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CENTERING CONDITIONS FOR PLANAR SEPTIC SYSTEMS

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Abstract. We find centering conditions for the following O-symmetric system of degree 7:

$$\begin{aligned}\dot{x} &= y + x(H_2(x,y) + H_6(x,y)),\\ \dot{y} &= -x + y(H_2(x,y) + H_6(x,y)),\end{aligned}$$

where $H_2(x,y)$ and $H_6(x,y)$ are homogeneous polynomials of degrees 2 and 6, respectively. In some cases, we can find commuting systems and first integrals for the original system. We also study the geometry of the central region.

1. Introduction

Consider the planar autonomous system of ordinary differential equations

$$\begin{aligned}\dot{x} &= y + xR_{n-1}(x,y),\\ \dot{y} &= -x + yR_{n-1}(x,y),\end{aligned} \tag{1.1}$$

where $R_{n-1}(x,y)$ is a polynomial in x and y, of degree $n-1$, and $R_{n-1}(0,0)=0$. This system has only one singular point at $O(0,0)$ which is the center of the linear part of the system. The orbits of this system move around the origin with constant angular speed, and the origin is so a uniformly isochronous singular point.

Such systems have been studied in many papers; see [1]–[5] and references therein. The following problem was stated as Problem 19.1. in [2]:

> Identify systems (1.1) of odd degree which are O-symmetric (not necessarily quasi homogeneous) having O as a (uniformly isochronous) center.

In this article we solve this problem for some systems of degree 7. In particular, we find necessary and sufficient conditions for system (1.1) with

$$\begin{aligned}R_{n-1}(x,y) = {}&a_0x^2 + a_1xy + a_2y^2 + c_0x^6 + c_1x^5y + c_2x^4y^2\\ &+ c_3x^3y^3 + c_4x^2y^4 + c_5xy^5 + c_6y^6,\end{aligned} \tag{1.2}$$

where $a_0, a_1, a_2, c_0, c_1, c_2, c_3, c_4, c_5, c_6$ are real numbers.

The plan for this paper is as follows: In Section 2, we present centering conditions. In Section 3, we investigate some properties of systems in the presence of a

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center. In particular, we discuss the question of existence of a polynomial commuting system. When this system exists, we give a first integral of the original system. We also study the geometry of the central region.

2. Results

Theorem 2.1. *The origin is a center of (1.2) if and only if one of the following two conditions is satisfied:*

(i) $a_0 = a_1 = a_2 = 0$, $5c_0 + c_2 + c_4 + 5c_6 = 0$;

(ii)

$$\begin{gathered} a_0 + a_2 = 0, \\ 5c_0 + c_2 + c_4 + 5c_6 = 0, \\ a_1(15c_0 + c_2 - c_4 - 15c_6) + 2a_0(5c_1 + 3c_3 + 5c_5) = 0, \\ (a_1^2 - 4a_0^2)(3c_0 - c_2 - c_4 + 3c_6) + 8a_0a_1(c_1 - c_5) = 0, \\ a_1(a_1^2 - 12a_0^2)(c_0 - c_2 + c_4 - c_6) + 2a_0(3a_1^2 - 4a_0^2)(c_1 - c_3 + c_5) = 0. \end{gathered}$$

Before proving this theorem, we consider the instance of (1.2) in which $a_0 = a_2 = 0$, that is

$$\begin{aligned} \dot{x} &= y + x(a_1xy + c_0x^6 + c_1x^5y + c_2x^4y^2 + c_3x^3y^3 + c_4x^2y^4 + c_5xy^5 + c_6y^6), \\ \dot{y} &= -x + y(a_1xy + c_0x^6 + c_1x^5y + c_2x^4y^2 + c_3x^3y^3 + c_4x^2y^4 + c_5xy^5 + c_6y^6). \end{aligned} \tag{2.1}$$

