Electronic Journal of Differential Equations, Vol. 2003(2003), No. 57, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

POSITIVE SOLUTIONS FOR INDEFINITE INHOMOGENEOUS NEUMANN ELLIPTIC PROBLEMS

YAVDAT IL'YASOV & THOMAS RUNST

Abstract. We consider a class of inhomogeneous Neumann boundary-value problems on a compact Riemannian manifold with boundary where indefinite and critical nonlinearities are included. We introduce a new and, in some sense, more general variational approach to these problems. Using this idea we prove new results on the existence and multiplicity of positive solutions.

1. Introduction and main results

Let (M, g) be a smooth connected compact Riemannian manifold of dimension $n \geq 2$ with boundary ∂M. In this paper we study the existence and multiplicity of positive solutions for the following class of inhomogeneous Neumann boundary-value problems with indefinite nonlinearities

$$-\Delta_p u - \lambda k(x)|u|^{p-2}u = K(x)|u|^{\gamma-2}u \quad \text{in } M, \tag{1.1}$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial n} + d(x)|u|^{p-2}u = D(x)|u|^{q-2}u \quad \text{on } \partial M, \tag{1.2}$$

where Δ_p, ∇ denotes the p-Laplace–Beltrami operator and the gradient in the metric g, respectively. $\frac{\partial}{\partial n}$ is the normal derivative with respect to the outward normal n on ∂M and the metric g. When $p = 2$ the problem corresponds to the classical Laplacian and also in this case the results are new. We study the problem (1.1)-(1.2) with respect to the real parameter λ. In what follows we assume that

$$p < \gamma \leq p^*, \quad \text{where } p^* = \begin{cases} \frac{pn}{n-p} & \text{if } p < n, \\ +\infty & \text{if } p \geq n, \end{cases} \tag{1.3}$$

$$p < q \leq p^{**}, \quad \text{where } p^{**} = \begin{cases} \frac{p(n-1)}{(n-p)} & \text{if } p < n, \\ +\infty & \text{if } p \geq n, \end{cases} \tag{1.4}$$

$$k(\cdot),\ K(\cdot) \in L_\infty(M), \quad d(\cdot), D(\cdot) \in L_\infty(\partial M). \tag{1.5}$$

Here p^* and p^{**} are the critical Sobolev exponents for the embedding $W_p^1(M) \subset L_{p^*}(M)$ and the trace-embedding $W_p^1(M) \subset L_{p^{**}}(\partial M)$, respectively. If $\gamma = p^*$

2000 *Mathematics Subject Classification.* 35J70, 35J65, 47H17.
Key words and phrases. p-Laplacian, nonlinear boundary conditions, indefinite and critical nonlinearities.

Submitted January 10, 2003. Published May 19, 2003.

and/or $q = p^{**}$, then one has a problem with critical exponents. When all non-linear terms are present both in the differential equation (1.1) and in the non-linear Neumann boundary condition (1.2), i.e. when $K \neq 0$ in M and $D \neq 0$ on ∂M one has a inhomogeneous problem. The nonlinearity $K(x)|u|^{\gamma-2}u$ ($D(x)|u|^{q-2}u$) is called indefinite if the function K on M (D on ∂M) changes the sign cf. [1, 2].

Problems like (1.1)-(1.2) arise in several contexts (see for example [4], [15]). In particular, when $p = 2$, $\gamma = p^*, q = p^{**}$, $n \geq 3$, the problem of the existence a positive solution for (1.1)-(1.2) is equivalent to the classical problem of finding a conformal metric g' on M with the prescribed scalar curvature K on M and the mean curvature D on ∂M [5, 9, 21]. For $p \neq 2$ we refer to [7] for background material and applications.

The case which is best known in the literature is the problem (1.1) with Dirichlet boundary condition and when nonlinearity has definite sign. The indefiniteness of the sign of nonlinearity changes essentially the structure of the solutions set. In this case, the dependence of the problem on the parameter λ is more complicate (cf. [1, 2]). The homogeneous cases with indefinite nonlinearity has been treated in several recent papers (in [2, 9, 10, 11, 16, 19, 22] for $p = 2$ and in [8] also for $p \neq 2$. An additional difficulty occurs if the problem is inhomogeneous or it involves multiple critical exponents. For instance, in applying of the constrained minimization method to the inhomogeneous problem, i.e. the finding of a suitable constraint or the finding of a suitable modification for the variational problem is not simple. The inhomogeneous cases of (1.1)-(1.2) for $p = 2$ with definite sign of nonlinearity have been considered in [15], [21]. In recent papers [12, 18] the authors investigated the inhomogeneous Neumann boundary value problem when one of the nonlinearities can be indefinite whereas the rest is with definite sign.

The main purpose of the present paper is a development of the fibering method of Pohozaev [17] for the investigation of the inhomogeneous Neumann boundary value problems (1.1)-(1.2) with indefinite nonlinearities and critical exponents.

Let us state our main results. To illustrate, we consider the case $d(x) \equiv 0$. Denote by $d\mu_g$ and $d\nu_g$ the Riemannian measure (induced by the metric g) on M and on ∂M, respectively. We consider our problem in the framework of the Sobolev space $W = W_p^1(M)$ equipped with the norm

$$\|u\| = \Big(\int_M |u|^p d\mu_g + \int_M |\nabla u|^p d\mu_g\Big)^{1/p}. \tag{1.6}$$

Define

$$\lambda^*(K) = \inf\Big\{\frac{\int_M |\nabla u|^p d\mu_g}{\int_M k(x)|u|^p d\mu_g} : \int_M K(x)|u|^\gamma d\mu_g \geq 0,\, u \in W\Big\},$$
$$\lambda^*(D) = \inf\Big\{\frac{\int_M |\nabla u|^p d\mu_g}{\int_M k(x)|u|^p d\mu_g} : \int_{\partial M} D(x)|u|^q d\nu_g \geq 0,\, u \in W\Big\}.$$

In the case when the set $\{u \in W_p^1(M) : \int_M K(x)|u|^\gamma d\mu_g \geq 0\}$ ($\{u \in W_p^1(M) : \int_{\partial M} D(x)|u|^q d\nu_g \geq 0\}$) is empty we put $\lambda^*(K) = +\infty$ ($\lambda^*(D) = +\infty$).

We denote by I_λ the Euler functional on $W_p^1(M)$ which corresponds to problem (1.1)-(1.2). Our main results on the existence and multiplicity of positive solutions for (1.1)-(1.2) are summarized in the following theorems.

Theorem 1.1. *Under the conditions of* (1.5), $k(x) \geq 0$ *on* M *and* $d(x) \equiv 0$, *we have the following:*

(I) *Let* $p < \gamma \le p^*$*; then* $\lambda^*(K) > 0$ *if and only if* $\int_M K(x)d\mu_g < 0$.
Let $p < q \le p^{**}$*; then* $\lambda^*(D) > 0$ *if and only if* $\int_{\partial M} D(x)d\nu_g < 0$.

(II) *Let* $p < \gamma < p^*$, $p < q < p^{**}$ *and* $q < \gamma$.

(1) *Suppose* $\int_M K(x)d\mu_g < 0$, $\int_{\partial M} D(x)d\nu_g < 0$. *Then for every* $\lambda \in (0, min\{\lambda^*(K),\ \lambda_*(D)\})$ *there exists a ground state* $u_1 \in W^1_p(M)$ *of* I_λ. *Furthermore,* $u_1 > 0$ *on* M *and* $I_\lambda(u_1) < 0$.

(2) *Suppose* $\int_M K(x)d\mu_g < 0$, *the set* $\{x \in M : K(x) > 0\}$ *is not empty and* $D(x) \le 0$ *on* ∂M. *Then for every* $\lambda < \lambda^*(K)$ *there exists a weak positive solution* $u_2 \in W^1_p(M)$ *of* (1.1)-(1.2) *such that* $u_2 > 0$ *on* M *and* $I_\lambda(u_2) > 0$.

Theorem 1.2. *Let* $\gamma = p^*,\ q = p^{**}$. *Under the conditions* (1.5), $k(x) \ge 0$ *on* M *and* $d(x) \equiv 0$, *we have the following: Suppose* $\int_M K(x)d\mu_g < 0$ *and* $D(x) \le 0$ *on* ∂M. *Then for every* $\lambda \in (0, \lambda^*(K))$ *there exists a ground state* $u_1 \in W^1_p(M)$ *of* I_λ. *Furthermore,* $u_1 > 0$ *on* M *and* $I_\lambda(u_1) < 0$.

