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EXISTENCE OF SOLUTIONS TO A SECOND ORDER PARTIAL DIFFERENTIAL EQUATION WITH NONLOCAL CONDITIONS

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Abstract. Using the cosine function theory, we prove the existence of mild and classical solutions for an abstract second-order Cauchy problem with nonlocal conditions.

1. Introduction

This paper concerns the second order nonlocal Cauchy problem

$$u''(t) = Au(t) + f(t, u(t), u'(t)), \quad t \in I = [0, a], \tag{1.1}$$

$$u(0) = x_0 + q(u, u'), \tag{1.2}$$

$$u'(0) = y_0 + p(u, u'), \tag{1.3}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators, $(C(t))_{t\in\mathbb{R}}$, on a Banach space X and $f : \mathbb{R} \times X^2 \to X$, $q, p : C(I : X)^2 \to X$ are appropriates continuous functions.

Motivated for numerous applications, Byszewski studied in [2] a first order evolution differential equation with nonlocal conditions modelled in the form

$$\begin{gathered} u' = Au(t) + f(t, u(t)), \quad t \in [0, a], \\ u(0) = x_0 + q(t_1, t_2, \dots, t_n, u(\cdot)), \end{gathered} \tag{1.4}$$

where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators on a Banach space X; $q : [0, a]^n \times X \to X$ is a continuous function and the symbol $q(t_1, t_2, \dots, t_n, u(\cdot))$ is used in the sense that in the place of " $\cdot$ " only the points t_i can be substituted; for instance $q(t_1, t_2, \dots, t_n, u(\cdot)) = \sum_{i=1}^{n} \alpha_i u(t_i)$. In the cited paper, Byszewski proved the existence of the mild, strong and classical solutions for (1.4) employing the contraction mapping principle and the semigroup theory. We refer the reader to [2]-[6] for a complementary literature respect first order differential equations with nonlocal conditions.

On the other hand, some second order partial differential equations with nonlocal conditions modelled using the cosine function theory has been considered in the

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literature, see for example [1, 13, 14]. In general the nonlocal conditions considered in these works are described in the form

$$x(0) = g(x) + x_0, \quad x'(0) = \eta,$$

where $g : C(I : X) \to X$ is appropriate and $\eta \in X$ is prefixed. It's relevant to observe that the problems studied in these papers do not consider "partial" evolution equations, since the authors proved their results under the assumption that the cosine function $(C(t))_{t\in\mathbb{R}}$, generated by A, is such that $C(t)$ is compact for every $t > 0$, which imply that $\dim(X) < \infty$, see Travis and Weeb [15, pp. 557], for details.

Our goal in this work is establish the existence of mild and classical solutions for the abstract nonlocal Cauchy problem (1.1)-(1.3) using the cosine function theory and the contraction mapping principle. The abstract results in this work are applicable to "partial" second order differential equations with nonlocal conditions, see the examples in Section 3.

In this paper henceforth, $C(\cdot) = (C(t))_{t\in\mathbb{R}}$ is a strongly continuous cosine function of bounded linear operators on a Banach space X with infinitesimal generator A. We refer the reader to [9, 15, 16] for the necessary concepts about cosine functions. Next, we only mention a few results and notations needed to establish our results. We denote by $S(t)$ the sine function associated with $C(t)$ which is defined by

$$S(t)x := \int_0^t C(s)x ds, \quad x \in X, \ t \in \mathbb{R}.$$

For a closed operator $B : D(B) \subset X \to X$ we denote by $[D(B)]$ the space $D(B)$ endowed with the graph norm $\|\cdot\|_B$. In particular, $[D(A)]$ is the space

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable}\},$$

endowed with the norm $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$. Moreover, in this work the notation E stands for the space formed by the vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1. We know from Kisiński [11], that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0\le t\le a} \|AS(t)x\|, \quad x \in E,$$

is a Banach space. The operator valued function $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A)\times E$. From this it follows that $AS(t) : E \to X$ is a bounded operator and that $AS(t)x \to 0$, as $t \to 0$, for each $x \in E$. Furthermore, if $x : [0,\infty) \to X$ is a locally integrable, then $y(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function, which is a consequence of the fact that

$$\int_0^t G(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\, ds \\ \int_0^t C(t-s)x(s)\, ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.

