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EXISTENCE OF SOLUTIONS FOR A CLASS OF ELLIPTIC SYSTEMS IN $\mathbb{R}^N$ INVOLVING THE p-LAPLACIAN

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Abstract. Using a variational approach, we study a class of nonlinear elliptic systems derived from a potential and involving the p-Laplacian. Under suitable assumptions on the nonlinearities, we show the existence of nontrivial solutions.

1. Introduction

In this paper, we deal with the nonlinear elliptic system

$$\begin{aligned} -\Delta_p u &= \frac{\partial F}{\partial u}(x,u,v) \quad \text{in } \mathbb{R}^N, \\ -\Delta_q v &= \frac{\partial F}{\partial v}(x,u,v) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{1.1}$$

The nonlinearities on the right hand side are the gradient of a C^1-functional F and $\triangle_p$ is the so-called p-Laplacian operator i.e. $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$; u and v are unknown real-valued functions defined in $\mathbb{R}^N$ and belonging to appropriate function spaces; $1 < p,\, q < N$. Many authors studied the existence of solutions for such problems (equations or systems) for which explicit solutions generally can not be given.

We observe that there exists a vast literature on the use of the Mountain Pass Theorem. Before stating our main theorem, we recall some work about some nonlinear problems: Rabinowitz [11] investigated a class of superlinear Schrödinger equations of the form $-\Delta u + q(x)u = f(x,u)$ defined in $\mathbb{R}^N$ using a variational approach based on a variant of the Mountain Pass Theorem. The nonlinear function verifies $|\partial f/\partial u| \le a|u|^{p-1} + b$; a and b are positive constants; $1 < p < 2^* - 1$ (in this case, the equation is said superlinear). Yu [14] obtained sufficient conditions on the nonlinearity for the existence of positive solutions for some nonlinear equations of the form $-\operatorname{div}(|a(x)\nabla u|^{p-2}\nabla u) = b(x)|u|^{p-2}u + f(x,u)$ defined on a smooth exterior domains; $a(x)$ and $b(x)$ are smooth functions. Costa [4] studied a class of elliptic systems $-\Delta u + a(x)u = f(x,u,v)$; $-\Delta v + b(x)v = g(x,u,v)$ in $\mathbb{R}^N$; with $(f,g) = \nabla F$, the potential F is nonquadratic at infinity. The partial derivatives satisfy the conditions $|\nabla f(x,U)| + |\nabla g(x,U)| \le c(1 + |U|^{p-1}); 1 < p < 2^* - 1$ if

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$N \geq 3$ (or $1 \leq p < +\infty$ if $N = 1, 2$). Do O [9] considered a non-autonomous perturbed eigenvalue problem involving the p-Laplacian of the form $-\Delta_p u = f(x, u)$ defined in $\mathbb{R}^N$; the nonlinearity f interacts with the first eigenvalue of a corresponding problem, and also verifies the following estimate $|f(x, u)| \leq \varphi(x)|u|^r + \psi(x)|u|^s$; $\varphi(x)$ and $\psi(x)$ are suitable functions; $0 < r \leq p - 1 \leq s < p^* - 1$.

In this work, we show the existence of nontrivial solutions for System (1.1) in homogeneous Sobolev spaces under mixed subcritical growth conditions; the primitive F being intimately connected with the first eigenvalue of an appropriate system. Using a weak version of the Palais-Smale condition, due to Cerami [5], we can apply the Mountain Pass Theorem to System (1.1). Our main goal in this article is to illustrate how the ideas introduced in [4, 9, 11] can be applied to handle the problem of existence of nontrivial solutions for System (1.1).

This paper is organized as follows. In section 2, we present some preliminary results and definitions; we also introduce precise assumptions under which our problem is studied. We reserve the section 3 for the proof of the main result.

2. Notations and Hypotheses

We first recall some standard definitions and notations. Let Z be a reflexive Banach space endowed with a norm $\|\cdot\|$. Let $I \in C^1(Z, \mathbb{R})$. We say that I satisfies the Cerami condition, denoted by (C) condition, if every $(w_n) \in Z$ such that

$$|I(w_n)| \leq c \quad \text{and} \quad (1 + \|w_n\|)I'(w_n) \to 0$$

contains a convergent subsequence in the norm of Z.

