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A multiplicity result for a class of superquadratic Hamiltonian systems *

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Abstract

We establish the existence of two nontrivial solutions to semilinear elliptic systems with superquadratic and subcritical growth rates. For a small positive parameter λ, we consider the system

$$\begin{gathered} -\Delta v = \lambda f(u) \quad \text{in } \Omega, \\ -\Delta u = g(v) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 1$. One solution is obtained applying Ambrosetti and Rabinowitz's classical Mountain Pass Theorem, and the other solution by a local minimization.

1 Introduction

The object of this paper is to establish the existence of two nontrivial solutions to a class of elliptic problems using a variational approach. In particular, we consider the problem for Hamiltonian systems

$$\begin{gathered} -\Delta v = \lambda f(u) \quad \text{in } \Omega, \\ -\Delta u = g(v) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ with $N \geq 1$, λ is a small positive parameter, and f, $g : \Omega \to \mathbb{R}$ are given continuous functions satisfying the following hypotheses:

(H1) There exists a positive constant C such that

$$|f(t)| \leq C(1 + |t|^r), \quad \text{for all } t \in \mathbb{R}.$$

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(H2) g is odd function which is increasing, $g(0) = 0$, and

$$\lim_{t\to+\infty} \frac{g(t)}{t^s} = 1,$$

where $r \geq 0$, $s > 0$, and

$$\frac{1}{r+1} + \frac{1}{s+1} > \frac{N-2}{N}\,, \tag{1.2}$$

when $N \geq 3$. However, for $N = 1, 2$ there is no restriction.

(H3) There are positive constants μ and R, with $(\mu - 1)s > 1$, such that for all $|u| \geq R$,

$$0 < \mu F(u) \leq u f(u)\,.$$

The inequality (1.2) expresses the subcritical character of system (1.1). Its superquadratic behavior is given by the Assumption (H3).

In recent years, various results on the existence of solutions for superlinear elliptic systems have been obtained. Among others, de Figueiredo and Felmer [7], and Hulshof and van der Vorst [9] study these problems by means of a variational approach that considers a Lagrangian formulation with strongly indefinite quadratic part and uses the generalized mountain pass theorem in its infinite dimensional setting due to Benci–Rabinowitz [2].

In system (1.1), we isolate v in the second equation obtaining the fourth order quasilinear scalar problem

$$\begin{gathered}\Delta(g^{-1}(\Delta u) = \lambda f(u) \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\\ \Delta u = 0 \quad \text{on } \partial\Omega\end{gathered} \tag{1.3}$$

whose two solutions may be obtained using variational techniques. Therefore, the solutions of system (1.1) may be obtained by means the critical points of the functional (1.4). One critical point is established using Ambrosetti and Rabinowitz's classical Mountain Pass Theorem (see for example [1]). To obtain the other critical point, we combine Ekeland's variational principle with local minimization.

It is well known that the solutions of equation (1.3) are the critical points of the associated functional

$$I_\lambda(u) = \frac{1}{p}\int_\Omega A(|\Delta u|^p)\,dx - \lambda\int_\Omega F(u)\,dx \tag{1.4}$$

defined on the reflexive Banach space $E = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm $\|u\| = \|\Delta u\|_{L^p}$ and with Fréchet derivative given by

$$\langle I'_\lambda(u), \phi\rangle = \int_\Omega a(|\Delta u|^p)|\Delta u|^{p-2}\Delta u \Delta\phi\,dx - \lambda\int_\Omega f(u)\phi\,dx, \quad \text{for all } \phi \in E, \tag{1.5}$$

where

$$A(t) = \int_0^t a(s)ds \quad \text{and} \quad F(t) = \int_0^t f(s)ds,$$

and where the function a is defined by

$$a(|t|^p)|t|^{p-2}t = g^{-1}(t), \tag{1.6}$$

with $p = (s+1)/s$. By hypotheses (H1) and (H2), and using standard arguments, we see that the expressions (1.4) and (1.5) are well defined, as well as that the functional I_λ is of class C^1. (See for example [6] and [11].) Note that the subcritical condition for equation (1.3) given by $r < p^{**} = Np/(N-2p)$ is equivalent to condition (1.2).