The existence of solutions of the second order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad t \in [0, a], \tag{1.5}$$

$$x(0) = x_0, \quad x'(0) = x_1, \tag{1.6}$$

where $h : [0, a] \to X$ is an integrable function, has been discussed in [15]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem has been treated in [16]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\, ds, \quad t \in [0, a], \tag{1.7}$$

is called a mild solution of (1.5)-(1.6) and that when $x_0 \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)x_0 + C(t)x_1 + \int_0^t C(t-s)h(s)\, ds. \tag{1.8}$$

Regularity of mild solutions of problem (1.5)-(1.6) was treated by Travis and Weeb in [16], by Bochenek in [7] and recently by Henriquez and Vasquez in [10].

This work contains three sections. In Section 2 we discuss existence of mild and classical solution for some second order abstract Cauchy problem with nonlocal conditions. In general the results are obtained using the contraction mapping principle and the ideas in [7], [10] and [16]. In the section 3, the "wave" equation with nonlocal conditions is studied.

The terminologies and notations are those generally used in functional analysis. In particular, if $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, we indicate by $\mathcal{L}(Z : Y)$ the Banach space of the bounded linear operators of Z in Y and we abbreviate this notation to $\mathcal{L}(Z)$ whenever $Z = Y$. $B_r(x : Z)$ denotes the closed ball with center at x and radius $r > 0$ in the space $(Z, \|\cdot\|_Z)$. Additionally, for a bounded function $\xi : [0, a] \to Z$ and $0 \le t \le a$ we will employ the notation $\xi_{Z,\,t}$ for

$$\xi_{Z,\,t} = \sup\{\|\xi(s)\|_Z : s \in [0, t]\}, \tag{1.9}$$

and we will write simply ξ_t when no confusion arises. Finally, we remark that the prefix $\mathcal{R}$ is used to indicate the image of a map.

2. Existence Results

In this section we discuss the existence of mild and classical solutions for some abstract second order partial differential equations with nonlocal conditions. Along of this section, $N \ge 1$ and $\tilde{N}$ are positive constants such that $\|C(t)\| \le N$ and $\|S(t)\| \le \tilde{N}$ for every $t \in I$. At first, we study the nonlocal Cauchy problem

$$u''(t) = Au(t) + f(t, u(t)), \quad t \in I, \tag{2.1}$$

$$u(0) = x_0 + q(u), \tag{2.2}$$

$$u'(0) = y_0 + p(u). \tag{2.3}$$

where $f : \mathbb{R} \times X \to X$ and $q, p : C(I : X) \to X$ are appropriates continuous functions.

By comparison with Travis [16], we introduce the followings definitions.

Definition 2.1. A function $u \in C(I : X)$ is a mild solution of the nonlocal Cauchy problem (2.1)-(2.3) if condition (2.2) is verified and

$$u(t) = C(t)(x_0 + q(u)) + S(t)(y_0 + p(u)) + \int_0^t S(t-s)f(s, u(s))ds, \quad t \in I. \tag{2.4}$$

Definition 2.2. A function $u(\cdot) \in C^2(I : X)$ is a classical solution of the nonlocal Cauchy problem (2.1)-(2.3), if $u(\cdot)$ is solution of the equation (2.1) and the conditions (2.2)-(2.3) are verified.

Now, we establish our first result.

Theorem 2.3. *Let $x_0, y_0 \in X$ and assume that there exist positive constants l_f, l_p, l_q such that*

$$\|f(t,x) - f(t,y)\| \le l_f \|x - y\|, \quad x, y \in X,$$
$$\|q(u) - q(v)\| \le l_q \|u - v\|_a, \quad u, v \in C(I : X),$$
$$\|p(u) - p(v)\| \le l_p \|u - v\|_a, \quad u, v \in C(I : X).$$

If $\theta = Nl_q + \tilde{N} l_p + \tilde{N} l_f a < 1$, then there exist a unique mild solution of (2.1)-(2.3).