The proof of these results is based on the fibering method of Pohozaev [17].

Remark 1.3. We refer to the Theorem 4.5, 4.10, Theorem 5.1, Theorem 5.2, for a more general version of the above results.

Remark 1.4. Symmetric results as in Theorem 1.1, Theorem 1.2 (Theorem 4.5, Theorem 4.10, Theorem 5.1 and Theorem 5.2) in more general cases) can be obtained when $\lambda = 0$ ($\lambda \le 0$) is fixed and the problem of the existence of positive solutions for (1.1) is considered with respect to parameter $\mu \in \mathbb{R}$ at the boundary condition

$$|\nabla u|^{p-2}\frac{\partial u}{\partial n} + \mu d(x)|u|^{p-2}u = D(x)|u|^{q-2}u \text{ on } \partial M,$$

instead of (1.2).

Remark 1.5. Some results in this paper have been announced in [13]. Since then, there has been some progress. This paper contains the details and extensions of [13] as well as other results.

Remark 1.6. In the paper [18] existence and multiplicity results for problem (1.1)-(1.2) when D has a definite sign whereas K may change one are proved by using the fibering method. However our approach and results are different then in [18].

The paper is organized as follows. In Section 2, based on the fibering strategy of Pohozaev we introduce an explicit process of construction of the constrained minimization problems associated with the given abstract functional on Banach spaces. In Section 3, we give the basic variational formulation for problem (1.1)-(1.2). In Section 4 we prove our main results on the existence and multiplicity of positive solutions in subcritical cases of nonlinearities. Finally, in Section 5 we prove the existence of positive solutions in critical cases of exponents.

2. The fibering scheme

A powerful tool of studying the existence of critical points for a functional given on Banach space is a constrained minimization method [2, 8, 9, 20]. The main difficulty in applying the method is to find suitable constraints on admissible functions and/or to find a suitable modification for the variational problem.

In this section, based on the fibering strategy of Pohozaev [17] we introduce an explicit scheme of construction of constrained minimization problems for arbitrary functional given on Banach spaces.

Let $(W, \|\cdot\|)$ be a real Banach space. Assume that the norm $\|\cdot\|$ defines a C^1-functional $u \to \|u\|$ on $W \setminus \{0\}$. In this case, the sphere $S^1 = \{v \in W | \ \|v\| = 1\}$ is a closed submanifold of class C^1 in W and $\mathbb{R}^+ \times S^1$ is C^1-diffeomorphic with $W \setminus \{0\}$. Thus we have the trivial principal fibre bundle $P(S^1, \mathbb{R}^+)$ over S^1 with structure group $\mathbb{R}^+$ and the bundle space $W \setminus \{0\}$ that C^1-diffeomorphic to $\mathbb{R}^+ \times S^1$.

Actually the way of construction of constrained minimization problems which we introduce below relies on the trivial principal fibre bundle $P(S^1, \mathbb{R}^+)$. In what follows, it is therefore reasonable to call this scheme as *the trivial fibering scheme with respect to fibre bundle* $P(S^1, \mathbb{R}^+)$ (in short *the trivial fibering scheme*).

Let $I(u)$ be a functional on W of class $C^1(W \setminus \{0\})$. Associate with I there exists a function $\tilde{I} : \mathbb{R}^+ \times S^1 \to \mathbb{R}$ defined by

$$\tilde{I}(t, v) = I(tv), \quad (t, v) \in \mathbb{R}^+ \times S^1. \tag{2.1}$$

Since $\mathbb{R}^+ \times S^1$ is C^1-diffeomorphic with $W \setminus \{0\}$ it follows that $\tilde{I}(t, v)$ is a C^1-functional on $\mathbb{R}^+ \times S^1$ and the set of critical points of the functional $\tilde{I}(t, v)$ on $\mathbb{R}^+ \times S^1$ as well as the set of critical points of the functional $I(u)$ on $W \setminus \{0\}$ are one-to-one. Moreover, we have the following statement.

Proposition 2.1 (Pohozaev [17]). *Let $(t_0, v_0) \in \mathbb{R}^+ \times S^1$ be a critical point of $\tilde{I}(t, v)$ then $u_0 = t_0 v_0 \in W \setminus \{0\}$ is a critical point of $I(u)$.*

We impose an additional condition on I

(RD) The first derivative $\frac{\partial}{\partial t}\tilde{I}(t, v)$ is a C^1-functional on $\mathbb{R}^+ \times S^1$.

We define

$$Q(t, v) = \frac{\partial}{\partial t}\tilde{I}(t, v), \quad L(t, v) = \frac{\partial^2}{\partial t^2}\tilde{I}(t, v), \ (t, v) \in \mathbb{R}^+ \times S^1. \tag{2.2}$$

Extract from $\mathbb{R}^+ \times S^1$ the sets

$$\Sigma^1 = \{(t, v) \in \mathbb{R}^+ \times S^1 | Q(t, v) = 0, \ L(t, v) > 0\}, \tag{2.3}$$

$$\Sigma^2 = \{(t, v) \in \mathbb{R}^+ \times S^1 | Q(t, v) = 0, \ L(t, v) < 0\}. \tag{2.4}$$

Lemma 2.2. *Assume that (RD) holds, and let $j = 1, 2$. Then the set Σ^j is a submanifold of class C^1 in $\mathbb{R}^+ \times S^1$ and it is local C^1-diffeomorphic with S^1.*

The proof of this lemma will follow directly from the next proposition.

Proposition 2.3. *Let $(t_0, v_0) \in \Sigma^j$, $j = 1, 2$. Then there exist a neighborhood $\Lambda(v_0) \subset S^1$ of $v_0 \in S^1$ and an uniqueness C^1-map $t^j : \Lambda(v_0) \to \mathbb{R}$ such that*

$$t^j(v_0) = t_0, \quad (t^j(v), v) \in \Sigma^j, \quad v \in \Lambda(v_0), \quad j = 1, 2. \tag{2.5}$$

Proof. Let $j = 1$, $j = 2$. Assume $(t_0, v_0) \in \Sigma^j$. Then $\partial Q(t_0, v_0)/\partial t = L(t_0, v_0) \neq 0$. It follows from the assumption (RD) that we have $Q \in C^1(\mathbb{R}^+ \times S^1)$. Hence, by the implicit function theorem we obtain the proof of the proposition. $\square$

Finally, we introduce the main constrained minimization problems associated with the given functional I.

Let $I(u)$ be a functional on W of class $C^1(W \setminus \{0\})$ and the assumption (RD) holds. The main constrained minimization problems by the trivial fibering scheme are the following

$$\hat{I}^j = \inf\{\tilde{I}(t,v) : (t,v) \in \Sigma^j\}, \quad j = 1,2, \tag{2.6}$$

where

$$\hat{I}^j = +\infty, \quad \text{if } \Sigma^j = \emptyset,\ j = 1,2. \tag{2.7}$$

Definition 2.4. A point $(t_0, v_0) \in \Sigma^j$ is said to be a solution of the problem (2.6), if $-\infty < \hat{I}^j = \tilde{I}(t_0, v_0) < \infty$, where $j = 1,2$.

Remark 2.5. It is reasonable to consider also the maximization problems like (2.6). However, the substitution $I' = -I$ reduces any maximization problem to the minimization one. Hence it suffices to study only minimization problems (2.6).

