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Nonlinear singular Navier problem of fourth order*

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Abstract

We present an existence result for a nonlinear singular differential equation of fourth order with Navier boundary conditions. Under appropriate conditions on the nonlinearity $f(t,x,y)$, we prove that the problem

$$\begin{gathered} L^2u = L(Lu) = f(.,u,Lu) \quad \text{a.e. in } (0,1), \\ u'(0) = 0, \quad (Lu)'(0) = 0, \quad u(1) = 0, \quad Lu(1) = 0. \end{gathered}$$

has a positive solution behaving like $(1-t)$ on $[0,1]$. Here L is a differential operator of second order, $Lu = \frac{1}{A}(Au')'$. For $f(t,x,y) = f(t,x)$, we prove a uniqueness result. Our approach is based on estimates for Green functions and on Schauder's fixed point theorem.

1 Introduction

Dalmasso [1] studied the existence of positive radial solutions for the Dirichlet problem

$$\begin{aligned} \Delta^2 u &= f(u) \quad \text{in } B_R, \\ u = \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial B_R, \end{aligned} \tag{1.1}$$

and for the Navier problem

$$\begin{aligned} \Delta^2 u &= f(u) \quad \text{in } B_R, \\ u = \Delta u &= 0 \quad \text{on } \partial B_R, \end{aligned} \tag{1.2}$$

where B_R denotes the ball of radius R centered at the origin in $\mathbb{R}^n$ $(n \geq 1)$, ∂B_R is the boundary of B_R, and $\frac{\partial}{\partial \nu}$ is the outward normal derivative.

Since only positive radial solutions are considered, problems (1.1) and (1.2) reduce to the one-dimensional equation

$$\Delta^2 u = f(u) \quad \text{in } [0,R),$$

with respective boundary conditions $u(R) = u'(R) = 0$ and $u(R) = \Delta u(R) = 0$, where Δ denotes the polar form of the Laplacian (i.e. $\Delta u = \frac{1}{t^{n-1}}(t^{n-1}u')'$).

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The main result of Dalmasso in [1], was an existence result when f is a positive sublinear function which is continuous and nondecreasing on $[0,\infty)$. If $f(u) = |u|^p$ ($p \in (0,1) \cup (1,\infty)$), Dalmasso proved uniqueness for (1.1) and (1.2).

In [3], we considered a more general type of equation having as a linear part the singular operator of second order

$$Lu = \frac{1}{A}(Au')',$$

where A satisfies some appropriate conditions. In fact, we were interested in the positive solutions of the nonlinear Dirichlet problem of fourth order

$$\begin{gathered} L^2u = L(Lu) = f(.,u) \quad \text{in } (0,1), \\ u'(0) = 0, \quad (Lu)'(0) = 0, \quad u'(1) = 0, \quad u(1) = 0. \end{gathered}$$

We proved an existence and a uniqueness result which generalize the result of Dalmasso [1] for problem (1.1).

In this paper, we study the existence for the Navier problem of fourth order related to the operator L. More precisely, we consider the nonlinear Navier problem

$$\begin{gathered} L^2u = L(Lu) = f(.,u,Lu) \quad \text{a.e. in } (0,1), \\ u'(0) = 0, \quad (Lu)'(0) = 0, \quad u(1) = 0, \quad Lu(1) = 0. \end{gathered} \tag{1.3}$$

Here, we use the following assumptions:

(H1) A is continuous on $[0,1]$, infinitely differentiable and positive on $(0,1]$.

(H2) The function $h : t \mapsto \frac{1}{A(t)}\int_0^t A(s)ds$ is continuously differentiable on $[0,1]$, with $h(0) = 0$.

(H3) $f : [0,1) \times (0,\infty) \times (-\infty,0) \to (0,\infty)$ is measurable and continuous with respect to the second and third variables.

(H4) f is non-increasing with respect to the second variable and nondecreasing with respect to the third variable.