For $1 < m < N$, let $m^* = \frac{Nm}{N-m}$ be the critical Sobolev exponent of m.

Let $D^{1,m}(\mathbb{R}^N)$ be the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{1,m} \equiv \|\nabla u\|_m = \left(\int_{\mathbb{R}^N} |\nabla u|^m dx\right)^{1/m}$. $D^{1,m}(\mathbb{R}^N)$ is a reflexive Banach space and may be written $D^{1,m}(\mathbb{R}^N) = \{u \in L^{m^*}(\mathbb{R}^N) : \nabla u \in (L^m(\mathbb{R}^N))^N\}$. Moreover, Sobolev imbedding holds; in fact there exists a positive constant c such that $\|u\|_{m^*} \leq c\|u\|_{1,m}$ for all $u \in D^{1,m}(\mathbb{R}^N)$ (see [13]).

Now we denote by Z the product space $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$; Z^* designates the dual space equipped with the dual norm $\|\cdot\|_*$.

For (u, v) in Z, we define the functionals I, J, K by

$$\begin{gathered} J(u, v) = \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|v\|_{1,q}^q, \\ K(u, v) = \int_{\mathbb{R}^N} F(x, u(x), v(x))dx, \\ I(u, v) = J(u, v) - K(u, v). \end{gathered}$$

In this article we use the following hypotheses:

(H1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ and $F(x, 0, 0) = 0$.

(H2) For all $U = (u, v) \in \mathbb{R}^2$ and for almost every $x \in \mathbb{R}^N$

$$\begin{aligned} |\frac{\partial F}{\partial u}(x, U)| &\leq a_1(x)|U|^{p_1-1} + a_2(x)|U|^{p_2-1}, \\ |\frac{\partial F}{\partial v}(x, U)| &\leq b_1(x)|U|^{q_1-1} + b_2(x)|U|^{q_2-1}. \end{aligned}$$

Here $1 < p_1,\ q_1 < \min(p,q),\ \max(p,q) < p_2,\ q_2 < \min(p^*,q^*),\ a_i \in L^{\alpha_i}(\mathbb{R}^N) \cap L^{\beta_i}(\mathbb{R}^N)$, $b_i \in L^{\gamma_i}(\mathbb{R}^N) \cap L^{\delta_i}(\mathbb{R}^N)$, $i = 1,2$.

$$\alpha_i = \frac{p^*}{p^* - p_i}, \quad \gamma_i = \frac{q^*}{q^* - q_i},$$
$$\beta_i = \frac{p^*q^*}{p^*q^* - p^*(p_i - 1) - q^*}, \quad \delta_i = \frac{p^*q^*}{p^*q^* - q^*(q_i - 1) - p^*}.$$

(H3) $U.\nabla F(x,U) - F(x,U) \le 0$, for all $(x,U) \in \mathbb{R}^N \times \mathbb{R}^2 - \{(0,0)\}$, where $\nabla F = (\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v})$. This type of condition has been introduced by Costa [4].

(H4) At last we suppose the existence of two positive and bounded functions $a \in L^{N/p}(\mathbb{R}^N)$ and $b \in L^{N/q}(\mathbb{R}^N)$ such that

$$\limsup_{|U|\to 0} \frac{pq|F(x,U)|}{qa(x)|u|^p + pb(x)|v|^q} < \lambda_1 < \liminf_{|U|\to+\infty} \frac{pq|F(x,U)|}{qa(x)|u|^p + pb(x)|v|^q}.$$

Putting

$$\Lambda = \Big\{(u,v) \in Z : \frac{1}{p}\int_{\mathbb{R}^N} a(x)|u|^p dx + \frac{1}{q}\int_{\mathbb{R}^N} b(x)|v|^q dx = 1\Big\},$$

$\lambda_1 = \inf_\Lambda J(u,v)$ is the first eigenvalue of the system

$$\begin{aligned} -\Delta_p u &= \lambda a(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ -\Delta_q v &= \lambda b(x)|v|^{q-2}v \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{2.1}$$

Remark The hypothesis (H4) is related with the interaction of the potential F and λ_1. Costa [4] was the first to introduce such assumption. A variant of this condition appeared in Do O [9]. An example of such functions is

$$F(x,u,v) = -a(x)|u|^\alpha|v|^\beta; \quad \max(p,q) < \alpha,\ \beta < \min(p^*,q^*).$$

It is easy to prove that F satisfies (H1), (H3) and (H4). In order to obtain (H2), we use Young's inequality.