Proof. On the space $Y = C(I : X)$ endowed with the sup norm, we define the mapping $\Phi : Y \to Y$, where

$$\Phi u(t) = C(t)(x_0 + q(u)) + S(t)(y_0 + p(u)) + \int_0^t S(t-s) f(s, u(s)) ds.$$

It is easy to see that Φ is well defined and with values in Y. Moreover, for $(u, w), (v, z) \in Y$ we get

$$\begin{aligned} \|\Phi u(t) - \Phi v(t)\| &\le Nl_q \|u - v\|_a + \tilde{N} l_p \|u - v\|_a + \tilde{N} l_f \int_0^t \|u - v\|_\theta d\theta, \\ &\le \big(Nl_q + \tilde{N} l_p + a\tilde{N} l_f\big) \|u - v\|_a, \end{aligned}$$

which imply that Φ is a contraction on Y. Thus, there exist a unique mild solution of (2.1)-(2.3). The proof is complete. □

Remark 2.4. In relation with the next result, we remark that a Banach space Y has the Radom Nikodym property, (abbreviated RNP), respect to a finite measure space (Ω, Σ, μ); if for each continuous vector measure $G : \Sigma \to Y$ of bounded variation, there exists $g \in L^1(\mu, Y)$ such that $G(E) = \int_E g d\mu$ for every $E \in \Sigma$. We refer to [8] for additional details respect of this matter.

Remark 2.5. In Theorem 2.6, below, $A^* : D(A^*) \to X^*$ is the adjoint operator of A which is well defined since $D(A)$ is dense in X.

Theorem 2.6. *Let the assumptions of Theorem 2.3 be satisfied. If $x_0 + \mathcal{R}q \subset D(A)$, $y_0 + \mathcal{R}p \subset E$ and any of the followings conditions is verified,*

(a) *The adjoint operator $A^* : D(A^*) \to X^*$ is such that $\overline{D(A^*)} = X^*$;*
(b) *The space X has the RNP property;*
(c) *$f(\cdot)$ is continuously differentiable;*

then the unique mild solution, $u(\cdot)$, of (2.1)-(2.3) is a classical solution.

Proof. From the preliminaries we know that $u(\cdot)$ is continuously differentiable and so that the function $t \to f(t, u(t))$ is Lipschitz on I. Let $y(\cdot) \in C(I : X)$ be the unique mild solution of

$$x''(t) = Ax(t) + f(t, u(t)), \quad t \in I, \tag{2.5}$$
$$x(0) = x_0 + q(u), \tag{2.6}$$
$$x'(0) = y_0 + p(u). \tag{2.7}$$

If (**a**) holds, it follows from [7, Theorem 1] that $y(\cdot)$ is a classical solution. On the other hand, if X has the *RNP*, then $s \to f(s, u(s)) \in W^{1,1}(I : X)$ which, from [10, Theorem 3.1], implies that $y(\cdot)$ is a classical solution of (2.5)-(2.7). When (**c**) is verified, from [16, Proposition 2.4] it follows that $y(\cdot)$ is also a classical solution.

Finally, from the uniqueness of solution of (2.5)-(2.7) we infer that $u(\cdot) = y(\cdot)$ and so that $u(\cdot)$ is a classical solution of (2.1)-(2.3). $\square$

Next, we study existence of solution for (1.1)-(1.3).

Definition 2.7. A function $u \in C^1(I : X)$ is a mild solution of the nonlocal Cauchy problem (1.1)-(1.3) if the conditions (1.2)-(1.3) are verified and

$$u(t) = C(t)(x_0+q(u,u'))+S(t)(y_0+p(u,u'))+\int_0^t S(t-s)f(s,u(s),u'(s))ds, \quad t \in I.$$

Definition 2.8. A function $u(\cdot) \in C^2(I : X)$ is a classical solution of the nonlocal Cauchy problem (1.1)-(1.3), if $u(\cdot)$ is solution of (1.1) and the conditions (1.2), (1.3) are satisfied.

