Electronic Journal of Differential Equations, Vol. **2003**(2003), No. 21, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Minimal and maximal solutions for two-point boundary-value problems *

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Abstract

In this article we consider a boundary-value problem for the equation $f(t,x,x',x'')=0$ with mixed boundary conditions. Assuming the existence of suitable barrier strips, and using the monotone iterative method, we obtain the minimal and maximal solutions.

1 Introduction

We apply the monotone iterative method to obtain minimal and maximal solutions to the nonlinear boundary-value problem (BVP)

$$\begin{gathered} f(t,x,x',x'')=0, \quad 0\le a\le t\le b,\\ x(a)=A, \quad x'(b)=B, \end{gathered} \tag{1.1}$$

where the scalar function $f(t,x,p,q)$ is continuous and has continuous first derivatives on suitable subsets of $[a,b]\times\mathbb{R}^3$. For results, which guarantee the existence of $C^2[a,b]$-solutions to BVPs for the equation $x''=f(t,x,x',x'')-y(t)$ with various linear boundary conditions, see [6, 7, 17, 18, 21, 22, 23]. Concerning the uniqueness results, we refer to [21]. A result, concerning the existence and uniqueness of $C^2[a,b]$-solutions to the BVP for the equation $x''=f(t,x,x',x'')$, with general linear boundary conditions, can be found in [27]. The results of [19] guarantee the existence of $W^{2,\infty}[a,b]$-solutions or of $C^2[a,b]$-solutions to the Dirichlet BVP for the equation $f(t,x,x',x'')=0$, where the function $f(t,x,p,q)$is defined on $[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\times Y$, and Y is a non-empty closed connected or locally connected subset of $\mathbb{R}^n$. Finally, the $C^2[a,b]$-solvability of BVPs for the equation $f(t,x,x',x'')=0$ with fully nonlinear boundary conditions is studied in [12].

Note that, in the literature, the monotone iterative method is applied on BVPs for equations of the forms $x''=f(t,x,x')$ and $(\phi(x'))'=f(t,x,x')$ with various boundary conditions (see, for example, [2, 3, 4, 5, 9, 10, 11, 13, 15, 20, 26, 28]). The sequences of iterates, considered in [2, 3, 4, 5, 10, 13, 28],

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* *Mathematics Subject Classifications:* 34B15.
Key words: Boundary-value problems, minimal and maximal solutions, monotone method, barrier strips.

Submitted December 12, 2002. Published February 28, 2003.

converge to the extremal solutions, while the sequences of iterates, considered in[9, 15, 19], converge to the unique solution. The first elements $u_0(t)$ and $v_0(t)$ of such sequences of iterates usually are lower and upper solutions respectively of the problems under consideration (see, for example, [2, 3, 4, 5, 10, 13, 28]. To derive the needed monotone iterates, the authors of [2, 3, 4, 5, 10, 13, 15, 28] use suitable growth conditions. For more applications of the monotone iterative method, see citeb1,l1,l3,s1,y1.

In this article, following citek1, we obtain the extremal solutions to (1.1) under assumption of the existence of suitable barrier strips (see Remarks 2.1 and 2.2 below), which immediately imply the first iterates $u_0(t)$ and $v_0(t)$. A version of [12, Theorem 5.1] implies the existence of the next iterates, and a suitable comparison result guarantees the monotone properties for the sequences of iterates. Finally, the Arzela-Askoli's theorem ensures the existence of the extremal solutions of the problem (1.1) as limits of the sequences of iterates.

2 Basic hypotheses

The following four hypotheses will be a tool for obtaining our results.

(H1) There are constants $K > 0$, F, F_1, L, L_1 such that

$$Fa \le A \le La, \quad F_1 < F \le B \le L < L_1\,.$$

For the set $T := \{(t,x) : a \le t \le b, Ft \le x \le Lt\}$, we assume that

$$f(t,x,p,q) + Kq \ge 0$$

on $\big\{(t,x,p,q) : (t,x) \in T,\ p \in [L, L_1],\ q \in (-\infty, 0)\big\}$, and

$$f(t,x,p,q) + Kq \le 0$$

on $\big\{(t,x,p,q) : (t,x) \in T,\ p \in [F_1, F],\ q \in (0, \infty)\big\}$.

