]ds < \frac{1}{e} \tag{4.17}$$

hold. Then (3.11) has a non-oscillatory solution.

Remark. The coefficient $\frac{1}{e}$ of $b(s)$ is not improvable. Indeed, for the equation

$$\dot{x}(t) + ax(t-\tau) - bx(t) = 0 \tag{4.18}$$

the inequality

$$a \le \frac{e^{b\tau}}{\tau e} \tag{4.19}$$

is necessary and sufficient for non-oscillation.

Theorem 4.9. *Suppose (A1)-(A3), (A5) hold for* (2.3) *and there exist nondecreasing for each* t *functions* $L(t,s), D(t,s)$*, such that*

$$D(t,s) \le T(t,s) \le R(t,s) \le L(t,s)$$

and there exist finite limits

$$\begin{aligned} B_{11} &= \lim_{t\to\infty} \int_{h(t)}^t L(s,s+)ds, \quad B_{12} = \lim_{t\to\infty} \int_{h(t)}^t D(s,s+)ds, \\ B_{21} &= \lim_{t\to\infty} \int_{g(t)}^t L(s,s+)ds, \quad B_{22} = \lim_{t\to\infty} \int_{g(t)}^t D(s,s+)ds. \end{aligned} \tag{4.20}$$

Suppose in addition that the system

$$\begin{aligned} \ln x_1 &> x_1 B_{11} - x_2 B_{12} \\ \ln x_2 &< x_1 B_{21} - x_2 B_{22} \end{aligned} \tag{4.21}$$

has a positive solution $\{x_1; x_2\}$ *such that eventually* $x_1 L(t,T+) \ge x_2 D(t,t+)$*. Then (2.3) has a non-oscillatory solution.*

Proof. Consider the function $u(t) = x_1 L(t,t+) - x_2 D(t,t+)$ which is eventually nonnegative. The system (4.21) yields

$$x_1 > \exp\{x_1 B_{11} - x_2 B_{12}\}, \quad x_2 < \exp\{x_1 B_{21} - x_2 B_{22}\}.$$

By definitions (4.20) there exists $t_1 \ge t_0$ such that for $t \ge t_1$

$$x_1 \ge \exp\Big\{x_1 \int_{h(t)}^t L(s,s+)ds - x_2 \int_{h(t)}^t D(s,s+)ds\Big\} = \exp\Big\{\int_{h(t)}^t u(s)ds\Big\},$$

$$-x_2 \ge -\exp\Big\{x_1 \int_{g(t)}^t L(s,s+)ds - x_2 \int_{g(t)}^t D(s,s+)ds\Big\} = -\exp\Big\{\int_{g(t)}^t u(s)ds\Big\}.$$

Similar to the definition of h, g in (h) let us define functions $H(t)$, $G(t)$ for $L(t,s), D(t,s)$. Then $H(t) \le h(t)$, $G(t) \ge g(t)$. Since $L(t,\cdot), D(t,\cdot)$ are nondecreasing for each t, then for any $t \ge t_1$

$$x_1 L(t,t+) = \int_{H(t)}^t x_1 d_s L(t,s) \ge \int_{h(t)}^t \exp\Big\{\int_{h(s)}^s u(\tau)d\tau\Big\} d_s L(t,s),$$

$$-x_2 D(t,t+) = -\int_{G(t)}^t x_2 d_s D(t,s) \ge -\int_{g(t)}^t \exp\Big\{\int_{g(s)}^s u(\tau)d\tau\Big\} d_s D(t,s).$$

The summation gives

$$u(t) \ge \int_{h(t)}^t \exp\Big\{\int_{h(s)}^s u(\tau)d\tau\Big\} d_s L(t,s) - \int_{g(t)}^t \exp\Big\{\int_{g(s)}^s u(\tau)d\tau\Big\} d_s D(t,s).$$

By Theorem 3.1 equation (2.3), where R, T are changed by L, D, respectively, has a non-oscillatory solution. Theorem 3.4 implis (2.3) also has a non-oscillatory solution. □

References

[1] R. Agarwal and S. Saker, Oscillation of solutions to neutral delay differential equations with positive and negative coefficients, *Int. J. Differ. Equ. Appl.* **2** (2001), no. 4, 449-465.

[2] N.V. Azbelev, V.P. Maksimov and L.F. Rakhmatullina, Introduction to the Theory of Linear Functional Differential Equations. World Federation Publishers Company, Atlanta, 1996.

[3] L. Berezansky and E. Braverman, On oscillation of equations with distributed delay, *Zeitschrift für Analysis und ihre Anwendungen* **20** (2001), No 2, 489-504.

[4] L. Berezansky, Y. Domshlak and E. Braverman, On oscillation properties of delay differential equations with positive and negative coefficients, *J. Math. Anal. Appl.* **274** (2002), 81–101.

[5] S.S. Cheng, X.-P. Guan, and J. Yang, Positive solutions of a nonlinear equation with positive and negative coefficients, *Acta Math. Hungar.* **86** (2000), 169-192.

[6] Q. Chuanxi and G. Ladas, Oscillation in differential equations with positive and negative coefficients, *Canad. Math. Bull.* **33** (1990), 442-451.

[7] Yu.I. Domshlak and A.I. Aliev, On oscillatory properties of the first order differential equations with one or two arguments, *Hiroshima Math. J.* **18** (1988), 31-46.

[8] N. Dunford and J.T. Schwartz, Linear Operators, Part 1: General Theory. New York: Interscience Publishers Inc., 1958.

[9] K. Farrell, E.A. Grove and G. Ladas, Neutral delay differential equations with positive and negative coefficients, *Appl. Anal.* **27** (1988), 181-197.

[10] W.S. He and W.T. Li, Nonoscillation and oscillation for neutral differential equations with positive and negative coefficients, *Int. J. Appl. Math.* **6** (2001), no. 2, 183-198.

[11] K. Kreith and G. Ladas, Allowable delays for positive diffusion processes, *Hiroshima Math. J.* **15** (1985), 437-443.

[12] G. Ladas, C.G. Philos and Y.G. Sficas, Oscillations of integro-differential equations. *Differential Integral Equations* **4** (1991), no. 5, 1113-1120.

[13] G. Ladas and Y.G. Sficas, Oscillation of delay differential equations with positive and negative coefficients. Proceedings of the International Conference on Qualitative Theory of Differential Equations, University of Alberta, June 18-20, 1984, 232-240.

[14] W. Li and H.S. Quan, Oscillation of higher order neutral differential equations with positive and negative coefficients, *Annals of Differential Eqs.* **11** (1995), 70-76.

[15] W. Li, H. Quan, and J.Wu, Oscillation of first order differential equations with variable coefficients, *Comm. in Appl. Anal.* **3** (1999), 1-13.

[16] W. Li and J. Yan, Oscillation of first order neutral differential equations with positive and negative coefficients, *Collect. Math.* **50** (1999), 199-209.

[17] L. Liao and A. Chen, Necessary and sufficient conditions for oscillations of neutral delay differential equations with several positive and negative coefficients, *Ann. Differential Equations* **12** (1996), 297-303.

[18] G.A. Wang, Oscillations of integro-differential equations with oscillating kernel, *Ann. Differential Equations* **8** (1992), no. 3, 361-366.

[19] J.S. Yu and J. Yan, Oscillation in first order differential equations with "integral smaller" coefficients, *J. Math. Anal. Appl.* **187** (1994), 361-370.

[20] X. Zhang and J. Yan, Oscillation criteria for first order neutral differential equations with positive and negative coefficients, *J. Math. Anal. Appl.* **253** (2001), 204-214.

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