Now we show that the trivial fibering scheme makes it possible to study of the existence of critical points of functionals. Denote by $\tilde{J}^j$ the restriction of $\tilde{I}$ on the submanifolds Σ^j, for $j = 1,2$:

$$\tilde{J}^j(t,v) = \tilde{I}(t,v),\ (t,v) \in \Sigma^j,\ j = 1,2.$$

Lemma 2.6. *Assume that hypothesis (RD) holds, and let $j = 1,2$. Let (t_0, v_0) be a critical point of the functional $\tilde{J}^j$ on the submanifolds Σ^j, i.e. holds*

$$d\tilde{J}^j(t_0, v_0)(h) = 0, \quad \forall h \in T_{(t_0,v_0)}(\Sigma^j). \tag{2.8}$$

Then (t_0, v_0) is a critical point for $\tilde{I}$ on $\mathbb{R}^+ \times S^1$, i.e.,

$$d\tilde{I}(t_0, v_0)(l) = 0,\ \forall l \in T_{(t_0,v_0)}(\mathbb{R}^+ \times S^1). \tag{2.9}$$

Here $d\tilde{J}^j(t_0, v_0)$ ($d\tilde{I}(t_0, v_0)$) is the differential of $\tilde{J}^i : \Sigma^j \to \mathbb{R}$ ($\tilde{I} : \mathbb{R}^+ \times S^1 \to \mathbb{R}$) at point (t_0, v_0), the set $T_{(t_0,v_0)}(\Sigma^j)$ ($T_{(t_0,v_0)}(\mathbb{R}^+ \times S^1)$) denotes the tangent space to Σ^j ($\mathbb{R}^+ \times S^1$) at (t_0, v_0).

Proof of Lemma 2.6. Let us prove this lemma for the case $j = 1$. Let (t_0, v_0) be a critical point of $\tilde{J}^1$ on Σ^1. Observe that

$$d\tilde{I}(t_0, v_0)(\tau, \phi) = \frac{\partial}{\partial t}\tilde{I}(t_0, v_0)(\tau) + \frac{\delta}{\delta v}\tilde{I}(t_0, v_0)(\phi) \tag{2.10}$$

for every $\tau \in T_{t_0}(\mathbb{R}^+)$ and $\phi \in T_{v_0}(S^1)$.

By virtue of (2.3) the first term on the right-hand side of (2.10) is equal zero. So to prove (2.9) it suffices to show that

$$\frac{\delta}{\delta v}\tilde{I}(t_0, v_0)(\phi) = 0, \quad \forall \phi \in T_{v_0}(S^1). \tag{2.11}$$

By Proposition 2.3 there exists a neighborhood $\Lambda(v_0) \subset S^1$ of $v_0 \in S^1$ and an uniqueness C^1-map $t^1 : \Lambda(v_0) \to \mathbb{R}$ such that (2.5) holds. Introduce $J^1(v) =: \tilde{I}(t^1(v), v)$, $v \in \Lambda(v_0)$. Then by the definition of $\tilde{J}^1$ we have

$$J^1(v) \equiv \tilde{J}^1(t^1(v), v), \quad v \in \Lambda(v_0). \tag{2.12}$$

Hence, taking into account that the submanifold Σ^j is local C^1 - diffeomorphic with S^1, we deduce that v_0 is a critical point of $J^1(v)$ on $\Lambda(v_0)$, i.e.

$$dJ^1(v_0)(\phi) = 0, \quad \forall \phi \in T_{v_0}(S^1).$$

Since $J^1(v) = \tilde{I}(t^1(v), v)$ as $v \in \Lambda(v_0)$ we get

$$0 = dJ^1(v_0)(h) = \frac{\partial}{\partial t}\tilde{I}(t^1(v_0), v_0)(dt^1(v_0))(h) + \frac{\delta}{\delta v}\tilde{I}(t^1(v_0), v_0)(h), \quad \forall h \in T_{v_0}(S^1). \tag{2.13}$$

By virtue of (2.3) the first term on the right-hand side of (2.13) is equal zero. Thus

$$\frac{\partial}{\partial v}\tilde{I}(t^1(v_0), v_0)(\phi) = 0, \quad \forall \phi \in T_{v_0}(S^1)$$

and we get (2.11). The proof of Lemma 2.6 is complete. □

From Lemma 2.6 and Proposition 2.1 we derive the following theorem.

Theorem 2.7. *Assume that $I(u) \in C^1(W \setminus \{0\})$ and (RD) hold. Let $(t_0^j, v_0^j) \in \Sigma^j$ be a solution of the variational problem (2.6), for $j = 1$ or $j = 2$, respectively. Then*

$$u_0^j = t_0^j v_0^j \in W \setminus \{0\} \tag{2.14}$$

is a critical point of I.

Let pr_2 be a canonical projection from $\mathbb{R}^+ \times S^1$ to S^1. Denote $\Theta^j = pr_2(\Sigma^j)$, $j = 1, 2$.

Recall that by Proposition 2.3 for every $v_0^j \in \Theta^j$, $j = 1, 2$, there exist a neighborhood $\Lambda(v_0^j) \subset \Theta^j$ and an uniqueness C^1-map $t^j : \Lambda(v_0^j) \to \mathbb{R}$ such that $(t^j(v), v) \in \Sigma^j$, $j = 1, 2$, respectively.

Definition 2.8. Let $j = 1, 2$. The trivial fibering scheme for I on W is said to be a solvable with respect to Σ^j if for every $v \in \Theta^j$ there exists a unique point $t^j(v) \in \mathbb{R}^+$ such that $(t^j(v), v) \in \Sigma^j$, respectively. In case when the trivial fibering scheme for I on W is solvable one with respect to both Σ^1 and Σ^2 then it is called a solvable.

If in addition the functional $t^j(v)$ can be found in exact form then the trivial fibering scheme is called exactly solvable.

We remark that in the papers [2, 9, 8, 20], it is used the constrained minimization method to homogeneous problems like (1.1)-(1.2) which is with respect to the trivial fibering scheme an exactly solvable one (see also below Remark 3.3).

We point out that in the present paper we are concerned with the applications of the trivial fibering scheme in cases of solvable but may be not exactly solvable.

Observe by Proposition 2.3 in case of the solvable trivial fibering scheme it can be defined the global functionals:

$$t^j : \Theta^j \to \mathbb{R}^+, \; j = 1, 2 \tag{2.15}$$

such that $(t^j(v), v) \in \Sigma^j$, $j = 1, 2$. Moreover in this case the sets Θ^j, $j = 1, 2$ are submanifolds of class C^1 in S^1 and $t_j(\cdot) \in C^1(\Theta_j)$, $j = 1, 2$. Hence we can define the following global functionals

$$J^1(v) = \tilde{I}(t^1(v), v), \quad v \in \Theta^1, \tag{2.16}$$

$$J^2(v) = \tilde{I}(t^2(v), v), \quad v \in \Theta^2. \tag{2.17}$$

Thus the variational problems (2.6) are reduced to the following equivalent, respectively

$$\hat{J}^j = \min\{J^j(v) : v \in \Theta^j\}, \quad j = 1, 2. \tag{2.18}$$

where

$$\hat{I}^j = +\infty, \quad \text{if } \Theta^j = \emptyset, \; j = 1, 2. \tag{2.19}$$

From Theorem 2.7we have the following statement.

Lemma 2.9. *Assume the trivial fibering scheme applying to the functional I is solvable. Let $j = 1, 2$ and $v_0^j \in \Theta_j$ is a solution of the problem (2.18). Then $u_0^j = t_j(v_0^j)v_0^j$ is a nonzero critical point of the functional I.*

Finally, we give a property for the constrained minimization problems (2.6) which also characterizes the trivial fibering scheme as basic.

Denote by Z a set of all nonzero critical points of I on space W. Then with respect to the trivial fibering scheme we have the following decomposition: $Z = Z_- \cup Z_+ \cup Z_0$, where

$$\begin{aligned} Z_+ &= \big\{u \in Z | (\|u\|, \frac{u}{\|u\|}) \in \Sigma_1\big\}, \\ Z_- &= \big\{u \in Z | (\|u\|, \frac{u}{\|u\|}) \in \Sigma_2\big\}, \\ Z_0 &= \big\{u \in Z | (\|u\|, \frac{u}{\|u\|}) \in \partial\sigma\big\}, \end{aligned}$$

with $\partial\sigma = \{(t, v) \in \mathbb{R}^+ \times S^1 | Q(t, v) = 0,\ L(t, v) = 0\}$.

For physical applications it is important to investigate ground states [6]. By the definition the nonzero critical point $u_g \in W$ is said to be a ground state if it is a point with the least level of I among all the nonzero critical points Z, i.e

$$\min\{I(u) : u \in Z\} = I(u_g). \tag{2.20}$$

We introduce in addition the following concept.

Definition 2.10. The nonzero critical point $u_g^- \in W$ ($u_g^+ \in W$) is said to be a ground state of type (-1) ((0)) for I if it holds:

$$\min\{I(u) | u \in Z_-\} = I(u_g^-), \quad (\min\{I(u) | u \in Z_+\} = I(u_g^+)). \tag{2.21}$$

The following lemma follows directly from the construction of constrained minimization problems (2.6).