3. Existence of solutions

Taking into account the above hypotheses, we have some assertions.

Lemma 3.1. *Under Hypotheses (H1) and (H2), the functional K is well defined and is of class C^1 on Z. Moreover, its derivative is*

$$K'(u,v)(w,z) = \int_{\mathbb{R}^N} (\frac{\partial F}{\partial u}(x,u,v)w + \frac{\partial F}{\partial v}(x,u,v)z)dx \quad \forall (u,v),(w,z) \in Z.$$

Proof. K is well defined on Z. Indeed, for all $(u,v) \in Z$ and by virtue of (H1) and (H2), we have

$$\begin{aligned} F(x,u,v) &= \int_0^u \frac{\partial F}{\partial s}(x,s,v)ds + F(x,0,v) \\ &= \int_0^u \frac{\partial F}{\partial s}(x,s,v)ds + \int_0^v \frac{\partial F}{\partial s}(x,0,s)ds + F(x,0,0) \end{aligned}$$

and

$$\begin{aligned} F(x,u,v) \le & c_1[a_1(x)(|u|^{p_1} + |v|^{p_1-1}|u|) + a_2(x)(|u|^{p_2} + |v|^{p_2-1}|u|) \\ & + b_1(x)|v|^{q_1} + b_2(x)|v|^{q_2}]. \end{aligned}$$

Using Hölder's inequality and Sobolev's imbedding and the fact that $a_i \in L^{\alpha_i}(\mathbb{R}^N)\cap L^{\beta_i}(\mathbb{R}^N)$, $b_i \in L^{\gamma_i}(\mathbb{R}^N)$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N} F(x,u,v)dx\\&\le c_2(\|a_1\|_{\alpha_1}\|u\|_{1,p}^{p_1} + \|a_1\|_{\beta_1}\|v\|_{1,q}^{p_1-1}\|u\|_{1,p} + \|a_2\|_{\alpha_2}\|u\|_{1,p}^{p_2}\\&\quad + \|a_2\|_{\beta_2}\|v\|_{1,q}^{p_2-1}\|u\|_{1,p} + \|b_1\|_{\gamma_1}\|v\|_{1,q}^{q_1} + \|b_2\|_{\gamma_2}\|v\|_{1,q}^{q_2}) < +\infty.\end{aligned}$$

Observe that $K'(u,v)$ is also well defined on Z since

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x,u,v)wdx \le & c_3\Big(\|a_1\|_{\alpha_1}\|u\|_{1,p}^{p_1-1} + \|a_1\|_{\beta_1}\|v\|_{1,q}^{p_1-1}\\&+ \|a_2\|_{\alpha_2}\|u\|_{1,p}^{p_2-1} + \|a_2\|_{\beta_2}\|v\|_{1,q}^{p_2-1}\Big)\|w\|_{1,p} < +\infty.\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x,u,v)zdx \le & c_4\Big(\|b_1\|_{\delta_1}\|u\|_{1,p}^{q_1-1} + \|b_1\|_{\gamma_1}\|v\|_{1,q}^{q_1-1}\\&+ \|b_2\|_{\delta_2}\|u\|_{1,p}^{q_2-1} + \|b_2\|_{\gamma_2}\|v\|_{1,q}^{q_2-1}\Big)\|z\|_{1,q} < +\infty.\end{aligned}$$

Now, we show that K is differentiable in sense of Fréchet at each point (u,v) of Z i.e. $\forall\varepsilon > 0$, there exists $\delta = \delta(\varepsilon,u,v) > 0$ such that $(\|w\|_{1,p} + \|z\|_{1,q}) \le \delta$ implies

$$|K(u+w,v+z) - K(u,v) - K'(u,v)(w,z)| \le \varepsilon(\|w\|_{1,p} + \|z\|_{1,q}).$$