Next we consider a technical condition concerned with the regularity of the solutions of equation (1.3).

(H4) Assume either that

$$s \le 2,$$
$$\frac{3N-2}{2N}\frac{1}{r+1} + \frac{1}{s+1} \ge \frac{N-2}{N} + \frac{1}{(r+1)(s+1)}$$

and that g is a differentiable function such that its derivative g' is a Lipschitz continuous function; or that

$$s > 2,$$
$$\frac{1}{r+1} + \frac{1}{s+1} \le \frac{N-2}{N} + \frac{1}{(r+1)(s+1)}$$

and that g is of class C^2 with $g''(t) = O(|t|^{s-2})$ at infinity.

To state our main result, which will be proved in Section 3, we assume the almost homogeneity condition at zero:

(H5) There are positive constants r_0 and s_0 such that $r_0 s_0 < 1$,

$$\lim_{t\to 0}\frac{f(t\sigma)}{f(t)} = \sigma^{r_0} \quad \text{and} \quad \lim_{t\to 0}\frac{g(t\sigma)}{g(t)} = \sigma^{s_0}.$$

Theorem 1.1 *Assume hypotheses (H1), (H2), (H3), and (H5). Then there exists a positive constant λ^* such that for any $0 < \lambda < \lambda^*$, there exist at least two nontrivial critical points $u_{\lambda,1}$, $u_{\lambda,2} \in E$ of the functional I_λ such that $\|u_{\lambda,1}\|_E \to +\infty$ and $\|u_{\lambda,2}\|_E \to 0$ as $\lambda \to 0$. Moreover, if we assume that condition (H4) holds, then $(u_{\lambda,1}, g^{-1}(\Delta u_{\lambda,1}))$ and $(u_{\lambda,2}, g^{-1}(\Delta u_{\lambda,2}))$ are strong solutions of system (1.1).*

Observe that the fact that system (1.1) is superquadratic does not imply that its two scalar equations are superquadratic. We have an analogous remark for the subcritical behavior of system (1.1).

We should mention that the idea of isolating one variable of system (1.1) to obtain a scalar equation is similar in spirit to the one proposed in [4] for the study of the existence of both positive periodic solutions and homoclinic solutions of a class of Hamiltonian systems. However, the functional analytic framework of [4] is different from ours because, for instance, the nonlinearity $g(v)$ is a power.

For work on superlinear elliptic systems using a priori estimates and degree theory arguments, see for example [3, 12]) and the references therein. Costa and Magalhães [5] discuss other results for systems with Hamiltonian form.

This paper is organized as follows. In Section 2, we give an abstract framework in which we establish an abstract critical point theorem used in Section 3 to prove the main result.

2 The Abstract Framework

This section establishes an abstract critical point theorem used in Section 3 to prove our main result.

We first recall standard definitions and notation. Let X be a reflexive Banach space endowed with a norm $\|\cdot\|$. We let $\langle\cdot,\cdot\rangle$ denote the duality pairing between X and its dual X^*. We denote the weak convergence in X by "$\rightharpoonup$ ", and denote the strong convergence by "$\to$ ".

We say a mapping $T : X \to X^*$ *satisfies the condition* (S_+) if for every sequence $(u_n) \subset X$, with $u_n \rightharpoonup u$ in X, and $\limsup_{n\to+\infty}\langle T(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in X.

Let $I \in C^1(X,\mathbb{R})$. We will say I *satisfies the Palais–Smale condition*, denoted (PS) condition, if every Palais–Smale sequence of I (or in other words a sequence $(u_n) \subset X$ such that $(I(u_n))$ is bounded and $I'(u_n) \to 0$ in the dual space X^*) is relatively compact.

Lemma 2.1 *Let $\Phi, \Psi : X \to \mathbb{R}$ be C^1 functionals satisfying for all $u \in X$,*

$$\begin{gathered} \mu\Phi(u) - \langle\Phi'(u), u\rangle \ge M\|u\|^p - N, \\ \mu\Psi(u) - \langle\Psi'(u), u\rangle \le Q \end{gathered} \tag{2.1}$$

with $\mu > p > 1$, and where M, N and Q are positive constants. Then every Palais–Smale sequence of the functional $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ is bounded.