Lemma 2.11. *Assume $I(u) \in C^1(W \setminus \{0\})$ and (RD) holds, where $j = 1$ or $j = 2$. Let $(t_0^j, v_0^j) \in \Sigma^j$ be a solution of the variational problem (2.6). Then $u^+ = t_0^1 v_0^1 \in W \setminus 0$ is a ground state of type (0) for I and $u^- = t_0^2 v_0^2 \in W \setminus 0$ is a ground state of type (-1) for I. Furthermore, if in addition $Z_0 = \emptyset$ then one of these solutions u^- or u^+ is a ground state for I, i.e.*

$$\min\{I(u) | u \in Z\} = \min\{I(u_g^-), I(u_g^+)\}. \tag{2.22}$$

For the case of the even functionals, $I(u) = I(|u|)$ with $u \in W$, we have the following statement.

Lemma 2.12. *Assume $I(u) \in C^1(W \setminus \{0\})$ is an even functional and (RD) holds. Suppose that there exists a solution of problem (2.6) $j = 1$ ($j = 2$). Then there exists a nonnegative on M ground state u^+ of type (0) for I (a nonnegative on M ground state u^- of type (-1) for I).*

Proof. As a particular case, consider $j = 1$. Since the functional I is even it follows that the functionals $\tilde{I}(t, v)$, $Q(t, v)$, $L(t, v)$ are also even with respect to $v \in S^1$. Hence the manifolds Σ^1 and Σ^2 are symmetric with respect to origin, i.e., if $(t, v) \in \Sigma^j$ then it follows that $(t, -v) \in \Sigma^j$.

Let us suppose that there exists a solution $(t_0^1, v_0^1) \in \Sigma^1$ of problem (2.6), $j = 1$. Then it follows that $(t_0^1, |v_0^1|) \in \Sigma^1$ where $t_0^1 > 0$ is also a solution of the problem (2.6), $j = 1$. Now, taking into account Lemma 2.12 we complete the proof. $\square$

3. Constrained minimization problems associated with (1.1)–(1.2).

In this section, we use the trivial fibering scheme to introduce the constrained minimization problems for (1.1)-(1.2). Let (M, g) be a connected compact Riemannian manifold with boundary ∂M of dimension $n \geq 2$. Let $g_{i,j}$ be the components of the given metric tensor $g = (g_{ij})$ with inverse matrix $(g^{i,j})$, and let $|g| = \det(g_{i,j})$. If (x^i) is a local system of coordinates, then we define the divergence operator div_g on the C^1 vector field $X = (X^i)$ by

$$\operatorname{div}_g X = \frac{1}{\sqrt{|g|}} \sum_i \frac{\partial}{\partial x_i} (\sqrt{|g|} X^i),$$

and the p-Laplace–Beltrami operator by $\Delta u = \operatorname{div}_g(|\nabla u|^{p-2} \nabla u)$. Here

$$\nabla u = \sum_i g^{i,j} \frac{\partial u}{\partial x_i}$$

denotes the gradient vector field of u. Let the Riemannian measure (induced by the metric g) on M and ∂M, respectively, be denoted by $d\mu_g$ and $d\nu_g$, respectively.

We consider our problems in the framework of the Sobolev space $W = W_p^1(M)$ equipped with the norm

$$\|u\| = \Big(\int_M |u|^p d\mu_g + \int_M |\nabla u|^p d\mu_g \Big)^{1/p}. \tag{3.1}$$

Let us introduce the following notation

$$\begin{gathered} f(u) = \int_M k(x)|u|^p d\mu_g, \quad F(u) = \int_M K(x)|u|^\gamma d\mu_g, \\ b(u) = \int_{\partial M} d(x)|u|^p d\nu_g, \quad B(u) = \int_{\partial M} D(x)|u|^q d\nu_g, \\ H_\lambda(u) = \int_M |\nabla u|^p d\mu_g + b(u) - \lambda f(u). \end{gathered} \tag{3.2}$$

We recall that there is a continuous embedding $W_p^1(M) \subset L_{p^*}(M)$ and a continuous trace-embedding $W_p^1(M) \subset L_{p^{**}}(\partial M)$, respectively. Using the hypotheses (1.3), (1.4), (1.5) and these embedding results it is easy to check that all functionals in (3.2) are well-defined on the Sobolev space W and belong to the class $C^1(W)$. The Euler functional I on W which corresponds to problem (1.1)-(1.2) is defined by

$$I_\lambda(u) = \frac{1}{p} H_\lambda(u) - \frac{1}{q} B(u) - \frac{1}{\gamma} F(u). \tag{3.3}$$

A function $u_0 \in W$ is called the *weak solution* for problem (1.1)-(1.2) if the following identity

$$\frac{\delta}{\delta u} I_\lambda(u_0)(\psi) = 0$$

holds for every function $\psi \in C^\infty(\overline{M})$. Hence the existence of weak solutions of problem (1.1)-(1.2) is equivalent to the existence of critical points for the Euler functional I_λ defined above.

Let us apply to functional (3.3) the trivial fibering scheme. It is easily verified that the norm (3.1) defines a C^1-functional $u \to \|u\|$ on $W \setminus \{0\}$. Hence the sphere $S^1 = \{v \in W | \ \|v\| = 1\}$ is a closed submanifold of class C^1 in W and $S^1 \times \mathbb{R}^+$ is C^1-diffeomorphic with $W \setminus \{0\}$.

Following the trivial fibering scheme, we associate with the original functional I_λ a new fibering functional $\tilde{I}_\lambda$ defined for $(t, v) \in \mathbb{R}^+ \times S^1$ by

$$\tilde{I}_\lambda(t, v) = I_\lambda(tv) = \frac{1}{p} t^p H_\lambda(v) - \frac{1}{q} t^q B(v) - \frac{1}{\gamma} t^\gamma F(v) \,. \tag{3.4}$$

For $(t, v) \in \mathbb{R}^+ \times S^1$, we define the functionals

$$Q_\lambda(t, v) = \frac{\partial}{\partial t} \tilde{I}_\lambda(t, v) = t^{p-1}(H_\lambda(v) - t^{q-p} B(v) - t^{\gamma-p} F(v)), \tag{3.5}$$

$$L_\lambda(t, v) = \frac{\partial^2}{\partial t^2} \tilde{I}_\lambda(t, v) = t^{p-2}((p-1)H_\lambda(v) - (q-1)t^{q-p} B(v) - (\gamma-1)t^{\gamma-p} F(v)). \tag{3.6}$$

Thus we can extract from $\mathbb{R}^+ \times S^1$ the sets

$$\Sigma^1_\lambda = \{(t, v) \in \mathbb{R}^+ \times S^1 | Q_\lambda(t, v) = 0, \ L_\lambda(t, v) > 0\}, \tag{3.7}$$

$$\Sigma^2_\lambda = \{(t, v) \in \mathbb{R}^+ \times S^1 | Q_\lambda(t, v) = 0, \ L_\lambda(t, v) < 0\}. \tag{3.8}$$

Thus in accordance to the trivial fibering scheme we have the following variational problems

$$\hat{I}^j_\lambda = \inf\{\tilde{I}_\lambda(t, v) | (t, v) \in \Sigma^j_\lambda\}, \quad j = 1, 2, \tag{3.9}$$

where

$$\hat{I}^j_\lambda = +\infty, \quad \text{if } \Sigma^j_\lambda = \emptyset, \quad j = 1, 2. \tag{3.10}$$

From (3.4) it follows that I_λ satisfies to condition (RD).

It is easy to verify that the equation $Q_\lambda(t, v) = 0$ can have, in dependent of $H_\lambda(v)$, $B(v)$ and $F(v)$, at most two solutions on $\mathbb{R}^+$. The conditions $L_\lambda(t, v) < 0$ and $L_\lambda(t, v) > 0$ separate them: the equation $Q_\lambda(t, v) = 0$ may have at most one solution $t^1(v) \in \mathbb{R}^+$ such that $Q_\lambda(t^1(v), v) = 0$, $(t^1(v), v) \in \Sigma^1_\lambda$, and at most one solution $t^2(v) \in \mathbb{R}^+$ such that $Q_\lambda(t^2(v), v) = 0$, $(t^2(v), v) \in \Sigma_2$, respectively. Moreover we have

$$t^j(\cdot) \in C^1(\Theta^j_\lambda), \quad j = 1, 2 \tag{3.11}$$

where $\Theta^j_\lambda = pr_2(\Sigma^j_\lambda)$, $j = 1, 2$ are submanifolds of class C^1 in S^1.