Let B_R be the ball of radius R, centered at the origin of $\mathbb{R}^N$. We put $B'_R = \mathbb{R}^N - B_R$ and we define a functional K_R on $D^{1,p}(B_R)\times D^{1,q}(B_R)$ by $K_R(u,v) = \int_{B_R} F(x,u(x),v(x))dx$. Taking (H1) and (H2) into account, it is well-known that $K_R \in C^1(D^{1,p}(B_R)\times D^{1,q}(B_R))$ and for any $(w,z) \in D^{1,p}(B_R)\times D^{1,q}(B_R)$, we have

$$K'_R(u,v)(w,z) = \int_{B_R}(\frac{\partial F}{\partial u}(x,u,v)w + \frac{\partial F}{\partial v}(x,u,v)z)dx.$$

Moreover, K'_R is compact from Z to Z^* (see [9, 10, 12]). On the other hand, for all (u,v) , $(w,z) \in Z$, we have

$$\begin{aligned}&\big|K(u+w,v+z) - K(u,v) - K'(u,v)(w,z)\big|\\&\le \big|K_R(u+w,v+z) - K_R(u,v) - K'_R(u,v)(w,z)\big|\\&+ \Big|\int_{B'_R}(F(x,u+w,v+z) - F(x,u,v) - \frac{\partial F}{\partial u}(x,u,v)w - \frac{\partial F}{\partial v}(x,u,v)z)dx\Big|.\end{aligned}$$

By the Mean-value theorem, we can write

$$F(x,u+w,v+z) - F(x,u,v) = \frac{\partial F}{\partial u}(x,u+\theta_1 w,v)w + \frac{\partial F}{\partial v}(x,u,v+\theta_2 z)z,$$

for $\theta_1,\theta_2 \in]0,1[$. By the growth condition (H2) and the fact that for $i = 1,2$,

$$\begin{aligned}&\|a_i\|_{L^{\alpha_i}(B'_R)} + \|a_i\|_{L^{\beta_i}(B'_R)} \to 0,\\&\|b_i\|_{L^{\gamma_i}(B'_R)} + \|b_i\|_{L^{\delta_i}(B'_R)} \to 0,\end{aligned} \tag{3.1}$$

as $R \to \infty$, we obtain for R sufficiently large that

$$\begin{aligned}&\Big|\int_{B'_R}(F(x,u+w,v+z)-F(x,u,v)-\frac{\partial F}{\partial u}(x,u,v)w-\frac{\partial F}{\partial v}(x,u,v)z)dx\Big|\\&\le \varepsilon\big(\|w\|_{1,p}+\|z\|_{1,q}\big).\end{aligned}$$

We have only to show that K' is continuous on Z. Let $(u_n,v_n)\to(u,v)$ in Z. For $(w,z)\in Z$, we have

$$\begin{aligned}&|K'(u_n,v_n)(w,z)-K'(u,v)(w,z)|\\&=|K'_R(u_n,v_n)(w,z)-K'_R(u,v)(w,z)|\\&\quad+\Big|\int_{B'_R}(\frac{\partial F}{\partial u}(x,u_n,v_n)+\frac{\partial F}{\partial u}(x,u,v))wdx\Big|\\&\quad+\Big|\int_{B'_R}(\frac{\partial F}{\partial v}(x,u_n,v_n)+\frac{\partial F}{\partial v}(x,u,v))zdx\Big|.\end{aligned}$$

Then K'_R is continuous on $D^{1,p}(B_R)\times D^{1,q}(B_R)$ (see [9, 11]). The first expression on the right hand side of the above equation tends to 0 as $n\to+\infty$; we use (H2) and (3.1) to prove that both the second and the third expressions tend also to 0 as R sufficiently large. □

Remark The functional J is of class C^1 on Z and its derivative is

$$J'(u,v)(w,z)=\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla wdx+\int_{\mathbb{R}^N}|\nabla v|^{q-2}\nabla v\nabla zdx.$$

Lemma 3.2. *Under assumptions (H1) and (H2), K' is compact from Z to Z^*.*

Proof. Let (u_n,v_n) be a bounded sequence in Z. Then there is a subsequence denoted again (u_n,v_n) weakly convergent to (u,v) in Z. As before, we write

$$\begin{aligned}&|K'(u_n,v_n)(w,z)-K'(u,v)(w,z)|\\&=|K'_R(u_n,v_n)(w,z)-K'_R(u,v)(w,z)|\\&\quad+\Big|\int_{B'_R}(\frac{\partial F}{\partial u}(x,u_n,v_n)-\frac{\partial F}{\partial u}(x,u,v))w\,dx\Big|\\&\quad+\Big|\int_{B'_R}(\frac{\partial F}{\partial v}(x,u_n,v_n)-\frac{\partial F}{\partial v}(x,u,v))z\,dx\Big|.\end{aligned}$$