Proof. Let $(u_n) \subset X$ be a Palais–Smale sequence, that is,

$$\Phi(u_n) - \lambda\Psi(u_n) \to c \tag{2.2}$$

and

$$|\langle\Phi'(u_n), v\rangle - \lambda\langle\Psi'(u_n), v\rangle| \le \epsilon_n\|v\| \tag{2.3}$$

where $\epsilon_n \to 0$ as $n \to +\infty$. Multiplying (2.2) by μ, subtracting (2.3), with $v = u_n$, from the expression obtained, using (2.1) we conclude that

$$\begin{aligned} 1 + \mu c + \epsilon_n\|u_n\| &\ge \mu\Phi(u_n) - \Phi'(u_n)u_n + \lambda(\Psi'(u_n)u_n - \mu\Psi(u_n)) \\ &\ge M\|u_n\|^p - N - \lambda Q. \end{aligned}$$

Consequently, (u_n) is bounded in X, since $p > 1$. $\square$

Lemma 2.2 *Let $\Phi, \Psi : X \to \mathbb{R}$ be functionals which satisfy the hypotheses of Lemma 2.1 such that Φ' belongs to the class $(S)_+$, and such that for every sequence (u_n) in X, with $u_n \rightharpoonup u$, we have $\lim_{n\to\infty}\langle \Psi'(u_n), u_n - u\rangle = 0$. Then the functional $I : X \to \mathbb{R}$ given by $I(u) = \Phi(u) - \lambda\Psi(u)$ satisfies the Palais-Smale condition.*

Proof. Let $(u_n) \subset X$ be a Palais-Smale sequence. According to Lemma 2.1, we have (u_n) is bounded in X. Hence we may take a subsequence, which is denoted by the same index, such that $u_n \rightharpoonup u$, for some u in X. Since $I'(u_n) \to 0$ in X^*, we have

$$|\langle \Phi'(u_n), u_n - u\rangle - \lambda\langle \Psi'(u_n), u_n - u\rangle| \le \epsilon_n \|u_n - u\|$$

where $\epsilon_n \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle = 0$, because $\lim_{n\to\infty}\langle \Psi'(u_n), u_n - u\rangle = 0$. Now since Φ' belongs to the class $(S)_+$, we conclude $u_n \to u$ in X. $\square$

The main result of this section, which will be proved in Subsection 2.1, is the following.

Theorem 2.3 *Let $\Phi, \Psi : X \to \mathbb{R}$ be functionals satisfying the hypotheses of Lemma 2.2. Further, suppose that the following three conditions are satisfied:*

(I1) $C_1\|u\|^p \le \Phi(u) \le C_2\|u\|^p + C_3$, *for all* $u \in X$.

(I2) $C_4\|u\|^\mu - C_5 \le \Psi(u) \le C_6\|u\|^r + C_7$, *for all* $u \in X$.

(I3) There is $v \in X - \{0\}$ such that

$$\lim_{t\to 0}\frac{\Psi(tv)}{\Phi(tv)} = +\infty,$$

where $r > \mu > p > 1$ and $C_1, \dots, C_7$ are positive constants. Then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, there exist two nontrivial critical points $\{u_\lambda, v_\lambda\}$ of the functional $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ such that $\|u_\lambda\| \to +\infty$ and $\|v_\lambda\| \to 0$ as $\lambda \to 0$.

Nest, we present three lemmas which lead to the proof of Theorem 2.3.

Lemma 2.4 *Let $\Phi, \Psi : X \to \mathbb{R}$ be functionals satisfying for all $u \in X$,*

$$\begin{gathered}\Phi(u) \ge C_1\|u\|^p,\\ \Psi(u) \le C_6\|u\|^r + C_7\end{gathered} \tag{2.4}$$

where $r > p > 1$ and C_1, C_6 and C_7 are positive constants. Then there exist positive constants α_λ, ρ_λ such that

$$\lim_{\lambda\to 0^+}\rho_\lambda = +\infty$$

and

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) > \alpha_\lambda, \quad \textit{if} \quad \|u\| = \rho_\lambda.$$