Thus we have deal with the solvable trivial fibering scheme and we can define

$$J^1_\lambda(v) = \tilde{I}_\lambda(t^1(v), v), \quad v \in \Theta^1_\lambda, \tag{3.12}$$

$$J^2_\lambda(v) = \tilde{I}_\lambda(t^2(v), v), \quad v \in \Theta^2_\lambda. \tag{3.13}$$

Thus problem (3.9) is reduced to the following problem

$$\hat{I}^j_\lambda = \min\{J^j_\lambda(v) : v \in \Theta^j_\lambda\}, \quad j = 1, 2. \tag{3.14}$$

From Theorem 2.7 we have the following statement.

Lemma 3.1. *Let $j = 1, 2$. Assume that $v^j_0 \in \Theta^j_\lambda$ is a solution of problem (3.14). Then $u^j_0 = t^j(v^j_0)v^j_0$ is a nonzero critical point of the functional I_λ.*

Remark 3.2. In the case when $p = 2$, $\gamma = 2^*$, $q = p^{**}$, $n \geq 3$, problems of type (1.1)-(1.2) have their root in Riemannian geometry. Let (M, g) be a Riemannian manifold of dimension $n \geq 3$ with the boundary ∂M, the scalar curvature $k(x)$ of M and the mean curvature $d(x)$ of ∂M. Let K be a given function on M and D be a fixed function on ∂M. One may ask the question: Can we find a new metric $\bar{g}$ on M such that K is the scalar curvature of $\bar{g}$ on M, D is the mean curvature of $\bar{g}$ on ∂M and $\bar{g}$ is conformal to g (i.e., it holds $\bar{g} = u^{4/(n-2)}g$ for some $u > 0$ on M)? This is equivalent (see Escobar [9, 10], Taira [21]) to the problem of finding positive solutions u of (1.1)-(1.2) with critical exponents $\gamma = 2^*$ and $q = p^{**}$, where k is the scalar. Thus, by the trivial fibering scheme we have also the variational statements (3.9) for this geometrical problem.

Remark 3.3. Observe, the variational definition (3.14) includes the formulations used by Escobar [9]-[11]. Indeed, let us consider the case $D(x) = 0$. This implies $B(\cdot) \equiv 0$ in (2.2). It is easy to verify that $L_\lambda(t(v), v) > 0$ and $L_\lambda(t(v), v) < 0$, respectively, holds, if $\operatorname{sgn}(F(v)) < 0$ and $\operatorname{sgn}(F(v)) > 0$, respectively. Hence we have $j = 1$ in the first case and $j = 2$ in the other one.

4. Existence and multiplicity for subcritical cases

In this section, we prove the main results of the paper, i.e., we show the existence and the multiplicity of positive solutions of (1.1)-(1.2). Define

$$\lambda^*(K) = \inf \Big\{ \frac{\int_M |\nabla u|^p d\mu_g + b(u)}{\int_M k(x)|u|^p d\mu_g} : F(u) \geq 0,\, u \in W \Big\}, \tag{4.1}$$

$$\lambda^*(D) = \inf \Big\{ \frac{\int_M |\nabla u|^p d\mu_g + b(u)}{\int_M k(x)|u|^p d\mu_g} : B(u) \geq 0,\, u \in W \Big\}, \tag{4.2}$$

where in case when the set $\{u \in W_p^1(M) : F(u) \geq 0\}$ ($\{u \in W_p^1(M) : B(u) \geq 0\}$) is empty we put $\lambda^*(K) = +\infty$ ($\lambda^*(D) = +\infty$). Remark that

$$\lambda_1 = \inf \Big\{ \frac{\int_M |\nabla u|^p d\mu_g + b(u)}{\int_M k(x)|u|^p d\mu_g} : u \in W_p^1(M) \Big\} \tag{4.3}$$

and λ_1 is the simple first eigenvalue of the Neumann boundary problem

$$\begin{gathered} -\Delta_p \phi_1 = \lambda_1 k(x)|\phi_1|^{p-2}\phi_1 \quad \text{in } M, \\ |\nabla \phi_1|^{p-2}\frac{\partial \phi_1}{\partial n} + d(x)|\phi_1|^{p-2}\phi_1 = 0 \quad \text{on } \partial M, \end{gathered} \tag{4.4}$$

where $\phi_1 > 0$ is a corresponding principal eigenfunction (see [23], [24]). Suppose that $k(x) \geq 0$ on M, $d(x) \geq 0$ on ∂M then it follows immediately from the definitions that $0 \leq \lambda_1 \leq \lambda^*(K)$ and $0 \leq \lambda_1 \leq \lambda^*(D)$.

Lemma 4.1. *Assume* (1.5) *holds and $k(x) \geq 0$ on M, $d(x) \geq 0$ on ∂M.*

1. *If $F(\phi_1) < 0$ and $p < \gamma \leq p^*$, then $\lambda_1 < \lambda^*(K)$*
2. *If $B(\phi_1) < 0$ and $p < q \leq p^{**}$, then $\lambda_1 < \lambda^*(D)$.*

Proof. First assertion: For our purpose it is important to prove separately some parts of the lemma in the following two cases: in subcritical cases of exponents and in critical cases of exponents, respectively.

Let us suppose that $F(\phi_1) < 0$. Assume to the contrary that $\lambda_1 = \lambda^*(K)$. Hence there exists a minimizing sequence $\{w_m\}$ for the problem (4.2) such that

$$E(w_m) = \frac{\int_M |\nabla w_m|^p d\mu_g + b(u)}{\int_M k(x)|w_m|^p d\mu_g} \to \lambda_1 = \lambda^*(K) \quad \text{as } m \to \infty,$$

where $F(w_m) \geq 0$, $m = 1, 2, \dots$, see (4.1). The functional $E(\cdot)$ is 0-homogeneous. Therefore we may assume without loss of generality that the sequence $\{w_m\}$ is bounded and that $w_m \rightharpoonup w$ weakly converges for some $w \in W$.

Since E is lower semi-continuous with respect to W we get $E(w) \leq \lambda_1$. But λ_1 is a minimum of E (see (4.3)) and therefore we get $E(w) = \lambda_1$.

Let us consider the subcritical cases; i.e., we assume that $p < \gamma < p^*$ holds. Then since W is compactly embedded in $L_s(M)$ for $p \leq s < p^*$ we may assume that $F(w_m) \to F(w)$ as $m \to \infty$. Hence $F(w) \geq 0$. Note that the eigenvalue λ_1 is simple. Hence it follows that $w = r\phi_1$ for some constant $r > 0$. This implies that we have $F(r\phi_1) \geq 0$, a contradiction to our assumption $F(r\phi_1) = r^\gamma F(\phi_1) < 0$.

Now let us consider also the critical case of the exponent. As it has been shown above it suffices to prove that $F(w) \geq 0$. Let us show that $w_m \to w$ strongly in W. Indeed, as it has been shown above we have $E(w) = \lambda_1$. This implies that

$$\int_M |\nabla w_m|^p d\mu_g \to \int_M |\nabla w|^p d\mu_g.$$

Now taking into account that $w_m \rightharpoonup w$ weakly in W we get $w_m \to w$ strongly in W. Thus we have $F(w) \geq 0$. Consequently, we have shown that $F(\phi_1) < 0$ implies $\lambda_1 < \lambda^*(K)$. □

Remark 4.2. The main difficulty in investigation of the solvability problem for the elliptic equations with critical exponents of nonlinearities is a "lack of compactness" (cf. [3], [20]). From the point of view of the overcoming this difficulty Lemma 4.1 plays the main role in our approach. Generally speaking, in our approach we reduce the problem of the lack of compactness mainly to the investigations at a bifurcation point λ_1.