Lemma 2.2. *The origin is a center of (2.1) if and only if one of the following two conditions is satisfied:*

$$a_1 = 0, \quad 5c_0 + c_2 + c_4 + 5c_6 = 0; \tag{2.2}$$

$$c_0 = c_2 = c_4 = c_6 = 0. \tag{2.3}$$

Proof. We used the software package *Mathematica* to find the first six Poincaré-Lyapunov constants of (2.1) (see more details about our method in [6]). Up to a positive scalar factor they are

$$\begin{aligned} &l_1 = 0, l_2 = 0, \\ &l_3 = 5c_0 + c_2 + c_4 + 5c_6, \\ &l_4 = -a_1(5c_0 + 3c_2 + 5c_4 + 35c_6), \\ &l_5 = a_1^2(-101c_0 - 17c_2 - 9c_4 + 19c_6), \\ &l_6 = 15a_1^3(621c_0 + 367c_2 + 565c_4 + 3367c_6) \\ &\qquad - 56(31c_1 + 20c_3 + 49c_5)(5c_0 + c_2 + c_4 + 5c_6). \end{aligned}$$

Necessity of conditions (2.2)–(2.3) result from solving the simultaneous equations $l_3 = l_4 = l_5 = l_6 = 0$.

In the case (2.2), the sufficiency part of the lemma follows from the fact that (2.1) is a quasi homogeneous system of degree 7 whose coefficients satisfy the equation $5c_0 + c_2 + c_4 + 5c_6 = 0$ representing a necessary and sufficient centering condition [1].

In the case (2.3), system (2.1) is reversible and its trajectories are symmetric with respect to both coordinate axes.

It is well known that if the linear part of a reversible system has a center at the origin, then the origin is also a center of the system itself. Thus, under conditions (2.2)-(2.3), the origin is a center of system (2.1). This completes the proof of the lemma. □

Proof of Theorem 2.1. In the previous lemma we considered a particular case. Now, we consider the general system (1.2). The first Poincaré-Lyapunov constant l_1 of (1.2) is

$$l_1 = 2(a_0 + a_2).$$

If $a_0 + a_2 = 0$ then the change of variables $x \mapsto x\cos\vartheta + y\sin\vartheta$, $y \mapsto -x\sin\vartheta + y\cos\vartheta$, with ϑ defined from the condition

$$a_0\cos^2\vartheta + a_1\sin\vartheta\cos\vartheta - a_0\sin^2\vartheta = 0, \tag{2.4}$$

reduces (1.2) to a system of the form (2.1):

$$\begin{aligned}
\dot{x} &= y + x(a_1'xy + c_0'x^6 + c_1'x^5y + c_2'x^4y^2 + c_3'x^3y^3 + c_4'x^2y^4 + c_5'xy^5 + c_6'y^6),\\
\dot{y} &= -x + y(a_1'xy + c_0'x^6 + c_1'x^5y + c_2'x^4y^2 + c_3'x^3y^3 + c_4'x^2y^4 + c_5'xy^5 + c_6'y^6)
\end{aligned}$$

whose coefficients are expressible in terms of the coefficients of (1.2). In particular, we have

$$\begin{aligned}
a_1' &= a_1\cos^2\vartheta - 4a_0\cos\vartheta\sin\vartheta - a_1\sin^2\vartheta,\\
c_0' &= (2d_0 - d_2 + 2d_4 - d_6)/32,\\
c_2' &= (6d_0 - d_2 - 10d_4 + 15d_6)/32,\\
c_4' &= (6d_0 + d_2 - 10d_4 - 15d_6)/32,\\
c_6' &= (2d_0 + d_2 + 2d_4 + d_6)/32,
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
d_0 &= 5c_0 + c_2 + c_4 + 5c_6,\\
d_2 &= (15c_0 + c_2 - c_4 - 15c_6)\cos 2\vartheta + (5c_1 + 3c_3 + 5c_5)\sin 2\vartheta,\\
d_4 &= (3c_0 - c_2 - c_4 + 3c_6)\cos 4\vartheta + 2(c_1 - c_5)\sin 4\vartheta,\\
d_6 &= (c_0 - c_2 + c_4 - c_6)\cos 6\vartheta + (c_1 - c_3 + c_5)\sin 6\vartheta,
\end{aligned}$$

and ϑ is defined in (2.4).