Proof. By assumptions (2.4) we have

$$I_\lambda(u) \geq C_1\|u\|^p - \lambda C_6\|u\|^r - \lambda C_7, \quad \text{for all } u \in X. \tag{2.5}$$

Choosing $u \in X$ such that

$$\|u\| = \lambda^{-s}, \quad \text{with } 0 < s(r-p) < 1, \tag{2.6}$$

and setting $\rho_\lambda = \lambda^{-s}$, we obtain

$$I_\lambda(u) \geq C_1\lambda^{-sp} - C_6\lambda^{1-sr} - \lambda C_7.$$

Taking $\alpha_\lambda = C_1\lambda^{-sp} - C_6\lambda^{1-sr} - \lambda C_7$, the Lemma results from the inequality $0 < s(r-p) < 1$. □

Lemma 2.5 *Let $\Phi, \Psi : X \to \mathbb{R}$ be functionals satisfying for all $u \in X$,*

$$\begin{aligned} \Phi(u) &\leq C_2\|u\|^p + C_3, \\ \Psi(u) &\geq C_4\|u\|^\mu - C_5 \end{aligned} \tag{2.7}$$

where $\mu > p > 1$ and C_2, C_3, C_4, and C_5 are positive constants. Then $I_\lambda(tu) = \Phi(tu) - \lambda\Psi(tu) \to -\infty$ as $t \to +\infty$, for all $u \in X - \{0\}$.

Proof. It follows easily from (2.7) that for all $t > 0$,

$$I_\lambda(tu) \leq C_2t^p\|u\|^p + C_3 - C_4t^\mu\lambda\|u\|^\mu + C_5, \quad \text{for all } u \in X.$$

Since $\mu > p$, the result follows. □

Lemma 2.6 *Let $\Phi, \Psi : X \to \mathbb{R}$ be C^1 functionals which satisfy the (PS) condition and the assumptions (2.4). Furthermore, suppose there exits $v \in X - \{0\}$ such that*

$$\lim_{t\to 0} \frac{\Psi(tv)}{\Phi(tv)} = +\infty. \tag{2.8}$$

Then there exits $\tilde\lambda > 0$ such that for all $\lambda \in (0, \tilde\lambda)$, the functional $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ has a nontrivial critical point v_λ such that $\|v_\lambda\| \to 0$ as $\lambda \to 0$, provided that $I_\lambda(0) = 0$.

Proof. Let $\alpha \in \mathbb{R}$ be such that $\alpha p < 1$, and let $u \in X$ such that $\|u\| = \lambda^\alpha$. According to inequality (2.5), there exists $\tilde\lambda > 0$ such that for all $\lambda \in (0, \tilde\lambda)$,

$$I_\lambda(u) \geq C_1\lambda^{\alpha p} - C_6\lambda^{1+\alpha r} - \lambda C_7 \geq 0.$$

It follows from the first inequality of (2.4) and the assumption (2.8) that there is $\delta > 0$ such that $\Psi(tv) > 0$, for all $|t| \leq \delta$. Therefore, given $\lambda \in (0, \tilde\lambda)$, there exists $t_\lambda \in (-\delta, \delta)$ such that

$$I_\lambda(t_\lambda v) = \Phi(t_\lambda v) - \lambda\Psi(t_\lambda v) = \Psi(t_\lambda v)\Big(\frac{\Phi(t_\lambda v)}{\Psi(t_\lambda v)} - \lambda\Big) < 0.$$

Hence the infimum of the functional I_λ in $B_X[0, \lambda^\alpha]$ is negative, where $B_X[0, R]$ denotes the closed ball with radius R centered at origin of X. Applying Ekeland's variational principle, we obtain a sequence $(u_n) \subset B_X[0, \lambda^\alpha]$ such that $I_\lambda(u_n) \to \inf_{B_X[0,\lambda^\alpha]} I_\lambda$ and $I'_\lambda(u_n) \to 0$. Since I_λ satisfies the (PS) condition, and since $I_\lambda(0) = 0$, we obtain a nontrivial minimizer u_λ, which completes the proof of the Lemma. □