Remark 4.3. Recall, if the set $\{u \in W : F(u) \geq 0\} = \emptyset$ ($\{u \in W : B(u) \geq 0\} = \emptyset$) then $\lambda^*(K) = +\infty$ ($\lambda^*(D) = +\infty$). Thus in this case Lemma 4.1 is trivial. Note that if the conditions $\{u \in W : F(u) \geq 0\} = \emptyset$ and $\{u \in W : B(u) \geq 0\} = \emptyset$ are satisfied then for $\lambda > 0$ problem (1.1)-(1.2) become coercive. Observe also the conditions $\{u \in W : F(u) \geq 0\} = \emptyset$ and $\{u \in W : B(u) \geq 0\} = \emptyset$ mean that $K(x) < 0$ on M and $D(x) < 0$ on M, respectively.

Proposition 4.4. *Let* (1.5) *and* $k(x) \geq 0$ *on* M , $d(x) \geq 0$ *on* ∂M *be satisfied. Then the following two statements hold*

(1) *If* $\lambda < \lambda^*(K)$ ($\lambda < \lambda^*(D)$) *and* $F(u) \geq 0$ ($B(u) \geq 0$) *for some* $u \in W$, *then* $H_\lambda(u) > 0$.
(2) *If* $\lambda < \lambda^*(K)$ ($\lambda < \lambda^*(D)$) *and* $H_\lambda(u) \leq 0$ *for some* $u \in W$, *then* $F(u) < 0$ ($B(u) < 0$).

The assertions in the above proposition follow immediately from the definitions, see (4.1), (4.2), (4.3).

Let us formulate our main theorem on the existence of positive solutions for the family of problems (1.1)-(1.2) in the subcritical cases.

Theorem 4.5. *Suppose that* (1.5) *and* $k(x) \geq 0$ *on* M, $d(x) \geq 0$ *on* ∂M, $p < q < p^{**}$, $p < \gamma < p^*$ *and* $q < \gamma$ *are satisfied.*

(1) *Assume* $F(\phi_1) < 0$ *and* $B(\phi_1) < 0$. *Then for every* $\lambda \in (\lambda_1, min\{\lambda^*(K), \lambda_*(D)\}$ *there exists a weak positive solution* u_1 *of* (1.1)-(1.2) *such that* $u_1 > 0$ *on* M *and* $u_1 \in W^1_p(M)$. *Furthermore, it holds* $I_\lambda(u_1) < 0$ *and* u^1 *is a ground state of type (0) for* I_λ.

(2) *Suppose that the set* $\{x \in M : K(x) > 0\}$ *is not empty and* $D(x) \leq 0$ *on* ∂M. *Assume* $F(\phi_1) < 0$ *holds. Then for every* $\lambda < \lambda^*(K)$ *there exists a weak positive solution* u_2 *of* (1.1)- (1.2) *such that* $u_2 > 0$ *on* M *and* $u_2 \in W^1_p(M)$. *Furthermore, we have* $I_\lambda(u_2) > 0$ *and* u^2 *is a ground state of type (-1) for* I_λ.

For the proof of this theorem, we use the following lemma.

Lemma 4.6. *Let* $k(x) \geq 0$ *on* M, $d(x) \geq 0$ *on* ∂M, $p < q \leq p^{**}$, $p < \gamma \leq p^*$ *and* $q < \gamma$.

(1) *Assume* $F(\phi_1) < 0$ *holds. Then for every* $\lambda \in (\lambda_1, \lambda^*(K))$
$$\Theta^o_{1,\lambda} := \{w \in W : H_\lambda(w) < 0\} \subseteq \Theta_{1,\lambda} \tag{4.5}$$
and the set $\Theta^o_{1,\lambda}$ *is not empty.*

(2) *Suppose that the set* $\{x \in M : K(x) > 0\}$ *is not empty and* $D(x) \leq 0$ *on* ∂M. *Then the set* $\Theta_{2,\lambda}$ *is not empty and*
$$\Theta_{2,\lambda} = \{w \in W : F(w) > 0\} \tag{4.6}$$
for every $\lambda < \lambda^*(K)$.

Proof. First assertion. Note that by Proposition 4.1. $\lambda_1 < \min\{\lambda^*(K), \lambda^*(D)\})$. At first we show (4.5). Let $\lambda \in (\lambda_1, \min\{\lambda^*(K), \lambda^*(D)\})$. We suppose that $w \in \Theta^o_{1,\lambda}$, i.e., $H_\lambda(w) < 0$ holds. Then by Proposition 4.4 we have that $F(w) < 0$ and $B(w) < 0$. These facts and (3.5) imply the existence of a number $t^1(w) > 0$ such that $Q(t^1(w), w) = 0$ and $L(t^1(w), w) > 0$ hold. Thus $w \in \Theta_{1,\lambda}$ and (4.5) is proved. Let us consider the first eigenvalue $\phi_1 \in S^1$ of problem (4.4). Then for any $\lambda > 0$ we have $H_\lambda(\phi_1) < 0$. Thus $\phi_1 \in \Theta^o_{1,\lambda}$, and therefore the set $\Theta^o_{1,\lambda}$ is not empty for $\lambda \in (\lambda_1, \lambda^*(K))$. The first assertion is proved.

We show the second part. Assume that the set $\{x \in M : K(x) > 0\}$ is not empty. Then there exists a function $v_0 \in W$ such that $F(v_0) > 0$. Applying Proposition 4.4 we deduce that $H_\lambda(v_0) > 0$ holds for any $\lambda < \lambda^*(K)$. Recall that we have $p < q < \gamma$. Hence we obtain from (3.5) the existence of a number $t^2(v) > 0$ such that $Q(t^2(v), v) = 0$ and $L(t^2(v), v) < 0$. This implies $v \in \Theta_{2,\lambda}$. Thus the set $\Theta_{2,\lambda}$ is not empty and
$$\{w \in W : F(w) > 0\} \subseteq \Theta_{2,\lambda}. \tag{4.7}$$
Suppose $F(w) \leq 0$ for some $w \in W$. By assumption we have $B(w) \leq 0$. Hence the equation $Q(t, w) = 0$ may have a solution $t^2(w)$ only in the case when $H_\lambda(w) < 0$ is satisfied. However, in this case, we have $L(t^2(w), w) > 0$ by (3.6). This fact yields $w \notin \Theta_{2,\lambda}$ and therefore $\{w \in W : F(w) \leq 0\} \cap \Theta_{2,\lambda} = \emptyset$. Using this and (4.7) we deduce (4.6). The proof is complete. □

For the proof of theorem 4.5, we restrict the functional J^1_λ on the set $\Theta^o_{1,\lambda}$. Therefore, instead of the minimization problem (2.6) for $j = 1$, we consider
$$\hat{I}^{1,o}_\lambda = \min\{J^1_\lambda(v) : v \in \Theta^o_{1,\lambda}\}. \tag{4.8}$$

To prove the existence of the solution u_1 and u_2 in W we apply Lemma 2.9. Therefore, we show that the variational problem (4.8) has a solution $v_1 \in W$ and (3.14) with $j = 2$ has a solution $v_2 \in W$.

Note that Lemma 4.6 implies also

$$J^{1,o}_\lambda(v) < 0, \quad \text{if } v \in \Theta^o_{1,\lambda}, \tag{4.9}$$

$$J^2_\lambda(v) > 0, \quad \text{if } v \in \Theta_{2,\lambda}. \tag{4.10}$$

Now we prove a mapping property of the functionals J^j_λ, $j = 1, 2$.

Lemma 4.7. *Let $k(x) \ge 0$ on M, $d(x) \ge 0$ on ∂M, $p < q < p^{**}$, $p < \gamma < p^*$ and $q < \gamma$.*

(1) *Assume that $F(\phi_1) < 0$, $B(\phi_1) < 0$. Let $\lambda \in (\lambda_1, \min\{\lambda^*(K), \lambda^*(D)\})$. Then the functional $J^1_\lambda(\cdot)$ defined on $\Theta^o_{1,\lambda}$ is bounded below, i.e., $-\infty < \inf_{\Theta^o_{1,\lambda}} J^1_\lambda(w)$.*

(2) *Suppose that the set $\{x \in M : K(x) > 0\}$ is not empty and $D(x) \le 0$ on ∂M. Let $\lambda < \lambda^*(K)$. Then the functional $J^2_\lambda(\cdot)$ defined on $\Theta_{2,\lambda}$ is bounded below, i.e., $-\infty < \inf_{\Theta_{2,\lambda}} J^2_\lambda(w)$.*

Proof. For the first assertions, observe that $\sup_{\Theta^o_{1,\lambda}} |J^1_\lambda(w)| = \infty$ if and only if there exists a sequence $v_m \in \Theta^o_{1,\lambda}$, $m = 1, 2, \dots$, such that $t^1(v_m) \to \infty$ as $m \to \infty$. By Proposition 4.4, if $H_\lambda(v) \le 0$ and $\lambda \in (\lambda_1, \min\{\lambda^*(K), \lambda^*(D)\})$ then we have $F(v) < 0$ and $B(v) < 0$. Hence and since $H_\lambda(w)$ is bounded on $\Theta^o_{1,\lambda} \subset S^1$ we deduce from the equation $Q_\lambda(t^1(v), v) = 0$ (cf. (3.5)) that is impossible if $t^1(v) \to \infty$.