Theorem 2.9. *Let $(x_0, y_0) \in E \times X$ and assume that the followings conditions hold:*

(a) *f is continuous and there exist positive constants l^i, $i = 1, 2$ such that*
$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \le l_f^1\|x_1 - x_2\| + l_f^2\|y_1 - y_2\|, \quad x_i, y_i \in X.$$

(b) *The functions $q(\cdot), p(\cdot) : C(I : X)^2 \to X$ are continuous, $q(\cdot)$ is E-valued and there exist positive constants l_p^i, l_q^i, $i = 1, 2$ such that*
$$\|q(u,w) - q(v,z)\|_1 \le l_q^1\|u - v\|_a + l_q^2\|w - z\|_a,$$
$$\|p(u,w) - p(v,z)\| \le l_p^1\|u - v\|_a + l_p^2\|w - z\|_a,$$
for every $u, v, w, z \in C(I : X)$.

Let $\Theta_1 = \max_{i=1,2}\{Nl_q^i + \tilde{N}(l_p^i + al_f^i)\}$ and $\Theta_2 = \max_{i=1,2}\{l_q^i + N(l_p^i + al_f^i)\}$. If $\Theta = \Theta_1 + \Theta_2 < 1$, then there exist a unique mild solution of (1.1)-(1.3).

Proof. On the space $Y = C(I : X)^2$, equipped with the norm
$$\|(u, v)\| = \|u\|_a + \|v\|_a,$$
we define the map $\Phi : Y \to Y$, where $\Phi(u, v) = (\Phi_1(u, v), \Phi_2(u, v))$ and

$$\Phi_1(u,v)(t) = C(t)(x_0 + q(u,v)) + S(t)(y_0 + p(u,v)) + \int_0^t S(t-s)f(s,u(s),v(s))ds,$$

$$\Phi_2(u,v)(t) = AS(t)(x_0+q(u,v))+C(t)(y_0+p(u,v))+\int_0^t C(t-s)f(s,u(s),v(s))ds.$$

It follows from the assumptions that each Φ_i is well defined and with values in $C(I : X)$. Moreover, for $(u, v), (w, z) \in Y$ we get

$$\|\Phi_1(u,v)-\Phi_1(w,z)\|_a \le \big(Nl_q^1+\tilde{N}(l_p^1+al_f^1)\big)\|u-w\|_a+\big(Nl_q^2+\tilde{N}(l_p^2+al_f^2)\big)\|v-z\|_a,$$

and so that

$$\|\Phi_1(u,v) - \Phi_1(w,z)\|_a \le \max_{i=1,2}\{Nl_q^i + \tilde{N}(l_p^i + al_f^i)\}\|(u,v) - (w,z)\|_a. \tag{2.8}$$

On the other hand, from the preliminaries and condition (b) we get

$$\begin{aligned}&\|\Phi_2(u,v) - \Phi_2(w,z)\|_a \\ &\le \|q(u,v) - q(w,z)\|_1 + N\big(l_p^1 + al_f^1\big)\|u - w\|_a + N\big(l_p^2 + al_f^2\big)\|v - z\|_a,\end{aligned}$$

and hence

$$\|\Phi_2(u,v) - \Phi_2(w,z)\|_a \le \max_{i=1,2}\{\, l_q^i + N(l_p^i + al_f^i)\,\}\|(u,v) - (w,z)\|_a. \tag{2.9}$$

Finally, from (2.8) and (2.9), it follows that

$$\|\Phi(u,v)-\Phi(w,z)\|_a \le \Theta\|(u,v)-(w,z)\|_a,$$

which imply that Φ is a contraction. Thus, there exists a unique mild solution of (1.1)-(1.3). The proof is complete. $\square$

To prove the next theorem we need the followings result.

Corollary 2.10. *Assume that the assumptions in Theorem 2.9 are verified and let $u(\cdot)$ be the mild solution of (1.1)-(1.3). Suppose, furthermore, that there exists l_f^3 such that*

$$\|f(t,x,y)-f(s,x,y)\| \le l_f^3|t-s|, \quad t,s\in I, \quad x,y\in X.$$

If $(x_0+q(u,u'), y_0+p(u,u')) \in D(A)\times E$, then $u'(\cdot)$ is Lipschitz on I.