(H5) For all $c > 0$, $\int_0^1 G(0,s)f(s,c(1-s),-c(1-s))ds < \infty$, where $G(t,s) = A(s)\Gamma(t,s) = A(s)\int_{t\vee s}^1 \frac{dr}{A(r)}$ and $t \vee s = \max(t,s)$.

Note that without loss of generality, we can assume that $\int_0^1 A(s)ds = 1$.

Our paper is organized as follows. In section 2, we give some estimates on the Green function $H(x,y)$ of the operator L^2 with Navier conditions, which enable us to establish the existence result for problem (1.3). The main result of the paper is proved in section 3. Namely, the existence of positive solutions $u \in C^2([0,1])$ of (1.3) behaving like $(1-t)$, for $t \in [0,1]$. In section 4, we give a uniqueness result of (1.3) with the special nonlinearity $f(t,x,y) = f(t,x)$.

Throughout this paper, the letter C will denote a generic positive constant which may vary from line to line and for a nonnegative measurable function f in $[0,1]$, we use the notation

$$Vf(t) = \int_0^1 G(t,s)f(s)ds = \int_t^1 \frac{1}{A(r)}\Big(\int_0^r A(s)f(s)ds\Big)dr$$

and

$$V^2 f(t) = V(Vf)(t) = \int_0^1 H(t,s)f(s)ds.$$

We point out that if f is a nonnegative function in $L^1_{\mathrm{loc}}([0,1])$, then

$$L(Vf) = -f \quad \text{a.e. in } [0,1]. \tag{1.4}$$

2 Estimates on the Green function

The Green function H of the operator L^2 with boundary conditions $u'(0) = 0, (Lu)'(0) = 0,\ u(1) = 0, Lu(1) = 0$ is explicitly determined in the following lemma.

Lemma 2.1 *Assume (H1) and (H2). Then for* t,s *in* $[0,1]$*, we have*

$$H(t,s) = \int_0^1 G(t,r)G(r,s)dr = A(s)\int_t^1 \frac{1}{A(\xi)}\Big(\int_0^\xi A(r)\Gamma(r,s)dr\Big)d\xi. \tag{2.1}$$

Moreover, H has the following properties

$$\Big|\frac{\partial^2}{\partial t^2}H(t,s)\Big| \le CG(0,s). \tag{2.2}$$

$$-CG(0,s) \le \frac{\partial}{\partial t}H(t,s) \le 0. \tag{2.3}$$

$$0 \le H(t,s) \le C(1-t)G(0,s). \tag{2.4}$$

Proof For $(t,s) \in (0,1] \times [0,1]$, we have

$$\begin{aligned}
\Big|\frac{\partial^2}{\partial t^2}H(t,s)\Big| &= \Big| - A(s)\Gamma(t,s) + A(s)\frac{A'(t)}{A^2(t)}\int_0^t A(r)\Gamma(r,s)ds\Big| \\
&\le A(s)\Gamma(0,s)|h'(t)|.
\end{aligned}$$

So from (H2), we obtain the inequality (2.2). To prove (2.3), we have

$$0 \le -\frac{\partial}{\partial t}H(t,s) = \frac{A(s)}{A(t)}\int_0^t A(r)\Gamma(r,s)ds \le G(0,s)h(t).$$

Now, since h is continuous on $[0,1]$, we deduce (2.3). Combining this with $H(1,s) = 0$, we obtain (2.4). ♢

Proposition 2.2 *Assume (H1) and (H2). Let $\delta \in (0,1]$, then there exists a positive constant $C(\delta)$ such that for all $t,s \in [0,1]$, we have*

$$G(t,s) \geq C(\delta)(1-t)G(\delta,s). \tag{2.5}$$

$$H(t,s) \geq C(\delta)(1-t)H(\delta,s). \tag{2.6}$$

Proof To prove (2.5), we distinguish the following cases:
Case 1: $0 \leq t \leq \delta \leq 1$. For any $s \in [0,1]$, the function $G(.,s)$ is non-increasing on $[0,1]$. So, we obtain the result with $C(\delta) = 1$.
Case 2: $0 < \delta \leq t \leq 1$. Since A is continuous and positive on $[\delta, 1]$, then there exist two positive constants a, b such that