Remark 2.1 Set $\Phi_1(t,x,p,q) \equiv f(t,x,p,q) + Kq$. Then, the strip $\Delta_1 = [a,b] \times [L, L_1]$, on which $\Phi_1(t,x,p,q) \ge 0$, and the strip $\Delta_2 = [a,b] \times [F_1, F]$, on which $\Phi_1(t,x,p,q) \le 0$, are such that the graph of the function $x'(t), t \in [a,b]$, does not cross Δ_1 and Δ_2, and is located between them. For this reason Δ_1 and Δ_2 are called barrier strips for $x'(t)$, $t \in [a,b]$.

(H2) There are constants G_i^-, G_i^+, H_i^-, H_i^+, $i = 1,2$, such that

$$G_2^+ > G_1^+ \ge 2C, \quad G_2^- > G_1^- \ge 2C,$$
$$H_2^+ < H_1^+ \le -2C, \quad H_2^- < H_1^- \le -2C,$$

where $C = \max\{|L|, |F|\}/(b-a)$, $f(t,x,p,q)$ and $f_q(t,x,p,q)$ are continuous and $f_q(t,x,p,q) < 0$ for

$$(t,x,p,q) \in [a,b] \times [m_1 - \varepsilon, M_1 + \varepsilon] \times [F - \varepsilon, L + \varepsilon] \times [m_2 - \varepsilon, M_2 + \varepsilon],$$

where $m_1 = \min\{Fa, Fb\}$ $M_1 = \max\{La, Lb\}$, $m_2 = \min\{H_2^+, H_2^-\}$, $M_2 = \max\{G_2^+, G_2^-\}$, and $\varepsilon > 0$ is fixed and such that

$$H_1^+ > H_2^+ + \varepsilon, \quad H_1^- > H_2^- + \varepsilon, \quad G_2^+ > G_1^+ + \varepsilon, \quad G_2^- > G_1^- + \varepsilon. \tag{2.1}$$

$f_t(t,x,p,q)$, $f_x(t,x,p,q)$ and $f_p(t,x,p,q)$ are continuous for (t,x,p,q) in $[a,b] \times [m_1, M_1] \times [F, L] \times [m_2, M_2]$;

$$f_t(t,x,p,q) + f_x(t,x,p,q)p + f_p(t,x,p,q)q \geq 0$$

for (t,x,p,q) in $[a,b] \times [m_1, M_1] \times [F, L] \times \big([H_2^+, H_1^+] \cup [G_1^+, G_2^+]\big)$, and

$$f_t(t,x,p,q) + f_x(t,x,p,q)p + f_p(t,x,p,q)q \leq 0$$

for (t,x,p,q) in $[a,b] \times [m_1, M_1] \times [F, L] \times \big([H_2^-, H_1^-] \cup [G_1^-, G_2^-]\big)$, where F and L are the constants of H1.

Remark 2.2 Set $\Phi_2(t,x,p,q) \equiv f_t(t,x,p,q) + f_x(t,x,p,q)p + f_p(t,x,p,q)q$. Then, the pair of strips $\Omega_1 = [a,b] \times ([H_2^+, H_1^+] \cup [G_1^+, G_2^+])$, where $\Phi_2(t,x,p,q) \geq 0$, and the pair of strips $\Omega_2 = [a,b] \times ([H_2^-, H_1^-] \cup [G_1^-, G_2^-])$, where $\Phi_2(t,x,p,q) \leq 0$, are such that the graph of the function $x''(t), t \in [a,b]$, can not cross the outer strips, of the four such ones, defined by Ω_1 and Ω_2. For this reason the outer strips of Ω_1 and Ω_2 are called barrier strips for $x''(t), t \in [a,b]$.

(H3) For $m_3 = \min\{H_1^+, H_1^-\}$ and $M_3 = \max\{G_1^+, G_1^-\}$

$$h(\lambda, t, x, p, m_3 - \varepsilon) h(\lambda, t, x, p, M_3 + \varepsilon) \leq 0$$

for (λ, t, x, p) in $[0,1] \times [a,b] \times [m_1 - \varepsilon, M_1 + \varepsilon] \times [F - \varepsilon, L + \varepsilon]$, where $h(\lambda, t, x, p, q) = (\lambda - 1)Kq + \lambda f(t,x,p,q)$, F, L, K are the constants of H1, and H_1^+, H_1^-, G_1^+, G_1^-, C, m_1, M_1, and ε are as in H2.

(H4) For (t,x,p,q) in $T \times [F, L] \times [\min\{H_1^+, H_1^-\}, \max\{G_1^+, G_1^-\}]$, $f_x(t,x,p,q) \geq 0$, where the trapezoid T and the constants F and L are as in H1, and H_1^+, H_1^-, G_1^+ and G_1^- are the constants in H2, and m_3 and M_3 are as in H3.