Proof. Let $t\in I$ and $h\in\mathbb{R}$ with $t+h\in I$. Using that $s\to u(s)$ is Lipschitz on I and that, for $t\in I$,

$$u'(t)=S(t)A(x_0+q(u,u'))+C(t)(y_0+p(u,u'))+\int_0^t C(t-s)f(s,u(s),u'(s))ds,$$

we obtain

$$\begin{aligned}
\|u'(t+h)-u'(t)\| &\le C_1h+\int_0^h \|C(t+h-s)f(s,u(s),u'(s))\|ds\\
&\quad + N\int_0^t [\,l_f^1\|u(s+h)-u(s)\|+l_f^2\|u'(s+h)-u'(s)\|+l_f^3h\,]ds\\
&\le C_2h+Nl_f^2\int_0^t \|u'(s+h)-u'(s)\|ds,
\end{aligned}$$

where C_i, $i=1,2$, are constants independents of h and $t\in I$. The assertion is now consequence of the Gronwall inequality. $\square$

In what follows, for the function $j:I\to X$ and $h\in\mathbb{R}$ we use the notation

$$\partial_h j(t) := \frac{j(t+h)-j(t)}{h}. \tag{2.10}$$

Moreover, if $j(\cdot): I\times X\to X$ is differentiable, we use the decomposition

$$\begin{aligned}
&j(t+s,y+y_1,w+w_1)-j(t,y,w)\\
&=(D_1j(t,y,w),D_2j(t,y,w),D_3j(t,y,w))(s,y_1,w_1)\\
&\quad+\|(s,y_1,w_1)\|_{I\times X^2}R(j(t,y,w),s,y_1,w_1),
\end{aligned} \tag{2.11}$$

where $\|R(j(t,y,w),s,y_1,w_1)\|\to 0$ when $\|(s,x_1,y_1)\|_{I\times X^2}=|s|+\|x_1\|+\|y_1\|\to 0$.

Theorem 2.11. *Let assumptions in Corollary 2.10 be satisfied and $u(\cdot)$ be the mild solution of (1.1)-(1.3). If $(x_0+q(u), y_0+p(u))\in D(A)\times E$ and any of the following conditions hold:*

- (a) *The adjoint operator $A^*: D(A^*)\to X^*$ is such that $\overline{D(A^*)}=X$;*
- (b) *The space X has the RNP property;*
- (c) *f is continuously differentiable,*

then $u(\cdot)$ is a classical solution of (1.1)-(1.3).

Proof. Firstly we remark that from Corollary 2.10 the function $t \to f(s, u(t), u'(t))$ is Lipschitz on I. When (a) or (b) are verified, the assertion follows using the steps in the proof of Theorem 2.6. Assume that condition (c) holds and let $v(\cdot) \in C(I : X)$ be the unique solution of the integral problem

$$\begin{aligned} v(t) &= C(t)A(x_0 + q(u,u')) + AS(t)(y_0 + p(u,u')) + f(0, u(0), u'(0)) \\ &\quad + \int_0^t C(t-s)D_1 f(w(s))ds + \int_0^t C(t-s)D_2 f(w(s))(u'(s))ds \\ &\quad + \int_0^t C(t-s)D_3 f(w(s))(v(s))ds, \quad t \in I, \end{aligned} \tag{2.12}$$

$$v(0) = A(x_0 + q(u,u')) + f(0, u(0), u'(0)), \tag{2.13}$$

where $\xi(t) = (t, u(t), u'(t))$. The existence and uniqueness of a solution of (2.12)-(2.13) follows from the contraction mapping principle; we omit additional details. Next, we prove that $u''(\cdot) = v(\cdot)$ on I. Let $t \in I$ and $h \in \mathbb{R}$ with $t + h \in I$. Since, for $t \in I$,

$$u'(t) = AS(t)(x_0 + q(u,u')) + C(t)(y_0 + p(u,u')) + \int_0^t C(t-s)f(s, u(s), u'(s))ds, \tag{2.14}$$

from (2.12), we obatin

$$\begin{aligned} &\|\partial_h u'(t) - v(t)\| \\ &\le \gamma_1(h) + \frac{1}{h}\int_0^h \|C(t+h-s)\|\|f(s,u(s),u'(s)) - f(0,u(0),u'(0))\|ds \\ &\quad + N\int_0^t \|\partial_h f(\xi(s)) - D_1 f(\xi(s)) - D_2 f(\xi(s))(u'(s)) - D_3 f(\xi(s))(v(s))\|ds \\ &\le \gamma_2(h) + N\int_0^t \|D_3 f(\xi(s))\|_{\mathcal{L}(X)}\|\partial_h u'(s) - v(s)\|ds \\ &\quad + N\int_0^t \|(1, \partial_h u(s), \partial_h u'(s))\|_{I\times X^2}\|R(f(\xi(s)), h, h\partial_h u(s), h\partial_h u'(s))\|ds, \end{aligned}$$