$$a \leq \frac{1}{A(r)} \leq b, \quad \text{for } r \in [\delta, 1].$$

We claim that $C(\delta) = a/b$. Indeed,

$$\begin{aligned} \Gamma(t,s) - \frac{a}{b}(1-t)\Gamma(\delta,s) &= \int_{t\vee s}^1 \frac{dr}{A(r)} - \frac{a}{b}(1-t)\int_{\delta\vee s}^1 \frac{dr}{A(r)} \\ &\geq a[(1-t\vee s) - (1-t)(1-\delta\vee s)] \\ &\geq 0. \end{aligned}$$

Then $G(t,s) \geq \frac{a}{b}(1-t)G(\delta,s)$, for $t,s \in [0,1]$ and (2.5) is deduced.

Now, we shall prove (2.6). For $t,s \in [0,1]$, we have

$$H(t,s) - \frac{a}{b}(1-t)H(\delta,s) = A(s)K(t,s),$$

where

$$K(t,s) = \int_s^1 \frac{1}{A(\theta)}\Big(\int_0^\theta G(t,r)dr\Big)d\theta - \frac{a}{b}(1-t)\int_s^1 \frac{1}{A(\theta)}\Big(\int_0^\theta G(\delta,r)dr\Big)d\theta.$$

So, from (2.5), we deduce that

$$\frac{\partial}{\partial s}K(t,s) = \frac{1}{A(s)}\int_0^s (\frac{a}{b}(1-t)G(\delta,r) - G(t,r))dr \leq 0,$$

which together with $K(t,1) = 0$ imply that K is nonnegative on $[0,1] \times [0,1]$. Thus (2.6) holds. $\diamondsuit$

3 Existence results

In this section, we prove existence of a positive solution for (1.3). We begin by stating an existence result for the nonlinear problem

$$\begin{gathered} L^2u = L(Lu) = f(.,u,Lu) \quad \text{a.e. in } (0,1), \\ u'(0) = 0, \quad (Lu)'(0) = 0, \quad u(1) = \alpha, \quad Lu(1) = -\beta. \end{gathered} \tag{3.1}$$

where $\alpha, \beta > 0$.

Lemma 3.1 *Assume (H1)–(H3). Let $\alpha, \beta \geq 0$ and $u \in C^2([0,1]) \cap C^3((0,1))$ be a solution of problem (3.1). Then the following properties hold*

(i) Lu is increasing and u is decreasing on $[0,1]$.

(ii) $u(t) = \alpha + (1-t)k(t)$, for $t \in [0,1]$, where $k \in C^1([0,1]) \cap C^2((0,1))$ and $k > 0$ on $[0,1]$.

Proof (i) Since u satisfies the differential equation $L^2 u = f(., u, Lu)$ with $(Lu)'(0) = 0$, it follows that

$$A(t)(Lu)'(t) = \int_0^t A(s) f(s, u(s), Lu(s)) ds.$$

Now, as f is a nonnegative function, we deduce that Lu is an increasing function on $[0,1]$. This together with $Lu(1) = -\beta$ and $u'(0) = 0$ imply that u is a decreasing function on $[0,1]$.

(ii) Since $u \in C^2([0,1]) \cap C^3((0,1))$ and $u(1) = \alpha$, then there exists a function $k \in C^1([0,1]) \cap C^2((0,1))$ such that

$$u(t) = \alpha + (1-t)k(t), \quad \text{for } t \in [0,1].$$

Moreover, since u is decreasing on $[0,1]$, k is positive on $[0,1)$. Furthermore, $k(1) = -u'(1) > 0$. ♢