Using Lemma 2.2, we see that the origin is a center of system (1.2) if and only if one of the following two conditions is satisfied:

$$a_0 + a_2 = 0, a_1' = 0, \quad 5c_0' + c_2' + c_4' + 5c_6' = 0; \tag{2.6}$$

$$a_0 + a_2 = 0, \quad c_0' = c_2' = c_4' = c_6' = 0. \tag{2.7}$$

In the case (2.6), $a_1' = 0$ and (2.4) amount to $a_0 = a_1 = 0$. Next, (2.5) yields $5c_0' + c_2' + c_4' + 5c_6' = d_0 = 5c_0 + c_2 + c_4 + 5c_6$. This proves the theorem in the case (i).

Next, $c_0' = c_2' = c_4' = c_6' = 0$ amounts to $d_0 = d_2 = d_4 = d_6 = 0$. Using (2.4), we eliminate ϑ from the last equations, thus arriving at the conditions of the case (2.7) expressed in terms of the coefficients of the original system (1.2). These conditions coincide with those in the case (ii) of Theorem 2.1 and the proof is complete. □

3. Properties of systems with center

Consider system (2.1) with coefficients that satisfy the centering conditions (2.2)–(2.3). For a planar polynomial system, the presence of an isochronous center is well known to be equivalent to the existence of a transverse analytic system commuting with it [7].

It is proved in [5] that in the case (2.2) system (2.1) commutes with a polynomial system of the form

$$\begin{aligned} \dot{x} &= x + xQ(x,y), \\ \dot{y} &= y + yQ(x,y), \end{aligned} \tag{3.1}$$

where

$$Q(x,y) = q_0x^6 + q_1x^5y + q_2x^4y^2 + q_3x^3y^3 + q_4x^2y^4 + q_5xy^5 + q_6y^6$$

which is a homogeneous polynomial of degree 6 satisfying

$$yQ_x(x,y) - xQ_y(x,y) = 6P_6(x,y) \tag{3.2}$$

with

$$P_6(x,y) = c_0x^6 + c_1x^5y + c_2x^4y^2 + c_3x^3y^3 + c_4x^2y^4 + c_5xy^5 + c_6y^6\,.$$

To satisfy this equation, we can take

$$\begin{gathered} q_0 = 0, \quad q_1 = -6c_0, \quad q_2 = -3c_1, q_3 = -2(5c_0 + c_2), \\ q_4 = -3(c_1 + c_3/2), \quad q_5 = 6c_6, \quad q_6 = -(c_1 + c_3/2 + c_5). \end{gathered} \tag{3.3}$$

Note that if we add $c(x^2+y^2)^3$ to the polynomial $Q(x,y)$ with the coefficients (3.3) then the resultant system of the form (3.1) commutes with (2.1).

Following [8], we say that a function $C : R^2 \to R$ and the curve $C = 0$ are invariants for a system $\dot{x} = p(x,y)$, $\dot{y} = q(x,y)$ if there is a polynomial L such that $\dot{C} = CL$, where $\dot{C} = C_xp + C_yq$. The polynomial L is called the cofactor of C.

Note that the functions

$$C_1 = x^2 + y^2, \quad C_2 = 1 + Q(x,y)$$

are invariants for (2.1). This enables us to find the first Darboux integral

$$H(x,y) = \frac{(x^2+y^2)^3}{1+Q(x,y)} \tag{3.4}$$

for this system. The Darboux method is presented, for instance, in [8, 9].

By [2], the center of (2.1) is of type B^k, and the boundary of the center domain is the union of k open unbounded trajectories ($1 \le k \le 6$). In the case under study, we can describe this boundary explicitly and indicate the possible values of k more precisely.

Passing to the polar coordinates $x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$ in (3.4), we can show that in the case (2.2), the boundary of the central region is defined by the equation

$$\varrho = \frac{1}{(c_0 - Q(\cos\varphi, \sin\varphi))^{1/6}}$$

with $c_0 = \max_{[0,2\pi]} Q(\cos\varphi, \sin\varphi)$.