To prove the second assertion, observe that from equation $Q_\lambda(t^2(v), v) = 0$ it follows

$$\tilde{I}_\lambda(t^2(v), v) = (t^2(v))^p[(\frac{1}{p} - \frac{1}{\gamma})H_\lambda(v) - (\frac{1}{q} - \frac{1}{\gamma})(t^2(v))^{q-p}B(v)]. \tag{4.11}$$

From Proposition 4.4 it follows that if $v \in \Theta_{2,\lambda}$ and $\lambda < \lambda^*(K)$ then $H_\lambda(v) > 0$ holds. Hence and since by assumption $B(v) \le 0$ we deduce from (4.11) that $J^2_\lambda(v) = \tilde{I}_\lambda(t^2(v), v) > 0$ for $v \in \Theta_{2,\lambda}$ and therefore the assertion 2) holds. □

Lemma 4.8. *Let $k(x) \ge 0$ on M, $d(x) \ge 0$ on ∂M, $p < q < p^{**}$, $p < \gamma < p^*$ and $q < \gamma$.*

(1) *Assume that $F(\phi_1) < 0$, $B(\phi_1) < 0$. Let $\lambda \in (\lambda_1, \min\{\lambda^*(K), \lambda^*(D)\})$. Then the functional $J^1_\lambda(\cdot)$ defined on $\Theta^o_{1,\lambda}$ is weakly lower semi - continuous with respect to W.*

(2) *Suppose that the set $\{x \in M : K(x) > 0\}$ is not empty and $D(x) \le 0$ on ∂M. Let $\lambda < \lambda^*(K)$. Then the functional $J^2_\lambda(\cdot)$ defined on $\Theta_{2,\lambda}$ is weakly lower semi-continuous with respect to W.*

Proof. Let $j = 1$ or $j = 2$ be fixed. We assume that $v_m \rightharpoonup v$ weakly with respect to W for some $v \in \Theta_j$. Recall that $\Theta_j \subset S^1$ and therefore $\{v_m\}$ is bounded in W. Thus we may assume that

$$B(v_m) \to B(v),\ F(v_m) \to F(v) \quad \text{as } m \to +\infty, \tag{4.12}$$

and

$$H_\lambda(v_m) \to \bar{H} \text{ as } m \to +\infty, \tag{4.13}$$

where $\bar{H}$ is finite. Since $H_\lambda(\cdot)$ is weakly lower semi-continuous with respect to W we get

$$H_\lambda(v) \le \bar{H}. \tag{4.14}$$

From (4.12), (4.13) it follows $\{t^j(v_m)\}$ is a convergent sequence. Furthermore it holds $t^j(v_m) \to \bar{t} < +\infty$ as $m \to +\infty$. Indeed, in both cases of assertions 1), 2) we have $F(v) \ne 0$ and $B(v) \ne \infty$, $|\bar{H}| \ne \infty$ for $v \in \Theta_j$, $j = 1, 2$, respectively. Hence and from (3.5) it follows that the contrary supposing $t^j(v_m) \to \bar{t} = +\infty$ as $m \to +\infty$ is impossible. Thus $t^j(v_m) \to \bar{t} < +\infty$ as $m \to +\infty$. Now we define

$$\bar{I}(t) = \frac{1}{p} t^p \bar{H} - \frac{1}{q} t^q B(v) - \frac{1}{\gamma} t^\gamma F(v)$$

for $t \in \mathbb{R}^+$. Then

$$J^j_\lambda(v_m) \to \bar{I}(\bar{t}) \quad \text{as } m \to +\infty. \tag{4.15}$$

Let us prove the assertion 1). It follows from (4.14) that $\bar{I}(\bar{t}) \ge \tilde{I}_\lambda(\bar{t}, v)$. It is easy to see that $t^1(v)$ is the minimization point of the function $\tilde{I}_\lambda(t, v)$ on $\mathbb{R}^+$. Therefore we have $\tilde{I}_\lambda(\bar{t}, v) \ge \tilde{I}_\lambda(t_1(v), v)$ and, consequently,

$$\lim_{m\to\infty} J^1_\lambda(v_m) = \bar{I}(\bar{t}) \ge J^1_\lambda(v).$$

Hence $J^1_\lambda(v)$ is weakly lower semi-continuous on $\Theta^o_{1,\lambda}$ with respect to W.

Now we prove the second assertion. Let us define

$$\bar{Q}(t) = \frac{1}{t^{p-1}} \frac{\partial}{\partial t} \bar{I}(t), \quad \bar{L}(t) = \frac{1}{t^{p-2}} \frac{\partial^2}{\partial t^2} \bar{I}(t)$$

for all $t \in \mathbb{R}^+$. Then it follows from (4.12), (4.13), (3.5) and (3.6) that

$$\bar{Q}(\bar{t}) = \bar{H} - \bar{t}^{q-p} B(v) - \bar{t}^{\gamma-p} F(v) = 0, \tag{4.16}$$

$$\bar{L}(\bar{t}) = (p-1)\bar{H} - (q-1)\bar{t}^{q-p} B(v) - (\gamma-1)\bar{t}^{\gamma-p} F(v) \le 0. \tag{4.17}$$

Assume that we have equality in (4.17). Then by (4.16) and (4.17) we get

$$(\gamma - p)\bar{H} - (\gamma - q)\bar{t}^{q-p} B(v) = 0.$$

Recall that $B(v) \le 0$ and $p < q < \gamma$ hold. Therefore, $\bar{H} \ge 0$ is only possible in the case when $\bar{H} = 0$. Then we deduce from (4.14) that $H_\lambda(v) \le 0$. By (4.6) we have $F(v) > 0$ for $v \in \Theta^2_\lambda$. Hence, since $\lambda < \lambda^*(K)$ we obtain by Proposition 4.4 a contradiction. Thus we have in (4.17) a strong inequality. This implies that the function $\bar{I}(t)$ defined on $\mathbb{R}^+$ attains a maximum at the point $\bar{t}$. Using (4.14) we infer that

$$\lim_{m\to\infty} J^2_\lambda(v_m) = \bar{I}(\bar{t}) \ge \bar{I}(t^2(v)) \ge \tilde{I}_\lambda(t^2(v), v) = J^2_\lambda(v),$$

i.e., the second case is proved. □

Now we complete the proof of our main theorem. We start with the first part of Theorem 4.5. Therefore we suppose that all corresponding assumptions are satisfied. We consider the minimization problem (4.8). Let $\{v_m\}$ be a minimizing sequence for this problem, i.e., we have $v_m \in \Theta^o_1$ and $J^1_\lambda(v_m) \to \hat{I}^{1,o}_\lambda$. Recall that

$$\|v_m\| = 1 \quad \text{for } m = 1, 2, \dots. \tag{4.18}$$

Thus v_m is bounded in W. Hence since W is reflexive, we may assume $v_m \rightharpoonup v^1$ weakly for some $v^1 \in W$. Let us suppose, for the moment, that

$$v^1 \in \Theta^o_1. \tag{4.19}$$

Then the blondeness and weakly lower semi-continuity of J^1_λ shows that

$$-\infty < J^1_\lambda(v^1) \le \hat{I}^1_\lambda.$$

Thus v^1 is solution of the problem (2.6).