Proposition 3.2 *Assume (H1)–(H5). Then for any $\alpha, \beta > 0$, problem (3.1) has at least one positive solution $u \in C^2([0,1]) \cap C^3((0,1))$, satisfying for any $t \in [0,1]$*

$$u(t) = \alpha + \beta(V1)(t) + V^2(f(., u, Lu))(t) \tag{3.2}$$

Proof Let $E = \{u \in C^1([0,1]) : u'(0) = 0 \text{ and } Lu \in C([0,1])\}$ endowed with the norm

$$\|u\| = \|Lu\|_\infty + |u(1)| = \sup_{t\in[0,1]} |Lu(t)| + |u(1)|,$$

and $C([0,1]) \times \mathbb{R}$ endowed with the norm

$$\|(g, \alpha)\|_1 = \|g\|_\infty + |\alpha|.$$

Then it is obvious to see that the map $(E, \|.\|) \to (C([0,1]) \times \mathbb{R}, \|.\|_1)$, defined as $u \mapsto (Lu, u(1))$ is an isometry. Thus $(E, \|.\|)$ is a Banach space.

Now, by (H5) and (2.4), we note that $V^2(f(., \alpha, -\beta))(0) < \infty$. So, in order to apply a fixed point argument, we consider the closed convex subset of E

$$\Lambda = \{u \in E : \alpha \leq u \leq \alpha + \beta V1(0) + V^2(f(., \alpha, -\beta))(0), Lu \leq -\beta.\}$$

Then we define the operator T on Λ by

$$Tu(t) = \alpha + \beta(V1)(t) + V^2(f(., u, Lu))(t), \text{ for } t \in [0,1].$$

First, we shall prove that T maps Λ into itself. Let $u \in \Lambda$. Then using (1.4), we have for $t \in [0,1]$,

$$L(Tu)(t) = -\beta - V(f(.,u,Lu))(t).$$

Using hypotheses (H4), we deduce from (2.3) that $Tu \in \Lambda$. Next, we prove the continuity of T in Λ. Let $(u_n)_n$ be a sequence in Λ such that

$$\|u_n - u\| = \|Lu_n - Lu\|_\infty + |u_n(1) - u(1)| \to 0 \quad \text{as } n \to \infty.$$

Then for any $t \in [0,1]$, we have $Tu_n(1) = Tu(1) = \alpha$ and

$$|L(Tu_n)(t) - L(Tu)(t)| \le \int_0^1 G(0,s)|f(s,u_n(s),Lu_n(s)) - f(s,u(s),Lu(s))|ds.$$

So, by hypotheses (H3) and (H5), we deduce that

$$\|Tu_n - Tu\| = \|L(Tu_n) - L(Tu)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Finally, we need to prove that $T\Lambda$ is relatively compact in $(E, \|.\|)$. From the continuity of the function $G(.,s)$, $s \in (0,1]$ and the hypotheses (H5), the family $\{L(Tu) : u \in \Lambda\}$ is equicontinuous on $[0,1]$. Moreover, $\{L(Tu) : u \in \Lambda\}$ is uniformly bounded. Now, using Ascoli's theorem, it follows that $\{L(Tu) : u \in \Lambda\}$ is relatively compact in $(C([0,1]), \|.\|_\infty)$, which implies that $T\Lambda$ is relatively compact in $(E, \|.\|)$. Hence, we conclude by Schauder's fixed point theorem, that T has a fixed point u in Λ, which satisfies the equation (3.2).

Now, by repeating differentiations in the integral equation (3.2) and using the statements (2.2)-(2.4), we show by (H5), that u is a positive solution of problem (3.1) and $u \in C^2([0,1]) \cap C^3((0,1))$. $\diamondsuit$

To prove the existence of positive solution for problem (1.3), we consider a sequence $(\alpha_n)_{n\ge 0}$ of positive real numbers, decreasing to zero and we put u_n the positive solution of (3.1) with α_n, α_n instead of α, β. Then we have the following Lemma.