3 Main result

For a function $y(t) \in C[a,b]$ bounded on $[a,b]$, we define a mapping

$$\mathcal{A}y = x,$$

where $x(t) \in C^2[a,b]$ is a solution to the BVP

$$\begin{gathered} f(t, y(t), x', x'') = 0, \quad t \in [a,b], \\ x(a) = A, \quad x'(b) = B. \end{gathered} \tag{3.1}$$

We will show that under the hypotheses H1, H2, and H3, the map $\mathcal{A}$ is uniquely determined. For this reason, we consider two sequences $\{u_n\}$ and $\{v_n\}$, $n = 0, 1, \dots$, defined by

$$u_{n+1} = \mathcal{A}u_n \quad \text{and} \quad v_{n+1} = \mathcal{A}v_n,$$

where $u_0 = Ft$, $v_0 = Lt$, $t \in [a,b]$, and F and L are as in H1. Now we formulate our main result.

Theorem 3.1 *Under hypotheses H1–H4, there are sequences $\{u_n\}$ and $\{v_n\}$, $n = 0, 1, \dots$, such that for $n \to +\infty$: $u_n \to u^m$, $v_n \to v^M$ and*

$$u_0 \le u_1 \le \dots \le u_n \le \dots \le u^m \le x \le v^M \le \dots \le v_n \le \dots \le v_1 \le v_0,$$

where $u^m(t)$ and $v^M(t)$ are the minimal and maximal solutions of the BVP (1.1) respectively, and $x(t) \in C^2[a,b]$ is a solution of (1.1).

The proof of this theorem can be found at the end of this article and is based on the auxiliary results, which we present in the next section.

4 Auxiliary results

We begin this section with an existence result, which is a modification of [8, Theorem 6.1, Chapter II]. Namely, we consider the family of BVPs

$$\begin{gathered} Kx'' = \lambda\big(Kx'' + f(t, y(t), x', x'')\big), \quad t \in [a,b], \\ x(a) = A, \quad x'(b) = B, \end{gathered} \tag{4.1}$$

where $\lambda \in [0,1]$ and $K > 0$.

Lemma 4.1 *Assume that there are constants Q_i, $i = 0, 1, \dots, 5$, independent of λ such that*

(i) For each solution $x(t) \in C^2[a,b]$ of (4.1) it holds

$$Q_0 < x(t) < Q_1, \; Q_2 < x'(t) < Q_3, \; Q_4 < x''(t) < Q_5, \quad t \in [a,b].$$

Also assume that:

(ii) $f(t,x,p,q)$ and $f_q(t,x,p,q)$ are continuous, and $f_q(t,x,p,q) < 0$ for all (t,x,p,q) in $[a,b] \times [Q_0,Q_1] \times [Q_2,Q_3] \times [Q_4,Q_5]$

(iii) $h(\lambda,t,x,p,Q_4)h(\lambda,t,x,p,Q_5) \le 0$ for (λ,t,x,p) in $\Lambda := [0,1] \times [a,b] \times [Q_0,Q_1] \times [Q_2,Q_3]$, where $h(\lambda,t,x,p,q) = (\lambda - 1)Kq + \lambda f(t,x,p,q)$.

Then the BVP (3.1) has a $C^2[a,b]$-solution for each $y(t) \in C[a,b]$ such that $Q_0 < y(t) < Q_1$, $t \in [a,b]$.

Proof In view of (ii) and (iii), we conclude that there is a unique function $G(\lambda, t, x, p)$ which is continuous on Λ and such that

$$q = G(\lambda, t, x, p) \quad \text{for} \quad (\lambda, t, x, p) \in \Lambda$$

is equivalent to the equation

$$h(\lambda, t, x, p, q) = 0 \quad \text{on} \quad \Lambda \times [Q_4, Q_5].$$

Note that $h(0, t, x, p, 0) = 0$ yields

$$G(0, t, x, p) = 0 \quad \text{for} \quad (t, x, p) \in [a, b] \times [Q_0, Q_1] \times [Q_2, Q_3]. \tag{4.2}$$