Proof of Theorem 2.3 By Lemma 2.2, the functional I_λ satisfies the (PS) condition. According to Lemmas 2.4 and 2.5, we may apply the Mountain Pass Theorem. It follows that there exists $\widehat{\lambda} > 0$ such that for all $\lambda \in (0, \widehat{\lambda})$, the functional I_λ has a critical point u_λ such that $I_\lambda(u_\lambda) > \alpha_\lambda > 0$ and $\|u_\lambda\| \geq \rho_\lambda = \lambda^{-s} \to +\infty$ as $\lambda \to 0$. Finally, since the functional I_λ satisfies the (PS) condition, by Lemma 2.6, we can take a suitable small λ^* such that, for all $\lambda \in (0, \lambda^*)$, the functional I_λ has another critical point v_λ such that $I_\lambda(v_\lambda) < 0$ and $\|v_\lambda\| \leq \lambda^\alpha \to 0$ as $\lambda \to 0$. The proof of Theorem 2.3 is now complete.

On the (S_+) condition

The next two lemmas are crucial results in our minimax argument.

Lemma 2.7 *Let $L : D(L) \to (L^p(\Omega))^k$ be a continuous, injective linear operator defined on the Banach space $D(L)$ endowed with the norm*

$$\|u\|^p = \int |Lu|^p,$$

where $|\cdot|$ denotes a norm of $\mathbb{R}^k$. Then the derivative of the functional $J : D(L) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p} \int_\Omega |Lu|^p \, dx$$

belongs to the class (S_+).

Proof. Let $(u_n) \subset D(L)$ be such that

$$u_n \rightharpoonup u \text{ in } D(L) \quad \text{and} \quad \lim_{n\to\infty} \sup \langle J'(u_n), u_n - u\rangle \leq 0.$$

Note that since $u_n \rightharpoonup u$ in $D(L)$, and since L is continuous, we have

$$\mathfrak{J}_n = \frac{1}{p} \int_\Omega |Lu|^{p-2} Lu(Lu_n - Lu)\, dx \to 0. \tag{2.9}$$

Now according to the Hölder inequality and using the elementary inequality

$$|x-y|^p \leq c(p)\{(|x|^{p-2}x - |y|^{p-2}y)(x-y)\}^{s/2}\{|x|^p + |y|^p\}^{1-s/2},$$

where $s = p$ if $p \in (1,2)$, $s = 2$ if $p \geq 2$, and $c(p)$ is a positive constant depending only on p, we see that

$$\begin{aligned}
\langle J'(u_n), u_n - u\rangle &= \int_\Omega |Lu_n|^{p-2} Lu_n (Lu_n - Lu)\, dx \\
&= \int_\Omega [|Lu_n|^{p-2}Lu_n - |Lu|^{p-2}Lu] L(u_n - u)\, dx + \mathfrak{J}_n \qquad (2.10)\\
&\geq C\Big\{\int_\Omega |Lu_n|^p + |Lu|^p\Big\}^{s/2-1} \int_\Omega |L(u_n - u)|^p + \mathfrak{J}_n.
\end{aligned}$$

Finally, we combine (2.9) and (2.10) with the fact that

$$\lim_{n\to\infty} \sup\langle J'(u_n), u_n - u\rangle \le 0$$

to obtain

$$\lim_{n\to\infty} \int_\Omega |L(u_n - u)|^p \, dx = 0,$$

which completes the proof of the lemma. $\square$

Next let $a \in C(\mathbb{R}^+, \mathbb{R})$ and define $A(s) = \int_0^s a(t)dt$ such that the following two conditions are satisfied

(A1) The function $h(t) = A(|t|^p)$ is strictly convex.

(A2) There are positive constants c_0, c_1, c_2, and c_4 such that

$$c_0 t - c_1 \le A(t) \le c_2 t - c_3, \quad \text{for all } t > 0.$$

Consider the functional $\Phi : D(L) \to \mathbb{R}$ given by

$$\Phi(u) = \frac{1}{p} \int_\Omega A(|Lu|^p) \, dx.$$

It is well known that Φ is well defined and that it is a C^1 functional [11], with Fréchet derivative given by

$$\langle \Phi'(u), v\rangle = \int_\Omega a(|Lu|^p)|Lu|^{p-2} Lu Lv \, dx.$$

The following is an important convergence criterion.