Lemma 3.3 *Assume (H1)–(H5). Then there exists $c > 0$ such that for each $n \in \mathbb{N}$ and $t \in [0,1]$, we have*

$$u_n(t) \ge c(1-t) \quad \text{and} \quad Lu_n(t) \le -c(1-t).$$

Proof Let $\delta \in (0,1)$. Since for each $n \in \mathbb{N}$, u_n verifies the equation (3.2), we obtain from (2.5) and (2.6), that for $t \in [0,1]$ and $n \in \mathbb{N}$,

$$\begin{aligned} u_n(t) &= \alpha_n + \alpha_n \int_0^1 G(t,s)ds + \int_0^1 H(t,s)f(s,u_n(s),Lu_n(s))ds \\ &\ge C(\delta)(1-t)(u_n(\delta) - \alpha_n) \end{aligned}$$

and

$$\begin{aligned} Lu_n(t) &= -\alpha_n - \int_0^1 G(t,s)f(s,u_n(s),Lu_n(s))ds \\ &\le C(\delta)(1-t)(Lu_n(\delta) + \alpha_n). \end{aligned}$$

Then for all $n \in \mathbb{N}$ and $t \in [0,1]$, we have

$$u_n(t) \geq aC(\delta)(1-t) \quad \text{and} \quad Lu_n(t) \leq -bC(\delta)(1-t),$$

where $a = \inf_{n\in\mathbb{N}}(u_n(\delta) - \alpha_n)$ and $b = \inf_{n\in\mathbb{N}}(-Lu_n(\delta) - \alpha_n)$. Note that, from Lemma 3.1, a and b are nonnegative constants. We claim that $c = C(\delta)\min(a,b) > 0$ and so the lemma is proved. To establish the claim, we consider a subsequence $((u_{n_k}(\delta) - \alpha_{n_k}), (-Lu_{n_k}(\delta) - \alpha_{n_k}))_k$, which converges to (a,b). Then for k large enough and $\delta \leq s \leq 1$,

$$0 \leq u_{n_k}(s) \leq u_{n_k}(\delta) \leq 1 + a + \alpha_0$$

and

$$0 \leq -Lu_{n_k}(s) \leq -Lu_{n_k}(\delta) \leq 1 + b + \alpha_0.$$

This implies from (H4), that for k large enough, we have

$$u_{n_k}(\delta) - \alpha_{n_k} \geq \int_\delta^1 H(\delta,s)f(s, 1+a+\alpha_0, -1-b-\alpha_0)ds > 0$$

and

$$-Lu_{n_k}(\delta) - \alpha_{n_k} \geq \int_\delta^1 G(\delta,s)f(s, 1+a+\alpha_0, -1-b-\alpha_0)ds > 0.$$

So the claim is proved. ♢

Now, we are ready to prove the main result of this section.

Theorem 3.4 *Assume (H1)–(H5). Then problem (1.3) has at least one positive solution $u \in C^2([0,1]) \cap C^3((0,1))$, such that for each $t \in [0,1]$,*

$$c_1(1-t) \leq u(t) \leq c_2(1-t), \tag{3.3}$$

where c_1, c_2 are positive constants.

Proof We aim to show the existence of $u \in C^2([0,1]) \cap C^3((0,1))$ such that $u = V^2(f(.,u,Lu))$. We first recall that for $n \in \mathbb{N}$, u_n satisfies the equation

$$u_n(t) = \alpha_n + \alpha_n(V1)(t) + V^2(f(.,u_n,Lu_n))(t), \text{ for } t \in [0,1]. \tag{3.4}$$

So using Lemma 3.3, (2.3) and (2.4), we have for $n \in \mathbb{N}$ and $t \in [0,1]$,

$$|u_n(t)| \leq \alpha_0 + \alpha_0 \int_0^1 G(0,s)ds + C\int_0^1 G(0,s)f(s, c(1-s), -c(1-s))ds$$

and

$$|u_n'(t)| \leq \alpha_0\|h\|_\infty + C\int_0^1 G(0,s)f(s, c(1-s), -c(1-s))ds.$$