Thus, the family (4.1) is equivalent to the family of BVPs

$$\begin{gathered} x'' = G(\lambda, t, y(t), x'), \quad t \in [a, b], \\ x(a) = A, \quad x'(b) = B, \end{gathered} \tag{4.3}$$

where $\lambda \in [0, 1]$. Now, define the set

$$U = \big\{x(t) \in C^2[a, b] : x(t) \in (Q_0, Q_1),\ x'(t) \in (Q_2, Q_3),\ x''(t) \in (Q_4, Q_5)\big\},$$

which is an open subset of the convex set $C^2_Q[a, b]$ of the Banach space $C^2[a, b]$ and consider the map $\mathbf{N} : C^2_Q[a, b] \to C[a, b]$, defined by

$$\mathbf{N}x = x'',$$

where $C^2_Q[a, b] = \{x \in C^2[a, b] : x(a) = A, x'(b) = B\}$. It is easy to see that the map $\mathbf{S} : C^2_{Q_0}[a, b] \to C[a, b]$, defined by

$$\mathbf{S}x = x''$$

with $C^2_{Q_0}[a, b] = \{x \in C^2[a, b] : x(a) = 0,\ x'(b) = 0\}$, is one-to-one and the problem $\mathbf{S}x = 0$, $x(a) = A$, $x'(b) = B$, has a unique solution l. Then $\mathbf{N}^{-1} : C[a, b] \to C^2_Q[a, b]$ exists, is continuous, and moreover

$$\mathbf{N}^{-1}s = \mathbf{S}^{-1}s + l.$$

Let $\mathbf{H}_\lambda : \overline{U} \to C^2_Q[a, b]$ be defined by

$$\mathbf{H}_\lambda x = \mathbf{N}^{-1}\mathbf{G}_\lambda \mathbf{j}(x), \quad \lambda \in [0, 1],$$

where $\mathbf{j} : C^2_Q[a, b] \to C^1[a, b]$ is defined by $\mathbf{j}x = x$, $\mathbf{G}_\lambda : C^1[a, b] \to C[a, b]$ is defined by

$$(\mathbf{G}_\lambda x)(t) = G\big(\lambda, t, y(t), x'(t)\big), \quad \lambda \in [0, 1].$$

Clearly, $\mathbf{H}_\lambda$ is a compact homotopy, because $\mathbf{j}$ is a completely continuous embeding, and $\mathbf{G}_\lambda$ and $\mathbf{N}^{-1}$ are continuous. Moreover, $\mathbf{H}_\lambda x = x$ implies

$$x = \mathbf{N}^{-1}\mathbf{G}_\lambda \mathbf{j}(x).$$

Hence, by the definition of $\mathbf{N}^{-1}$, we have

$$x = \mathbf{S}^{-1}\mathbf{G}_\lambda \mathbf{j}(x) + l.$$

Finally, since $\mathbf{S}l = 0$, it follows that

$$\mathbf{S}x = \mathbf{G}_\lambda \mathbf{j}(x).$$

Thus, the fixed points of $\mathbf{H}_\lambda$ are solutions to (4.3) and obviously$\mathbf{H}_\lambda$ has no fixed points on ∂U.In view of (4.2), the map $\mathbf{H}_0$, which has the form $\mathbf{H}_0 x = l$, is constant. Moreover, l, as the unique solution of $(4.1)_0$, belongs to the set U. Hence, by [8, Theorem 2.2], the map $\mathbf{H}_0$ is essential. The topological transversality theorem of [8] implies that $\mathbf{H}_1$ is also essential, i.e. for $\lambda = 1$ (4.3) has a solution. Moreover, for $\lambda = 1$ (4.3) coincides with (3.1). Therefore, the problem (3.1) has a solution. The proof of the lemma is complete. $\diamondsuit$

To obtain our next auxiliary results, we introduce the following two sets

$$\begin{aligned} V &= \{y(t) \in C[a,b] : Ft \le y(t) \le Lt,\, t \in [a,b]\},\\ V_1 &= \{y(t) \in C^1[a,b] : Ft \le y(t) \le Lt,\, F \le y'(t) \le L,\, t \in [a,b]\}, \end{aligned}$$

where the constants L and F are as in H1. Then we formulate the following results.

Lemma 4.2 *Let H1 hold and $x(t) \in C^2[a,b]$ be a solution to (4.1) with $y(t) \in V$. Then the following statements hold:*

(i) If there is an interval $T_1 \subseteq [a,b]$ such that

$$L \le x'(t) \le L_1 \quad \textit{for} \quad t \in T_1, \tag{4.4}$$

then $x''(t) \ge 0$ for $t \in T_1$.

(ii) If there is an interval $T_2 \subseteq [a,b]$ such that $F_1 \le x'(t) \le F$ for $t \in T_2$, then $x''(t) \le 0$ for $t \in T_2$.