where $\gamma_i(h) \to 0$ as $h \to 0$. It follows, from the Gronwall-Bellman inequality and Corollary 2.10, that $\partial_h u'(\cdot) \to v(\cdot)$ when $h \to 0$ and so that $u''(\cdot) = v(\cdot)$ on I.

From these remarks and Proposition 2.4 in [16], we infer that the mild solution, $y(\cdot)$, of the abstract Cauchy problem

$$\begin{aligned} x''(t) &= Ax(t) + f(t, u(t), u'(t)), \quad t \in I, \\ x(0) &= x_0 + q(u,u'), \\ x'(0) &= y_0 + p(u,u'), \end{aligned} \tag{2.15}$$

is a classical solution, which from the uniqueness solution of (2.15) permit conclude that $y(\cdot) = u(\cdot)$ and that $u(\cdot)$ is a classical solution of (1.1)-(1.3). The proof is complete. □

3. The wave equation with nonlocal conditions

In this section we illustrate some of the results of this work with the wave equation. On the space $X = L^2([0,\pi])$ we consider the operator $Af(\xi) = f''(\xi)$ with domain $D(A) = \{f(\cdot) \in H^2(0,\pi) : f(0) = f(\pi) = 0\}$. It's well known that A

is the generator of strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$ on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$, with corresponding normalized eigenvectors $z_n(\xi) := (\frac{2}{\pi})^{1/2}\sin(n\xi)$ and the following conditions hold :

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.
(b) If $\varphi \in D(A)$ then $A\varphi = -\sum_{n=1}^{\infty} n^2 < \varphi, z_n > z_n$.
(c) For $\varphi \in X$, $C(t)\varphi = \sum_{n=1}^{\infty}\cos(nt) < \varphi, z_n > z_n$. Moreover, from these expression, it follows that $S(t)\varphi = \sum_{n=1}^{\infty}\frac{\sin(nt)}{n} < \varphi, z_n > z_n$, that $S(t)$ is compact for every $t > 0$ and that $\|C(t)\| \le 1$ and $\|S(t)\| \le 1$ for every $t \in [0,\pi]$.
(d) If G denotes the group of translations on X defined by $G(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(G(t) + G(-t))$. Hence it follows, see [9], that $A = B^2$, where B is the infinitesimal generator of the group G and that $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$.

Now, we consider the boundary-value problem with nonlocal conditions

$$\frac{\partial^2 w(t,\xi)}{\partial t^2} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + F(t,\xi,w(t,\xi)), \quad t \in I = [0,\pi], \tag{3.1}$$

$$w(t,0) = w(t,\pi) = 0, \quad t \in I, \tag{3.2}$$

$$w(0,\xi) = x_0(\xi) + \sum_{i=1}^{n}\alpha_i w(t_i,\xi), \quad \xi \in I, \tag{3.3}$$

$$\frac{\partial w(0,\xi)}{\partial t} = y_0(\xi) + \sum_{i=1}^{k}\beta_i w(s_i,\xi), \quad \xi \in I, \tag{3.4}$$

where $x_0, y_0 \in X$; $F : I^2 \times \mathbb{R} \to \mathbb{R}$ is continuous and $0 < t_i, s_j < \pi$, α_i, β_j are prefixed numbers. Under the previous conditions, the nonlocal differential problem (3.1)-(3.4) can be modelled as the abstract nonlocal Cauchy problem

$$u''(t) = Au(t) + f(t,u(t)), \quad t \in I, \tag{3.5}$$

$$u(0) = x_0 + q(u), \tag{3.6}$$

$$u'(0) = y_0 + p(u), \tag{3.7}$$

where $f(t,x)(\xi) = F(t,\xi,x(\xi))$, $x \in X$, and $p, q : C(I : X) \to X$ are defined by

$$q(u)(\xi) = \sum_{i=1}^{n}\alpha_i u(t_i,\xi), \quad p(u)(\xi) = \sum_{i=1}^{k}\beta_i u(s_i,\xi), \quad u \in C(I : X).$$