Hence from (H5) and Ascoli's theorem, it follows that the family of functions $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $C([0,1])$. On the other hand, from Lemma 3.3 and (H4), we have for each $n \in \mathbb{N}$ and $t, t' \in [0,1]$,

$$|Lu_n(t) - Lu_n(t')| \leq \alpha_0 + C\int_0^1 |G(t,s) - G(t',s)| f(s, c(1-s), -c(1-s))ds$$

and

$$|Lu_n(t)| \leq \alpha_0 + \int_0^1 G(0,s) f(s, c(1-s), -c(1-s))ds.$$

Now, since for each $s \in (0,1]$, the function $t \to G(t,s)$ is continuous on $[0,1]$ and using (H5), we deduce from Ascoli's theorem that $(Lu_n)_{n\in\mathbb{N}}$ is relatively compact in $C([0,1])$. So, let $(u_{n_k})_{k\in\mathbb{N}}$ and $(Lu_{n_k})_{k\in\mathbb{N}}$ be the subsequences which converge uniformly to functions $u \in C([0,1])$ and $v \in C([0,1])$, respectively.

We claim that $v = Lu$. Indeed, letting $n \to \infty$ in (3.4) we deduce by the dominated convergence theorem, that

$$u = V^2(f(., u, v)).$$

Then from (1.4), we have $Lu = -V(f(., u, v))$. On the other hand, from (3.4) we have

$$Lu_n = -\alpha_n - V(f(., u_n, Lu_n)), \quad \text{for } n \in \mathbb{N}.$$

Consequently, by the dominated convergence theorem, we get

$$v = -V(f(., u, v)) = Lu.$$

We conclude that $u = V^2(f(., u, Lu))$. So using the statements (2.2)-(2.4), we deduce that $u \in C^2([0,1]) \cap C^3((0,1))$ is a positive solution of problem (1.3). Finally, (3.3) follows immediately from Lemma 3.1. $\diamondsuit$

Remark 3.5 The result of Theorem 3.4 is also valid to the more general problem of high order

$$\begin{gathered} L^n u = (-1)^n f(., u, -Lu, \dots, (-1)^{n-1} L^{n-1} u) \quad \text{a.e. in } (0,1) \\ (L^k u)'(0) = 0, \quad (L^k u)(1) = 0, \quad k \in \{0, 1, \dots, n-1\}, \end{gathered} \tag{3.5}$$

where $n \geq 2$, the nonlinear term $f(t, y_1, \dots, y_n)$ is assumed to have singularities at $t = 1$ and $y_i = 0$ $(1 \leq i \leq n)$ and to satisfy the following conditions

(H6) $f : [0,1) \times ((0,\infty))^n \to (0,\infty)$ is measurable,
f is continuous and nondecreasing with respect to each y_i, $1 \leq i \leq n$,
For all $c > 0$, $\int_0^1 G(0,s) f(s, c(1-s), \dots, c(1-s))ds < \infty$.

In fact, we give in Propositions 3.6 and 3.7 some estimates for the Green function of the operator $u \to (-1)^n L^n u$ with boundary conditions $(L^k u)'(0) = 0$,

$(L^k u)(1) = 0$, for $k \in \{0, 1, \dots, n-1\}$, which is given by the following iterated relation

$$
\begin{gathered}
G_1(t,s) = G(t,s) = A(s)\int_{t\vee s}^{1}\frac{dr}{A(r)},\\
G_n(t,s) = \int_t^1 \frac{1}{A(\xi)}\Big(\int_0^\xi A(r)G_{n-1}(r,s)dr\Big)d\xi, \quad \text{for } n \geq 2.
\end{gathered}
$$

Proposition 3.6 *Assume (H1) and (H2). Let $n \geq 2$, then there exists a constant $C_n > 0$, such that for each $t, s \in [0,1] \times (0,1]$, we have*

(i) $|\frac{\partial^2}{\partial t^2} G_n(t,s)| \leq C_n G(0,s)$.

(ii) $-C_n G(0,s) \leq \frac{\partial}{\partial t} G_n(t,s) \leq 0$.

(iii) $0 \leq G_n(t,s) \leq C_n(1-t)G(0,s)$.