Proof Since the proofs of (i) and (ii) are similar, it is sufficient to show that (4.4) implies $x''(t) \ge 0$ for $t \in T_1$. Indeed, the assertion is true for $\lambda = 0$. Now, let $\lambda \in (0,1]$ and assume that there is a $t_0 \in T_1$ such that $x''(t_0) < 0$. Then

$$0 > Kx''(t_0) = \lambda\,[Kx''(t_0) + f(t_0, x(t_0), x'(t_0), x''(t_0))] \ge 0.$$

This contradiction proves the assertion. $\diamondsuit$

Lemma 4.3 *Let H1 hold, and $x(t) \in C^2[a,b]$ be a solution to (4.1) with $y(t) \in V$. Then*

$$Ft \le x(t) \le Lt, \quad F \le x'(t) \le L \quad \textit{for} \quad t \in [a,b].$$

Proof Consider the sets

$$Y_0 = \{t \in [a,b] : L < x'(t) \le L_1\} \quad \text{and} \quad Y_1 = \{t \in [a,b] : F_1 \le x'(t) < F\}$$

and suppose that they are not empty. Then, using the continuity of $x'(t)$ and the inequality $F \le x'(b) \le L$, we easily conclude that there are closed intervals $[t_0, \tau_0] \subseteq Y_0$ and $[t_1, \tau_1] \subseteq Y_1$ such that

$$x'(t_0) > x'(\tau_0) \quad \text{and} \quad x'(t_1) < x'(\tau_1). \tag{4.5}$$

On the other hand, by Lemma 4.2, we have

$$x''(t) \ge 0 \quad \text{for} \quad t \in [t_0, \tau_0] \quad \text{and} \quad x''(t) \le 0 \quad \text{for } t \in [t_1, \tau_1]$$

and therefore, we have

$$x'(t_0) \le x'(\tau_0) \quad \text{and} \quad x'(t_1) \ge x'(\tau_1).$$

But this contradicts (4.5). The obtained contradiction shows that Y_0 and $Y_1$are empty, and so we see that

$$F \le x'(t) \le L \quad \text{for} \quad t \in [a,b].$$

Integrating this expression from a to t and using the fact that $Fa \le A \le La$, we get

$$Ft \le x(t) \le Lt, \quad t \in [a,b]$$

which concludes the proof. ♢

Remark 4.4 Let $x(t) \in C^2[a,b]$ be a solution to (1.1). Then, in view of Lemma 4.3, if $F = L$, it follows that$x'(t) = B$, $t \in [a,b]$. Now, using $Fa \le A \le La$, we see that $x(t) = Bt$, $t \in [a,b]$, is the unique $C^2[a,b]$-solution to the problem (1.1).

Lemma 4.5 *Let H1 and H2 hold, and $x(t) \in C^2[a,b]$ be a solution to* (4.1) *with $y(t) \in V_1$. Then*

$$m_3 \le x''(t) \le M_3, \quad t \in [a,b],$$

and there is a constant D independent of λ such that

$$|x'''(t)| \le D \quad \textit{for } t \in [a,b].$$

Proof By the mean value theorem, there is a $\xi \in (a,b)$ such that $x''(\xi) = [x'(b) - x'(a)]/(b-a)$. Since Lemma 4.3 implies

$$F \le x'(t) \le L \quad \text{for} \quad t \in [a,b], \tag{4.6}$$

we see that

$$x''(\xi) \le 2C \le G_1^+, \tag{4.7}$$

where $C = \max\{|L|, |F|\}/(b-a)$. Now suppose that the set

$$Y = \{t \in [a, \xi] : G_1^+ < x''(t) \le G_2^+\}$$

is not empty. The continuity of $x''(t)$ and (4.7) imply that there is a closed interval $[t_0, \tau_0] \subseteq Y$ such that

$$x''(t_0) > x''(\tau_0). \tag{4.8}$$

Since (4.6) holds for $t \in [t_0, \tau_0]$ and

$$\begin{gathered} G_1^+ < x''(t) \le G_2^+ \quad \text{for } t \in [t_0, \tau_0], \\ m_1 \le Ft \le y(t) \le Lt \le M_1 \quad \text{for } t \in [t_0, \tau_0], \\ F \le y'(t) \le L \quad \text{for } t \in [t_0, \tau_0], \end{gathered} \tag{4.9}$$

in view of H2, we have

$$\Psi_1(t) \equiv f_q\big(t, y(t), x'(t), x''(t)\big) < 0, \quad t \in [t_0, \tau_0],$$

and for $t \in [t_0, \tau_0]$,

$$\begin{aligned} \Psi_2(t) \equiv & f_t\big(t, y(t), x'(t), x''(t)\big) + f_x\big(t, y(t), x'(t), x''(t)\big)y'(t) \\ & + f_p\Big(t, y(t), x'(t), x''(t)\Big)x''(t) \ge 0. \end{aligned}$$