Lemma 2.8 *The derivative of the functional Φ belongs to the class (S_+).*

Proof. Let $(u_n) \subset D(L)$ be such that

$$u_n \rightharpoonup u \text{ in } D(L) \quad \text{and} \quad \lim_{n\to\infty} \sup\langle \Phi'(u_n), u_n - u\rangle \le 0.$$

We will show that

$$\lim_{n\to\infty} \sup \int_\Omega |Lu_n|^{p-2} Lu_n \cdot (Lu_n - Lu) \, dx = 0, \tag{2.11}$$

and the proof will then follow from Lemma 2.7. First note that

$$\begin{aligned} &\langle \Phi'(u_n), u_n - u\rangle \\ &= \int_\Omega [a(|Lu_n|^p)|Lu_n|^{p-2} Lu_n - a(|Lu|^p)|Lu|^{p-2} Lu] \cdot (Lu_n - Lu) \, dx + \mathfrak{K}_n, \end{aligned}$$

where

$$\mathfrak{K}_n = \int_\Omega a(|Lu|^p)|Lu|^{p-2} Lu] \cdot (Lu_n - Lu) \, dx.$$

Since $u_n \rightharpoonup u$ in $D(L)$, and since a and L are continuous functions, we have $\mathfrak{K}_n \to 0$. Also, since the function $h(t) = A(|t|^p)$ is strictly convex, and since $\lim_{n\to\infty} \sup\langle \Phi'(u_n), u_n - u\rangle \le 0$, it follows that $Lu_n(x) \to Lu(x)$ almost everywhere in Ω.

Now observe that if $x \in \Omega$ is such that

$$Lu_n(x)(Lu_n(x) - Lu(x)) \le 0,$$

then

$$Lu_n(x)Lu_n(x) \le Lu_n(x)Lu(x) \le |Lu_n(x)||Lu(x)|,$$

which implies

$$|Lu_n(x)| \le |Lu(x)|.$$

On the other hand, if $x \in \Omega$ is such that

$$Lu_n(x)(Lu_n(x) - Lu(x)) \ge 0,$$

then by (A2), there exist positive constants C and M such that

$$a(|Lu_n|^p)|Lu_n|^{p-2}Lu_n(Lu_n - Lu) \ge C|Lu_n|^{p-2}Lu_n(Lu_n - Lu),$$

when $|Lu_n(x)| > M$. Set

$$\eta_n(x) = |Lu_n(x)|^{p-2}Lu_n(x) \cdot (Lu_n(x) - Lu(x))$$

and consider the sets

$$\begin{aligned} \mathcal{A}_n &= \{x \in \Omega : |Lu_n(x)| \le M\}, \\ \mathcal{B}_n &= \{x \in \Omega : |Lu_n(x)| > M\}, \\ \mathcal{C}_n &= \{x \in \Omega : \eta_n(x) \ge 0\}, \\ \mathcal{D}_n &= \{x \in \Omega : \eta_n(x) < 0\}. \end{aligned}$$

Then

$$\int_\Omega \eta_n(x)\,dx = \int_\Omega \eta_n \chi_{\mathcal{A}_n}\chi_{\mathcal{C}_n}\,dx + \int_\Omega \eta_n \chi_{\mathcal{B}_n}\chi_{\mathcal{C}_n}\,dx + \int_\Omega \eta_n \chi_{\mathcal{D}_n}\,dx,$$