Since the restriction operator is continuous, we have $(u_n,v_n)\rightharpoonup(u,v)$ in $D^{1,p}(B_R)\times D^{1,q}(B_R)$. Because of the compactness of K'_R, the first expression on the right hand side of the equation tends to 0 as $n\to+\infty$; as above both the second and the third expressions tend also to 0 as R sufficiently large. □

Lemma 3.3. *If (H1), (H2), and (H3) hold then $I=J-K$ satisfies the condition (C).*

Proof. Let $(u_n,v_n)\subset Z$ such that

(i) $|I(u_n,v_n)|\le c$.
(ii) $(1+\|u_n\|_{1,p}+\|v_n\|_{1,q})I'(u_n,v_n)\to 0$ in Z^*, as $n\to+\infty$.

From (ii), we have $I'(u_n,v_n)(w,z) \le \varepsilon_n \to 0$ as $n \to +\infty$, $\forall\ (w,z) \in Z$. In particular, for $(w,z) = (u_n,v_n)$, we get

$$\begin{aligned} & I'(u_n,v_n)(u_n,v_n) \\ & = \|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q - \int_{\mathbb{R}^N} (\frac{\partial F}{\partial u}(x,u_n,v_n)u_n + \frac{\partial F}{\partial v}(x,u_n,v_n)v_n)dx \le \varepsilon_n. \end{aligned}$$

On the other hand,

$$I(u_n,v_n) = \frac{1}{p}\|u_n\|_{1,p}^p + \frac{1}{q}\|v_n\|_{1,q}^q - \int_{\mathbb{R}^N} F(x,u_n,v_n)dx \le c.$$

Then, taking (H2) into account, we get

$$\begin{aligned} \varepsilon_n + c &\ge I'(u_n,v_n)(u_n,v_n) - I(u_n,v_n) \\ &= (1-\frac{1}{p})\|u_n\|_{1,p}^p + (1-\frac{1}{q})\|v_n\|_{1,q}^q \\ &\quad + \int_{\mathbb{R}^N} (F(x,u_n,v_n) - \frac{\partial F}{\partial u}(x,u_n,v_n)u_n - \frac{\partial F}{\partial v}(x,u_n,v_n)v_n)dx \\ &\ge (1-\frac{1}{p})\|u_n\|_{1,p}^p + (1-\frac{1}{q})\|v_n\|_{1,q}^q. \end{aligned}$$

Hence, (u_n,v_n) is bounded in Z. There is a subsequence denoted again (u_n,v_n) weakly convergent in Z. Since K' is compact, $K'(u_n,v_n)$ is a Cauchy's sequence in Z^*. We have $J'(u,v) = I'(u,v) + K'(u,v), \forall (u,v) \in Z$ and

$$\begin{aligned} & \big(J'(u_n,v_n) - J'(u_m,v_m)\big)(u_n-u_m,0) \\ & = \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)(\nabla u_n - \nabla u_m)dx. \end{aligned}$$

Observe that for all $\lambda, \mu \in \mathbb{R}^N$,

$$|\lambda-\mu|^p \le \begin{cases} (|\lambda|^{p-2}\lambda - |\mu|^{p-2}\mu)(\lambda-\mu) & \text{if } p \ge 2, \\ \big[(|\lambda|^{p-2}\lambda - |\mu|^{p-2}\mu)(\lambda-\mu)\big]^{p/2} (|\lambda|+|\mu|)^{(2-p)p/2} & \text{if } 1 < p < 2, \end{cases}$$

(see [6, 15]). Substituting λ and μ by ∇u_n and ∇u_m respectively and integrating over $\mathbb{R}^N$, we obtain

$$\|u_n - u_m\|_{1,p}^p \le (J'(u_n,v_n) - J'(u_m,v_m))(u_n-u_m,0), \text{ if } p \ge 2.$$

and

$$\|u_n - u_m\|_{1,p}^2 \le |(J'(u_n,v_n) - J'(u_m,v_m))(u_n-u_m,0)|(\|u_n\|_{1,p}^p + \|u_m\|_{1,p}^p)^{\frac{2-p}{2}},$$

if $1 < p < 2$.