Thus, using the last two inequalities and the continuity of f_t, f_x, f_p and f_q on $[t_0, \tau_0]$, we conclude that x''' is continuous on $[t_0, \tau_0]$ and

$$x'''(t) = \lambda\Psi_2\ (t)/[K(1-\lambda) - \lambda\Psi_1(t)] \ge 0 \quad \text{for } t \in [t_0, \tau_0]. \tag{4.10}$$

Consequently, $x''(t_0) \le x''(\tau_0)$, which contradicts (4.8). Thus,

$$x''(t) \le G_1^+ \quad \text{for } t \in [a, \xi].$$

The inequality $H_1^- \le x''(t)$, $t \in [a, \xi]$ can be obtained in the same manner. Similarly, it is easy to show that

$$H_1^+ \le x''(t) \le G_1^-, \quad t \in [\xi, b].$$

Finally, using (4.6), (4.9), the fact that $x''$is bounded on $[a, b]$ and the continuity of the partial derivatives of $f(t, x, p, q)$ on the set $[a, b] \times [m_1, M_1] \times [F, L] \times [m_3, M_3]$, from (4.10) it follows that there is a constant D independent of λ such that

$$|x'''(t)| \le D \quad \text{for } t \in [a, b].$$

The proof of the lemma is complete. $\diamondsuit$

Lemma 4.6 *Suppose that H1, H2 and H3 hold. Then the BVP (3.1) has a $C^2[a, b]$-solution, if $y(t) \in V_1$.*

Proof Let $x(t) \in C^2[a,b]$ be a solution to $(4.1)_\lambda$. Then, by Lemma 4.3, we have

$$F - \varepsilon < x'(t) < L + \varepsilon \quad \text{for } t \in [a,b]$$
$$m_1 - \varepsilon < x(t) < M_1 + \varepsilon \quad \text{for } t \in [a,b],$$

while, by Lemma 4.5,

$$m_3 - \varepsilon < x''(t) < M_3 + \varepsilon \quad \text{for } t \in [a,b],$$

where $\varepsilon > 0$ is as in H2. Thus, the condition (i) of Lemma 4.1 holds for $Q_0 = m_1 - \varepsilon$, $Q_1 = M_1 + \varepsilon$, $Q_2 = F - \varepsilon$, $Q_3 = L + \varepsilon$, $Q_4 = m_3 - \varepsilon$ and $Q_5 = M_3 + \varepsilon$. Moreover, from (2.1) and H3 it follows that the conditions (ii) and (iii) of Lemma 4.1 are satisfied. Also,

$$m_1 - \varepsilon < y(t) < M_1 + \varepsilon \quad \text{for } t \in [a,b].$$

So, we can apply Lemma 4.1 to conclude that the problem (3.1) has a solution in $C^2[a,b]$. The proof of the lemma is complete. ♢

We need the following two lemmas which are adopted from [24].

Lemma 4.7 ([24, Chapter I, Theorem 1]) *Suppose $\phi(t)$ satisfies the differential inequality*

$$\phi'' + g(t)\phi' \geq 0 \quad \text{for } a < t < b, \tag{4.11}$$

with $g(t)$ a bounded function. If $\phi(t) \leq M$ in (a,b) and if the maximum M of ϕ is attained at an interior point c of (a,b), then $\phi \equiv M$.

Lemma 4.8 ([24, Chapter I, Theorem 2]) *Suppose $\phi(t)$ is a nonconstant function which satisfies the inequality (4.11) and has one-sided derivatives at a and b, and suppose g is bounded on every closed subinterval of (a,b). If the maximum of ϕ occurs at $t = a$ and g is bounded below at $t = a$, then $\phi'(a) < 0$. If the maximum occurs at $t = b$ and g is bounded above at $t = b$, then $\phi'(b) > 0$.*

Lemma 4.9 *Suppose that $\phi \in C^2(a,b) \cap C^1[a,b]$ satisfies the inequality*

$$\phi''(t) + g(t)\phi'(t) \geq 0 \quad \text{for } t \in (a,b),$$

where $g(t)$ is bounded on (a,b). If $\phi(a) \leq 0$ and

$$\phi'(b) \leq 0, \tag{4.12}$$

then

$$\phi(t) \leq 0 \quad \text{for } t \in [a,b]. \tag{4.13}$$

Proof First, assume that $\phi(t)$achieves its maximum at $t_0 \in (a,b)$. By Lemma 4.7, for $t \in [a,b]$ we obtain $\phi(t) \equiv \phi(t_0) = \phi(a) \le 0$ and so (4.13) holds.