The central region is a curvilinear k-polygon whose vertices are points at infinity in the intersection of the equator of the Poincaré sphere with the rays $x =$

$r\cos\varphi_i, y = r\sin\varphi_i, r > 0$, where the values of φ_i are determined from the conditions

$$Q(\cos\varphi_i, \sin\varphi_i) = c_0, 0 \le \varphi_i < 2\pi.$$

Note that the values φ_i are solutions of the equation $P_6(\cos\varphi, \sin\varphi) = 0$. Indeed, from (3.2) we deduce:

$$\begin{aligned} 0 &= \frac{d}{d\varphi} Q(\cos\varphi, \sin\varphi)|_{\varphi=\varphi_i} \\ &= -Q_x(\cos\varphi_i, \sin\varphi_i)\sin\varphi_i + Q_y(\cos\varphi_i, \sin\varphi_i)\cos\varphi_i \\ &= 6P_6(\cos\varphi_i, \sin\varphi_i). \end{aligned}$$

The trigonometric polynomial $Q(\cos\varphi, \sin\varphi)$ of degree 6 satisfies the condition $Q(\cos(\varphi+\pi), \sin(\varphi+\pi)) = Q(\cos\varphi, \sin\varphi)$ and takes its every value, c_0 inclusively, on the interval $[0, 2\pi)$ an even number of times. Thus, in the case (2.2) the central region is symmetric about the origin and its boundary is the union of an even number of unbounded trajectories. Therefore, the center is of type B^k, where $k = 2, 4, 6$. Moreover, a "generic" system has a center of type B^2. For the center to be of type B^4 or B^6, the trigonometric polynomial $Q(\cos\varphi, \sin\varphi)$ must take its greatest value c_0 on the interval $[0, 2\pi)$ more than twice. This requires extra restrictions on the coefficients of the system.

In the case (2.3) system (2.1) takes the form

$$\begin{aligned} \dot{x} &= y + x(a_1xy + c_1x^5y + c_3x^3y^3 + c_5xy^5), \\ \dot{y} &= -x + y(a_1xy + c_1x^5y + c_3x^3y^3 + c_5xy^5). \end{aligned}$$

If $a_1 = 0$ then we arrive at the case (2.2). If $a_1 \ne 0$ then we may assume that $a_1 = 1$. The general case is reduced to this by the change of variables $x \mapsto x/\sqrt{a}, y \mapsto y/\sqrt{a}$ for $a > 0$ or $x \mapsto y/\sqrt{-a}, y \mapsto x/\sqrt{-a}, t \mapsto -t$ for $a < 0$.

According to [5], the system

$$\begin{aligned} \dot{x} &= y + x(xy + c_1x^5y + c_3x^3y^3 + c_5xy^5) \equiv y + xP(x, y), \\ \dot{y} &= -x + y(xy + c_1x^5y + c_3x^3y^3 + c_5xy^5) \equiv -x + yP(x, y) \end{aligned} \tag{3.5}$$

commutes with an analytic system of the form

$$\dot{x} = xQ(x, y), \quad \dot{y} = yQ(x, y), \tag{3.6}$$

where the function $Q(x, y)$ meets the equation

$$\begin{aligned} &x(Q_y(x, y) + P_x(x, y)Q(x, y) - P(x, y)Q_x(x, y)) \\ &+ y(-Q_x(x, y) + P_y(x, y)Q(x, y) - P(x, y)Q_y(x, y)) = 0. \end{aligned} \tag{3.7}$$

Suppose that the function $Q(x, y)$ is a polynomial in the variables x and y which has degree N in y: $Q(x, y) = Q_0(x) + Q_1(x)y + \ldots + Q_N(x)y^N$. After insertion of $Q(x, y)$ in (3.7), the left-hand side becomes a polynomial of degree at most $N + 5$ and the coefficient of y^{N+5} is

$$c_5x((6 - N)Q_N(x) - xQ'_N(x)).$$

Therefore, we must have $xQ'_N(x) = (6 - N)Q_N(x)$ or $c_5 = 0$. If $c_5 \ne 0$, this yields $N \le 6$. The same bound of N can be shown to be true also in the case when $c_5 = 0$. Likewise, we can show that the degree of $Q(x, y)$ in x is at most 6.