Proposition 3.1. *Assume that the previous conditions are verified and that there exists a function $\eta(\cdot) \in \mathrm{L}^1(I : \mathrm{L}^\infty(I : R))$ such that*

$$|F(t,\xi,x_1) - F(t,\xi,x_2)| \le \eta(t,\xi)|x_1 - x_2|, \quad t,\xi \in I, x_i \in \mathbb{R}.$$

If

$$\Theta = \sum_{i=1}^{n}|\alpha_i| + \sum_{i=1}^{k}|\beta_i| + \int_0^{\pi}\eta(s,\cdot)_\pi ds < 1,$$

then there exists a unique mild solution, $u(\cdot)$, of (3.5)-(3.7). If in addition $x_0 + \sum_{i=1}^{n}\alpha_i u(t_i) \in D(A)$ and $y_0 + \sum_{i=1}^{k}\beta_i u(s_i) \in E$, then $u(\cdot)$ is a classical solution.

For the proof of this proposition: the existence follows from Theorem 2.3, and the regularity assertion is consequence of Theorem 2.6 since X has the *RNP* property.

To complete this section we consider the nonlocal Cauchy problem

$$\frac{\partial^2 w(t,\xi)}{\partial t^2} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + F(t,\xi,w(t,\xi),\frac{\partial w(t,\xi)}{\partial t}), \quad t \in I, \tag{3.8}$$

$$w(t,0) = w(t,\pi) = 0, \quad t \in I, \tag{3.9}$$

$$w(0,\xi) = x_0(\xi) + \int_0^\pi Q(w(s,\cdot))(\xi)ds, \quad \xi \in I, \tag{3.10}$$

$$\frac{\partial w(0,\xi)}{\partial t} = y_0(\xi) + \int_0^\pi P(\frac{\partial w(s,\cdot)}{\partial s})(\xi)ds, \quad \xi \in I, \tag{3.11}$$

where $x_0, y_0 \in X$ and $P : X \to X$, $Q : X \to E$ are Lipschitz continuous. We refer the reader to [12] for examples of operators with these properties. Under the previous conditions, problem (3.8)-(3.11) can be modelled as the abstract nonlocal Cauchy problem

$$u''(t) = Au(t) + f(t,u(t),u'(t)), \quad t \in I, \tag{3.12}$$

$$u(0) = x_0 + q(u,u'), \tag{3.13}$$

$$u'(0) = y_0 + p(u,u'), \tag{3.14}$$

where the substituting operators $f : I \times X \to X$ and $p, q : C(I : X)^2 \to X$ are defined by $f(t,x)(\xi) = F(\xi,t,x(\xi))$ and

$$p(u,v) = \int_0^\pi P(v(s))(\xi)ds \quad \text{and} \quad q(u,v) = \int_0^\pi Q(u(s))(\xi)ds, \quad u,v \in C(I : X).$$

Proposition 3.2. *Assume that the followings conditions are satisfied.*

(a) *There exist a continuous function $\eta : I^3 \to \mathbb{R}$ such that*

$$|F(t,\xi,x_1,x_2) - F(s,\xi,y_1,y_2)| \le \eta(t,s,\xi)(\,|t-s| + \sum_{i=1}^{2} |x_i - y_i|)$$

for every $t, s, \xi \in I, x_i, y_i \in \mathbb{R}$.

(b) *There exist constants L_P, L_Q such that*

$$\|P(x) - P(y)\| \le L_P \|x - y\|, \quad x, y \in X,$$
$$\|Q(x) - Q(y)\|_1 \le L_Q \|x - y\|, \quad x, y \in X.$$

If $\Theta = 2(L_P + L_Q)\pi + 2\int_0^\pi \eta(s,s,\cdot)_\pi ds < 1$, then there exist a unique mild solution, $u(\cdot)$, of (3.12)-(3.14). If $(x_0 + q(u,u'), y_0 + p(u,u')) \in D(A) \times E$, then $u(\cdot)$ is a classical solution.

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