Next, suppose that $\phi(t)$ achieves its maximum at the ends of the interval $[a,b]$. If we assume $\phi(t) \le \phi(b)$, $t \in [a,b]$, the application of Lemma 4.8 shows that $\phi'(b) > 0$, which contradicts (4.12). Thus, by our assumtions, $\phi(t) \le \phi(a) \le 0$, $t \in [a,b]$, and so (4.13) follows. The proof is complete. ♢

In the last two lemmas we use the map $\mathcal{A}$ defined in the section 3.

Lemma 4.10 *Under assumptions H1, H2, and H3, for any $y \in V_1$, the image x by the map $\mathcal{A}$ exists and it is unique.*

Proof The existence of the image of x follows from Lemma 4.6. In order to see that x is unique, fix y and assume that$z$is an other image of y by $\mathcal{A}$ and consider the function $\phi(t) = x(t) - z(t)$, $t \in [a,b]$. Then, it is evident that

$$f\big(t, y(t), x'(t), x''(t)\big) - f\big(t, y(t), z'(t), z''(t)\big) = 0, \quad t \in [a,b].$$

Next, we construct the equality

$$\begin{gathered} f\big(t, y(t), x'(t), x''(t)\big) - f\big(t, y(t), z'(t), x''(t)\big) \\ +f\big(t, y(t), z'(t), x''(t)\big) - f\big(t, y(t), z'(t), z''(t)\big) = 0, \end{gathered}$$

which can be rewritten in the form $I_1(t)\phi'(t) + \ I_2(t)\phi''(t) = 0$, where

$$\begin{aligned} I_1(t) &= \int_0^1 f_p\big(t, y(t), z'(t) + \theta(x'(t) - z'(t)), x''(t)\big) d\theta, \\ I_2(t) &= \int_0^1 f_q\big(t, y(t), z'(t), z''(t) + \theta(x''(t) - z''(t))\big) d\theta. \end{aligned}$$

Hence, the function $\phi(t)$ is a solution to the BVP

$$\begin{gathered} \phi''(t) + \frac{I_1(t)}{I_2(t)}\phi'(t) = 0, \quad t \in [a,b], \\ \phi(a) = 0, \quad \phi'(b) = 0. \end{gathered}$$

Moreover, it is easy to conclude that $\phi(t) \equiv 0$, $t \in [a,b]$, is the unique solution of the above BVP. Consequently, $x(t) \equiv z(t)$, $t \in [a,b]$. The proof of the lemma is complete. ♢

Lemma 4.11 *Under the hypotheses H1–H4, if $y_1(t), y_2(t) \in V_1$ are such that $y_1(t) \le y_2(t)$ for $t \in [a,b]$, then*

$$x_1(t) \le x_2(t) \ \ \textit{for} \quad t \in [a,b],$$

where $x_i = \mathcal{A}y_i, i = 1,2$.

Proof Observe that, by Lemma 4.3, we have $F \le x_1'(t) \le L$, $t \in [a,b]$, and, by Lemma 4.5,
$$m_3 \le {x_1}''(t) \le M_3, \quad t \in [a,b].$$
Moreover,
$$Ft \le y_1(t) \le y_2(t) \le Lt, \quad t \in [a,b].$$
Thus, from $f_x(t,x,p,q) \ge 0$ for (t,x,p,q) in $T \times [F,L] \times [m_3, M_3]$ it follows that
$$0 = f\big(t, y_1(t), x_1'(t), x_1''(t)\big) \le f\big(t, y_2(t), x_1'(t), x_1''(t)\big), \quad t \in [a,b].$$
Hence, for $t \in [a,b]$ we have
$$f\big(t, y_2(t), x_2'(t), x_2''(t)\big) - f\big(t, y_2(t), x_1'(t), x_1''(t)\big) \le 0$$
and then, as in Lemma 4.10, we construct the inequality
$$\begin{aligned} &f\big(t, y_2(t), x_1'(t), x_1''(t)\big) - f\big(t, y_2(t), x_2'(t), x_1''(t)\big) \\ &+ f\big(t, y_2(t), x_2'(t), x_1''(t)\big) - f\big(t, y_2(t), x_2'(t), x_2''(t)\big) \ge 0 \end{aligned}$$
from which for $\phi(t) = x_1(t) - x_2(t)$, $t \in [a,b]$, we find
$$\phi''(t) + \frac{J_1(t)}{J_2(t)}\phi'(t) \ge 0, \quad t \in [a,b],$$
where
$$\begin{aligned} J_1(t) &= \int_0^1 f_p\big(t, y_2(t), x_2'(t) + \theta(x_1'(t) - x_2'(t)), x_1''(t)\big) d\theta, \\ J_2(t) &= \int_0^1 f_q\big(t, y_2(t), x_2'(t), x_2''(t) + \theta(x_1''(t) - x_2''(t))\big) d\theta. \end{aligned}$$
Furthermore, $\phi(a) = 0$, $\phi'(b) = 0$. Finally, applying Lemma 4.9, we see that $\phi(t) \le 0$ for $t \in [a,b]$, which completes the proof. $\diamondsuit$