Substituting the polynomial $Q(x, y) = \sum_{i,j=0}^{6} q_{ij}x^iy^j$ in (3.7) yields a system of linear equations in the coefficients q_{ij}. We derived this system and investigated its properties by using the software package *Mathematica*. In this way we found out that the necessary solvability condition is

$$c_3^2 - 4c_1c_5 = 0. \tag{3.8}$$

This condition is also sufficient.

As an example, assume that $c_1 = \alpha^2$, $c_5 = \beta^2$, and $c_3 = 2\alpha\beta$. Then (2.1) takes the form

$$\begin{aligned} \dot{x} &= y + x^2y(1 + (\alpha x^2 + \beta y^2)^2), \\ \dot{y} &= -x + xy^2(1 + (\alpha x^2 + \beta y^2)^2). \end{aligned} \tag{3.9}$$

Straightforward calculations show that (3.9) commutes with the polynomial system

$$\begin{aligned} \dot{x} &= x(\alpha - \beta + (\alpha x^2 + \beta y^2) + (\alpha x^2 + \beta y^2)^3), \\ \dot{y} &= y(\alpha - \beta + (\alpha x^2 + \beta y^2) + (\alpha x^2 + \beta y^2)^3). \end{aligned}$$

Likewise, we can study the case when $c_1 = \alpha^2$, $c_5 = \beta^2$, $c_3 = -2\alpha\beta$ and the case when $c_1 = -\alpha^2$, $c_5 = -\beta^2$, and $c_3 = \pm 2\alpha\beta$.

We have thus proved that system (3.5) commutes with a polynomial system of the form (3.6) if and only if (3.8) holds.

Recall that every uniformly isochronous O-symmetric quintic system satisfying the center conditions commutes with some polynomial system of the same degree [6]. At the same time, an arbitrary uniformly isochronous (not necessarily O-symmetric) quintic system with a center may fail to commute with any polynomial system [4].

The functions

$$C_1 = x^2 + y^2, C_2 = \alpha - \beta + (\alpha x^2 + \beta y^2) + (\alpha x^2 + \beta y^2)^3$$

are invariants for (3.9) with the respective cofactors

$$L_1 = 2xy(1 + (\alpha x^2 + \beta y^2)^2), \quad L_2 = 2xy(1 + 3(\alpha x^2 + \beta y^2)^2).$$

Moreover, if $\alpha \neq \beta$ then the function

$$C_3 = \exp\Big(\int_0^{\alpha x^2 + \beta y^2} \frac{dt}{\alpha - \beta + t + t^3}\Big)$$

is an invariant with the cofactor $L_3 = 2xy$. We have $3L_1 - L_2 - 2L_3 = 0$. In this case the function

$$\begin{aligned} H(x,y) &= \frac{C_1^3}{C_2C_3^2} \\ &= \frac{(x^2+y^2)^3}{\alpha - \beta + (\alpha x^2 + \beta y^2) + (\alpha x^2 + \beta y^2)^3} \\ &\quad \times \frac{1}{\exp(2\int_0^{\alpha x^2 + \beta y^2} dt/(\alpha - \beta + t + t^3)} \end{aligned}$$

is a first Darboux integral of (3.9).

When $\alpha = \beta$, the change of variables $x = \sqrt{r}\cos\varphi$, $y = \sqrt{r}\sin\varphi$ reduces (3.9) to the system

$$\dot{r} = r^2(1 + \alpha^2 r^2)\sin 2\varphi, \dot{\varphi} = -1$$

for which the function

$$G(x,y) = \frac{1}{r} + \sin^2\varphi + \arctan\alpha r$$

is a first integral. Therefore, the function

$$H(x,y) = \frac{x^2+y^2}{1+x^2+\alpha(x^2+y^2)\arctan\alpha(x^2+y^2)}$$

is a first integral of (3.9) for $\alpha = \beta$.

As in the case (2.2), system (3.9) has a center of type B^k $(1 \le k \le 6)$. Since the system under study is reversible, the central region is symmetric with respect to both coordinate axes. Therefore, the center is of type B^2, B^4, or B^6. The singular points on the equator of the Poincaré sphere, vertices of the symmetric boundary of the central region, are saddle points. They have common separatrices only in exceptional cases. So in the case (2.6) the center is "generically" of type B^2. Clearly, all our results about system (2.1) can be translated to system (1.2).

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