where $\chi_{\mathcal{U}}$ denotes the characteristic function of the a set $\mathcal{U} \subset \mathbb{R}^N$. By the Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} \int_\Omega \eta_n \chi_{\mathcal{A}_n}\chi_{\mathcal{C}_n}\,dx = \int_\Omega \eta_n \chi_{\mathcal{D}_n}\,dx = 0. \tag{2.12}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega \eta_n \chi_{\mathcal{B}_n} \chi_{\mathcal{C}_n}\,dx \\
&\le C \int_\Omega a(|Lu_n|^p)\eta_n \chi_{\mathcal{B}_n} \chi_{\mathcal{C}_n}\,dx \\
&= C \int_\Omega a(|Lu_n|^p)\eta_n \chi_{\mathcal{B}_n}(1-\chi_{\mathcal{D}_n})\,dx \\
&= C \int_\Omega a(|Lu_n|^p)\eta_n \chi_{\mathcal{B}_n}\,dx - C\int_\Omega a(|Lu_n|^p)\eta_n \chi_{\mathcal{B}_n}\chi_{\mathcal{D}_n}\,dx \\
&= C \int_\Omega a(|Lu_n|^p)\eta_n (1-\chi_{\mathcal{A}_n})\,dx - C\int_\Omega a(|Lu_n|^p)\eta_n \chi_{\mathcal{B}_n}\chi_{\mathcal{D}_n}\,dx.
\end{aligned}$$

These estimates together with the Lebesgue dominated convergence theorem and the fact that $\lim_{n\to\infty}\sup\langle \Phi'(u_n), u_n - u\rangle \le 0$ imply

$$\lim_{n\to\infty}\sup \int_\Omega \eta_n \chi_{\mathcal{B}_n}\chi_{\mathcal{C}_n}\,dx = 0. \tag{2.13}$$

The equality (2.11) now follows from (2.12) and (2.13). The Lemma results from (2.11) and the fact that $u_n \to u$ in $D(L)$. □

3 Proof of Theorem 1.1

Existence of critical point for functional I_λ in (1.4)

This part of the proof is an application of Theorem 2.3. Consider the functionals

$$\Phi(u) = \frac{1}{p}\int_\Omega A(|\Delta u|^p)\,dx \quad \text{and} \quad \Psi(u) = \int_\Omega F(u)\,dx$$

defined on the Banach space $E = W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. Next we check that the conditions of Theorem 2.3 are satisfied. By our assumption (H2) and (1.6), it is standard to prove that assumptions (A1) and (A2) of Lemma 2.8 hold with $c_1 = 0$ and, of course, the condition (I1) of Theorem 2.3 holds with $L = \Delta$. Thus by Lemma 2.8, the derivative of the functional Φ belongs to the class (S_+). Furthermore, from assumptions (H1), (H2), and (H3) it is easy to see that conditions (2.1) and (I2) hold, and using the Sobolev imbedding theorem we have

$$\lim_{n\to\infty}\langle \Psi'(u_n), u_n - u\rangle = \lim \int_\Omega f(x,u_n)(u_n-u) = 0,$$

for every sequence (u_n) in E such that $u_n \rightharpoonup u$.

Then, using assumption (H4), we show that condition (I3) holds, i.e., there exists $v \in E$ such that

$$\lim_{t\to 0}\frac{\Psi(tv)}{\Phi(tv)} = +\infty. \tag{3.1}$$

First note that by assumption (H5),

$$\lim_{t\to 0}\frac{F(t)}{A(t^p)}=+\infty. \tag{3.2}$$

Let $v\in C_0^\infty(\Omega,[0,1])\backslash\{0\}$ such that $0\le \Delta v\le 1$. We have

$$\frac{\Psi(tv)}{\Phi(tv)}=\frac{p\int_\Omega \frac{F(tv)}{F(t)}\,dx}{\int_\Omega \frac{A(t^p|\Delta v|^p)}{A(t^p)}\,dx}\frac{F(t)}{A(t^p)}. \tag{3.3}$$

Also, from (H5), by the Lebesgue dominated convergence theorem we get

$$\lim_{t\to 0^+}\int_\Omega \frac{F(tv)}{F(t)}\,dx=\int_\Omega v^{r_0+1}\,dx$$

and

$$\lim_{t\to 0^+}\int \frac{A(t^p|\Delta v|^p)}{A(t^p)}\,dx=\int_\Omega |\Delta v|^p\,dx.$$

Therefore, passing to the limit in (3.3) and using (3.2) we obtain (3.1).

Regularity and the existence of solutions for system (1.1)

Here we prove that critical points of the functional I_λ are indeed strong solutions of problem (1.3).