Since (u_n) is bounded in $D^{1,p}(\mathbb{R}^N)$ and $(J'(u_n,v_n) - J'(u_m,v_m))(u_n-u_m,0) \to 0$ as $n,m \to +\infty$, then (u_n) is a Cauchy's sequence in $D^{1,p}(\mathbb{R}^N)$. Hence (u_n) converges in $D^{1,p}(\mathbb{R}^N)$. In the same way, we prove that (v_n) converges in $D^{1,q}(\mathbb{R}^N)$. □

Moreover the functional $I = J - K$ verifies the geometric conditions of the Mountain Pass Theorem, summarized in the following lemma.

Lemma 3.4. *Under Assumptions (H1), (H2), (H3), and (H4), the functional I satisfies*

(I1) *There exist ρ, $\sigma > 0$ such that $\|u\|_{1,p} + \|v\|_{1,q} = \rho$ implies $I(u,v) \ge \sigma > 0$.*
(I2) *There exists $E \in Z$ such that $\|E\|_Z > \rho$ and $I(E) \le 0$.*

Proof. By (H4), there exists $\rho > 0$ such that

$$\|u\|_{1,p} + \|v\|_{1,q} = \rho \Longrightarrow F(x,u,v) < \lambda_1(\frac{1}{p}a(x)|u|^p + \frac{1}{q}b(x)|v|^p).$$

The variational characterization of λ_1 (see [7]) gives

$$\int_{\mathbb{R}^N} F(x,u,v)dx < \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|v\|_{1,q}^q.$$

Then there exist $\rho, \sigma > 0$ such that $\|u\|_{1,p} + \|v\|_{1,q} = \rho$ implies $I(u,v) \geq \sigma > 0$. Let (φ, ψ) be an eigenfunction associated with λ_1. In view of (H4), we get for $\varepsilon > 0$ and t sufficiently large,

$$F(x, t^{\frac{1}{p}}\varphi, t^{\frac{1}{q}}\psi) \geq (\lambda_1 + \varepsilon)(\frac{t}{p}a(x)|\varphi|^p + \frac{t}{q}b(x)|\psi|^p).$$

Hence

$$\begin{aligned} I(t^{\frac{1}{p}}\varphi, t^{\frac{1}{q}}\psi) &= \frac{t}{p}\int_{\mathbb{R}^N} |\nabla\varphi|^p dx + \frac{t}{q}\int_{\mathbb{R}^N} |\nabla\psi|^q dx - \int_{\mathbb{R}^N} F(x, t^{\frac{1}{p}}\varphi, t^{\frac{1}{q}}\psi)dx \\ &\leq \frac{t}{p}\int_{\mathbb{R}^N} |\nabla\varphi|^p dx + \frac{t}{q}\int_{\mathbb{R}^N} |\nabla\psi|^q dx \\ &\quad - (\lambda_1 + \varepsilon)\Big(\frac{t}{p}\int_{\mathbb{R}^N} a(x)|\varphi|^p dx + \frac{t}{q}\int_{\mathbb{R}^N} b(x)|\psi|^p dx\Big) \\ &\leq -t\varepsilon(\frac{1}{p}\int_{\mathbb{R}^N} a(x)|\varphi|^p dx + \frac{1}{q}\int_{\mathbb{R}^N} b(x)|\psi|^p dx). \end{aligned}$$

we deduce that $\lim_{t\to+\infty} I(t^{\frac{1}{p}}\varphi, t^{1/q}\psi) = -\infty$. So, for t large, $I(t^{1/p}\varphi, t^{1/q}\psi) \leq 0$.

Consequently, the functional I has a critical value. Note that the critical points of I are precisely the weak solutions of System (1.1). □

Now, we can state the main theorem.

Theorem 3.5. *System* (1.1) *has at least one nontrivial solution* (u, v).

Proof. In view of Lemmas 3.3 and 3.4, we can apply the Mountain-Pass theorem (see [8, 10, 13]) to conclude that system (1.1) has a nontrivial weak solution. □

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