Proposition 3.7 *Assume (H1) and (H2) and let $\delta \in (0,1]$, then there exists a positive constant $C(\delta)$ such that for all $t, s \in [0,1]$ and $n \in \mathbb{N}^*$, we have*

$$
G_n(t,s) \geq C(\delta)(1-t)G_n(\delta,s).
$$

Using the same argument as in the proof of Theorem 3.4, we easily obtain the following more general result.

Theorem 3.8 *Assume (H1), (H2) and (H6). Then* (3.5) *has at least one positive solution $u \in C^2([0,1]) \cap C^{2n-1}((0,1))$, satisfying for each $t \in [0,1]$,*

$$
c_1(1-t) \leq u(t) \leq c_2(1-t),
$$

where c_1, c_2 are positive constants.

4 Uniqueness result

In this section, we assume that $f(t,x,y) \equiv f(t,x)$ and we aim to prove a uniqueness result for problem (1.3). We need the following lemma.

Lemma 4.1 *Assume (H1)–(H5) and let $u, v \in C^2([0,1]) \cap C^3((0,1))$ be two solutions of problem (1.3) satisfying* (3.3). *Then the following identity holds*

$$
\int_0^1 A(t)(u-v)(t)L^2(u-v)(t)dt = \int_0^1 A(t)(L(u-v))^2(t)dt.
$$

Proof Two integrations by parts yield

$$\begin{aligned}&\int_0^t A(s)(u-v)(s)L^2(u-v)(s)ds\\&= A(t)(u-v)(t)(L(u-v))'(t) - A(t)(u-v)'(t)L(u-v)(t)\\&\quad+\int_0^t A(s)(L(u-v))^2(s)ds,\end{aligned}$$

for all $t\in[0,1)$. Then, since $A(1)(u-v)'(1)L(u-v)(1)=0$, we need only to prove that

$$\lim_{t\to 1} A(t)(u-v)(t)(L(u-v))'(t)=0.$$

Using (3.3), there exist two constants $c_1, c_2>0$ such that

$$c_1(1-t)\le u(t)\le c_2(1-t)\quad\text{and}\quad c_1(1-t)\le v(t)\le c_2(1-t)$$

Let $\delta\in(0,1)$. Then by (H4), for $t\in[\delta,1]$, we have

$$\begin{aligned}&|A(t)(u-v)(t)(Lu)'(t)|\\&=|(u-v)(t)|\int_0^t A(s)f(s,u(s))ds\\&\le (c_2-c_1)(1-t)\int_0^t A(s)f(s,c_1(1-s))ds\\&\le (c_2-c_1)\big(\inf_{r\in[\delta,1]}\frac{1}{A(r)}\big)^{-1}\int_t^1\frac{1}{A(\xi)}\big(\int_0^1 A(s)f(s,c_1(1-s))ds\big)d\xi.\end{aligned}$$

Then using (H5), the result holds. ♢

Theorem 4.2 *Assume (H1)–(H5). Then problem (1.3) has a unique positive solution* $u\in C^2([0,1])\cap C^3((0,1))$*, satisfying* (3.3).

Proof The existence result is establised in Theorem 3.4. We shall prove the uniqueness. Let $u,v\in C^2([0,1])\cap C^3((0,1))$ be two solutions of problem (1.3) satisfying (3.3). From (H4), it follows that

$$(u-v)L^2(u-v)=(u-v)(f(.,u)-f(.,v))\le 0.$$

So, by Lemma 4.1, we deduce that $L(u-v)=0$. This together with $(u-v)'(0)=0$ and $(u-v)(1)=0$ imply that $u=v$. ♢

Remark 4.3 Let q be a nonnegative and continuous function on $[0,1]$, infinitely differentiable on $(0,1)$. Then the result of Theorem 4.2 is also valid for the following more general Navier problem

$$\begin{gathered}(L-q)^2u=f(.,u)\quad\text{a.e. in }(0,1),\\ u'(0)=0,\quad((L-q)u)'(0)=0,\quad u(1)=0,\quad((L-q)u)(1)=0,\end{gathered}\tag{4.1}$$

where f satisfies (H3)–(H5). Indeed, let $\varphi \in C^2([0,1]) \cap C^\infty((0,1))$ be the unique solution of the problem

$$\begin{gathered} Lu - qu = 0 \quad \text{in } (0,1), \\ u'(0) = 0, \quad u(0) = 1. \end{gathered}$$