Proposition 4 *Let $u\in E=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ be a critical point of the functional I_λ. Then $u\in W^{4,l}(\Omega)$ for some $l>1$.*

As a consequence of Proposition 4 we see that $u\in W^{4-(1/l),l}(\partial\Omega)$. Hence, integration by parts shows that u satisfies the second boundary condition of problem (1.3). Therefore the pair $(u,g^{-1}(\Delta u))$ is a strong solution of system (1.1).

Proof of Proposition 4. Assume that $N>2p$ (the other case is easier). Using the continuous imbedding $W^{2,p}(\Omega)\hookrightarrow L^q(\Omega)$ for $q=Np/(N-2p)$ and assumption (H1) we see that $f(u)\in L^{q/r}(\Omega)$. Thus, from standard regularity argument we get that $v=g^{-1}(\Delta u)$ belongs to $W^{2,q/r}(\Omega)$ which is continuously imbedded in the Sobolev space $W^{1,r_1}(\Omega)$ with $r_1=Nq/(Nr-q)$ (see [8]).

To obtain $u\in W^{4,l}(\Omega)$ for some $l>1$, it is enough to prove that $g(v)\in W^{2,l}(\Omega)$, that is, $g'(v)\partial v/\partial x_i\in W^{1,l}(\Omega)$. We separate the proof into two cases.

Case 1. Assume that

$$\begin{gathered} s\le 2,\\ \frac{3N-2}{2N}\frac{1}{r+1}+\frac{1}{s+1}\ge\frac{N-2}{N}+\frac{1}{(r+1)(s+1)} \end{gathered} \tag{4.1}$$

and that g is a differentiable function such that its derivative g' is a Lipschitz continuous function. In this case, it is well known that $g'(v) \in W^{1,r_1}(\Omega)$. This fact together with (4.1) implies that $g'(v)\partial v/\partial x_i \in W^{1,l}(\Omega)$ for some $l > 1$. Hence, Proposition 4 is proved in case 1.

Case 2. Assume that

$$\begin{gathered} s > 2, \\ \frac{1}{r+1} + \frac{1}{s+1} \leq \frac{N-2}{N} + \frac{1}{(r+1)(s+1)} \end{gathered} \tag{4.2}$$

and that g is of class C^2 with $g''(t) = O(|t|^{s-2})$ at infinity. In this case, we use the following result concerning superposition mapping on Sobolev space, due to Marcus and Mizel [10].

Let $\mathfrak{M}(\Omega)$ denote the space of real measurable functions in Ω. Given a Borel function $h : \mathbb{R} \to \mathbb{R}$ we define the superposition mapping $T_h : \mathfrak{M}(\Omega) \to \mathfrak{M}(\Omega)$ by $T_h u \doteq h \circ u$.

Proposition 5 *Assume that η, ξ are two numbers such that $1 < \eta \leq \xi < N$. Then T_h maps $W^{1,\xi}(\Omega)$ into $W^{1,\eta}(\Omega)$ if and only if the following conditions hold*

1. *The function h is locally Lipschitz in $\mathbb{R}$;*
2. *the first order derivative of h satisfies the inequality*

$$|h'(t)| \leq C(1 + |t|^R) \quad \textit{almost everywere in } \mathbb{R},$$

where C is a positive constant and $R = \dfrac{N(\xi - \eta)}{\eta(N - \xi)}$.

We know that $v \in W^{1,\xi}(\Omega)$ where $\xi = Nq/(Nr - q)$, thus using that g is of class C^2 and $g''(t) = O(|t|^{s-2})$ at infinity, we get that $g'(v) \in W^{1,\eta}(\Omega)$ where

$$s - 2 = \frac{N(\xi - \eta)}{\eta(N - \xi)},$$

thus

$$\eta = \frac{N\xi}{(N - \xi)(s - 2) + N}.$$

We must have that $\eta \leq \xi$, i.e., $(N - \xi)(s - 2) \geq 0$, since $s > 2$. This condition is equivalent to

$$N \geq \xi = \frac{Nq}{Nr - q},$$

that is,

$$\frac{N-2}{N} + \frac{1}{(r+1)(s+1)} \geq \frac{1}{r+1} + \frac{1}{s+1}.$$

This completes the proof of Proposition 4. $\square$

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