5 Proof of Theorem 3.1

Consider the sequences $\{u_n\}$ and $\{v_n\}$, defined by
$$u_{n+1} = \mathcal{A}u_n \quad \text{and} \quad v_{n+1} = \mathcal{A}v_n, \quad n = 0, 1, \dots$$
In view of Lemma 4.6, from Lemma 4.3 it follows that $Ft = u_0 \le u_1$ and $v_1 \le v_0 = Lt$. Moreover, Lemma 4.11 and induction arguments imply that
$$u_{n-1} \le u_n, \quad v_n \le v_{n-1}, \quad n = 1, 2, \dots$$
On the other hand, since $u_0 \le v_0$, by Lemma 4.11 and induction arguments, we conclude that $u_n \le v_n$, $n = 0, 1, \dots$. ¿From this observation it follows that
$$u_0 \le u_n \le v_0, \quad n = 0, 1, \dots.$$

Therefore, $\{u_n\}$ is uniformly bounded. Furthermore, since, by Lemma 4.3, $\{u'_n\}$ is uniformly bounded, we see that $\{u_n\}$ is equicontinuous. Finally, since, by Lemma 4.5, $\{u'''_n\}$ is uniformly bounded, it follows that the sequence $\{u''_n\}$ is uniformly bounded and equicontinuous. Thus, we can apply the Arzela-Ascoli theorem to conclude that there are a subsequence $\{u_{n_i}\}$ and a function $u \in C^2[a,b]$ such that $\{u_{n_i}\}$, $\{u'_{n_i}\}$ and $\{u''_{n_i}\}$ are uniformly convergent on $[a,b]$ to u, u' and u'' respectively. Now, using the fact that $u_{n_i} = \mathcal{A}u_{n_{i-1}}$ can be rewritten equivalently in the form

$$\begin{aligned} u_{n_i}(t) = \frac{1}{K}\int_a^t \Big(\int_b^r \big(Ku''_{n_i}(s) + f(s, u_{n_{i-1}}(s), u'_{n_i}(s), u''_{n_i}(s))\big)ds\Big)dr \\ + B(t-a) + A, \end{aligned}$$

letting $i \to +\infty$, we obtain

$$u(t) = \frac{1}{K}\int_a^t \Big(\int_b^r \big(Ku''(s) + f(s, u(s), u'(s), u''(s))\big)ds\Big)dr + B(t-a) + A,$$

from which it follows that $u(t)$ is a solution to the BVP (1.1).

Remark that, if $x(t)$ is a solution of (1.1), then, by Lemma 4.3, we have $u_0(t) \le x(t)$ for $t \in [a,b]$. Applying Lemma 4.11 (it is possible, because $x = \mathcal{A}x$), by induction we obtain

$$u_n(t) \le x(t), \quad t \in [a,b], \quad n = 0, 1, \dots,$$

and then $u(t) \le x(t)$, $t \in [a,b]$, which holds for each solution $x(t) \in C^2[a,b]$ of the problem (1.1). Consequently, it follows that

$$u(t) \equiv u^m(t), \quad t \in [a,b].$$

By similar arguments, we conclude that $\lim v_n = v^M(t)$, $t \in [a,b]$. Thus, the proof is complete. ♢

Acknowledgement P. S. Kelevedjiev would like to thank the Ministry of National Economy of Helenic Republic for providing the NATO Science Fellowship (No. DOO 850/02) and the University of Ioannina for its hospitality.

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