From [2], φ is nondecreasing on $[0,1]$ and for any $t \in [0,1]$,

$$1 \le \varphi(t) \le \exp\Big(\int_0^t \frac{1}{A(s)}\Big(\int_0^s A(r)q(r)dr\Big)ds\Big) \le \exp(\|q\|_\infty \|h\|_\infty). \tag{4.2}$$

Now, we consider the differential operator $L_{A\varphi^2}$ defined by

$$L_{A\varphi^2} u = \frac{1}{A\varphi^2}(A\varphi^2 u')'.$$

So, for each $v \in C^\infty((0,1))$, we have $(L-q)(\varphi v) = \varphi L_{A\varphi^2} v$, which implies that

$$(L-q)^2(\varphi v) = \varphi (L_{A\varphi^2})^2 v.$$

Then it is obvious to see that $u = \varphi v$ is a solution of (4.1) if and only if v satisfies

$$\begin{gathered} L^2_{A\varphi^2} v = g(.,v) \quad \text{a.e. in } (0,1), \\ v'(0) = 0, \quad (L_{A\varphi^2} v)'(0) = 0, \quad v(1) = 0, \quad L_{A\varphi^2} v(1) = 0, \end{gathered} \tag{4.3}$$

where $g(t,x) = \frac{f(t,\varphi(t)x)}{\varphi(t)}$, for $(t,x) \in [0,1) \times (0,\infty)$. On the other hand, using (4.2), we remark that the assumption (H5) is equivalent to

$$\forall c > 0, \quad \int_0^1 A(s)\varphi^2(s)\big(\int_s^1 \frac{dr}{A(r)\varphi^2(r)}\big)g(s, c(1-s))ds < \infty.$$

So applying Theorem 4.2, we deduce that problem (4.3) has a unique solution $v \in C^2([0,1]) \cap C^3((0,1))$ satisfying (3.3). Hence $u = \varphi v$ is obviously the unique positive solution of (4.1) satisfying (3.3) and which is in $C^2([0,1]) \cap C^3((0,1))$.

Example 4.4 Let $\alpha, \beta \ge 0$ and k be a positive measurable function on $[0,1)$, which satisfies

$$\int_0^1 (1-s)^{1-(\alpha\vee\beta)} k(s)ds < \infty.$$

Then the Navier problem

$$\begin{gathered} u^{(4)}(t) = k(t)\big(u^{-\alpha}(t) + (-u'')^{-\beta}(t)\big), \quad t \in (0,1), \\ u'(0) = 0, \quad u^{(3)}(0) = 0, \quad u(1) = 0, \quad u''(1) = 0, \end{gathered}$$

has a positive solution $u \in C^2([0,1]) \cap C^3((0,1))$, satisfying (3.3).

Example 4.5 Let $A(t) = t^\gamma (\gamma \geq 0)$, for $t \in [0,1]$. Let $\alpha, \beta \geq 0$ and k be a positive measurable function on $[0,1)$, which satisfies

$$\int_0^1 (1-s)^{1-\alpha-\beta} k(s)\, ds < \infty.$$

We consider the problem

$$\begin{gathered} L^2 u = k(t) u^{-\alpha}(t)(-Lu)^{-\beta}(t), \quad t \in (0,1), \\ u'(0) = 0, \quad (Lu)'(0) = 0, \quad u(1) = 0, \quad Lu(1) = 0. \end{gathered}$$

Since $G(0,s) \leq (1-s)$ then (H5) is satisfied and the above problem has at least one positive solution $u \in C^2([0,1]) \cap C^3((0,1))$, satisfying (3.3). Moreover, if $\beta = 0$ then the solution u is unique.

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