Now we prove (4.19). First of all we observe from (4.18) that $v^1 \neq 0$. Indeed, assume to the contrary that $v^1 = 0$. Since $W^1_p(M)$ is compactly embedded in the space $L_p(M)$ and also compactly trace - embedded in the space $L_p(\partial M)$, we may assume $b(v_m) \to 0$ and $f(v_m) \to 0$ as $m \to \infty$. These and (4.18) imply $H_\lambda(v_m) > 0$ for m large enough. Therefore we get a contradiction to the fact that $H_\lambda(v_m) < 0$ for $v_m \in \Theta^o_1$.

Now we show $v^1 \notin \partial\Theta^o_1$. It is sufficient to prove that the following strong inequality

$$H_\lambda(v^1) < 0 \tag{4.20}$$

holds. Using the weakly lower semi-continuity of H_λ it follows from the definition of v^1 that $H_\lambda(v^1) \le 0$. Assume to the contrary that $H_\lambda(v^1) = 0$. Since $\lambda < \min\{\lambda^*(K), \lambda^*(D)\}$ we conclude by Proposition 4.4, ii) that $F(v^1) < 0$, $B(v^1) < 0$. This fact, the continuity of F on $L_\gamma(M)$ and B on $L_q(\partial M)$ imply that $t^1(v_m) \to 0$ as $m \to \infty$. Applying now (3.4) we obtain that $\bar{I}_\lambda(t^1(v_m), v_m) \to 0$ as $m \to \infty$. On the other hand it is easy to see that $J^1_\lambda(v) < 0$ for all $v \in \Theta^o_{1,\lambda}$. Therefore we have a contradiction to the assumption that $\{v_m\}$ is minimizing sequence. Thus we have proved (4.20). Hence (4.19) is true.

Now we prove the second statement of Theorem 4.5. Suppose that the corresponding assumptions of Theorem 4.5 hold. We consider the minimization problem (2.6) with $j = 2$. Let $\{v_m\}$ be a minimizing sequence for this problem, i.e., we have $v_m \in \Theta_2$ and $J^2_\lambda(v_m) \to \hat{I}^2_\lambda$. As above in the proof of the first part of Theorem 4.5 it can be shown that $v_m \rightharpoonup v^2$ weakly with some $v^2 \in W$. Therefore, the proof is finished if

$$v^2 \in \Theta_2\,. \tag{4.21}$$

By the second part of Lemma 4.6 it is sufficient to show that the strong inequality

$$F(v^2) > 0 \tag{4.22}$$

holds. Assume to the contrary that $F(v^2) = 0$. Since $\lambda < \lambda^*(K)$ we conclude by Proposition 4.4, i) that $H_\lambda(v^2) > 0$. Hence using the continuity of F on $L_\gamma(M)$, supposing $B(v_m) \le 0$ we derive that $t^2(v_m) \to \infty$ as $m \to \infty$. Observe that by (3.8), (3.4), (3.5) we have

$$\tilde{I}_\lambda(t^2(v_m)v_m) = (t^2(v_m))^p[(\frac{1}{p} - \frac{1}{\gamma})H_\lambda(v_m) - (\frac{1}{q} - \frac{1}{\gamma})(t^2(v_m))^{q-p}B(v_m)].$$

This fact, the lower semi-continuity of H_λ and since $B(v_m) \le 0$, $m = 1, 2, \dots$, imply that $\tilde{I}_\lambda(t^2(v_m), v_m) \to \infty$ as $m \to \infty$. Therefore we get a contradiction to the assumption that $\{v_m\}$ is minimizing sequence. Thus (4.21) is proved.

By Lemma 3.1 the functions $u_j = t^j(v^j)v^j$, $j = 1, 2$, are weak solutions of (1.1) and (1.2). It follows from Lemma 2.12, since the functional I_λ is even, that $u_j \ge 0$ in M. By the maximum principle [23], since $u_j \not\equiv 0$, we see that $u_j > 0$ in M. Finally, it follows from (4.9) and (4.10), respectively, that $I_\lambda(u_1) > 0$ and $I_\lambda(u_2) < 0$. By Lemma 2.11 we have that u_2 is a ground state of type (-1) and u_1 is a ground state of type (0) for I_λ. The proof of Theorem 4.5 is finished. □

Next, we prove a lemma on the existence of ground states.

Lemma 4.9. *Suppose* (1.5)*,* $k(x) \geq 0$ *on* M*,* $d(x) \geq 0$ *on* ∂M*,* $p < \gamma < p^*$*,* $p < q < p^{**}$ *and* $q < \gamma$ *are satisfied. Furthermore, we assume that*

(1) $F(\phi_1) < 0$
(2) $D(x) \leq 0$ *on* ∂M*.*

Then for every $\lambda \in (\lambda_1, \lambda^*(K))$ *there exists a ground state* $u_1 \in W^1_p(M)$ *of* I_λ*. Furthermore,* $u_1 > 0$*,* $I_\lambda(u_1) < 0$*.*

Proof. First let us remark that under the additional assumption $D(x) \leq 0$ on ∂M we have

$$\Theta^o_{1,\lambda} = \Theta_{1,\lambda}. \tag{4.23}$$

Indeed, suppose $H_\lambda(w) \geq 0$ for some $w \in W$. By assumption we have $B(w) \leq 0$. Hence the equation $Q(t,w) = 0$ may have a solution $t^1(w) \neq 0$ only in the case when $F(w) > 0$ is satisfied. However, in this case, we have $L(t^1(w), w) < 0$ by (3.6). This fact yields $w \notin \Theta_{1,\lambda}$ and therefore $\{w \in W : H_\lambda(w) \leq 0\} \cap \Theta_{1,\lambda} = \emptyset$. Using this and Lemma 4.6 we deduce (4.23).

It follows from the proof of Theorem 4.5 and from (4.23) that there exists a positive solution $u_1 \in W^1_p(M)$ of variational problem (3.14), $j = 1$ such that $I_\lambda(u_1) < 0$.

Now let us show that u_1 is a ground state for I_λ. First note that for the solution u_2 of (3.14), $j = 2$ we have $I_\lambda(u_2) > 0$. Hence

$$\min\{I_\lambda(u_1), I_\lambda(u_2)\} = I_\lambda(u_1).$$

Therefore by Lemma 2.11 to prove our assertion it remains to show that the set

$$\partial\sigma = \{(t,v) \in \mathbb{R}^+ \times S^1 | Q(t,v) = 0,\ L(t,v) = 0\},$$

is empty. Assume the converse. Then by (3.5), (3.6) there exists $(t,v) \in \mathbb{R}^+ \times S^1$ such that it holds the following system of equations

$$\begin{gathered} H_\lambda(v_0) - t^{q-p}B(v_0) - t^{\gamma-p}F(v_0) = 0, \\ (p-1)H_\lambda(v_0) - (q-1)t^{q-p}B(v_0) - (\gamma-1)t^{\gamma-p}F(v_0) = 0. \end{gathered} \tag{4.24}$$

From here we derive

$$(q-p)H_\lambda(v) + (\gamma-q)t^{\gamma-p}F(v) = 0.$$

However, this is impossible since by Proposition 4.4 we have for $\lambda < \lambda^*(K)$ if $F(v) \geq 0$ then $H_\lambda(v) > 0$ and if $H_\lambda(u) \leq 0$ then $F(u) < 0$. The contradiction proves the lemma. □

From Theorem 4.5 and Lemma 4.9 we can derive the following multiplicity results.

Theorem 4.10. *Suppose that* (1.5)*,* $k(x) \geq 0$ *on* M*,* $d(x) \geq 0$ *on* ∂M*,* $p < \gamma < p^*$*,* $p < q < p^{**}$ *and* $q < \gamma$ *are satisfied. Furthermore, we assume*

(1) $F(\phi_1) < 0$ *holds*
(2) *The set* $\{x \in M : K(x) > 0\}$ *is not empty*
(3) $D(x) \leq 0$ *on* ∂M*.*

Then for every $\lambda \in (\lambda_1, \lambda^*(K))$ *there exists at least two weak positive solutions* u_1 *and* u_2 *of* (1.1)*-*(1.2) *such that* $u_1 > 0$ *and* $u_2 > 0$ *on* M*. Furthermore, we have* $u_1, u_2 \in W^1_p(M)$*,* $I_\lambda(u_1) < 0$*,* $I_\lambda(u_2) > 0$*.* u_1 *is a ground state and* u_2 *is a ground state of type (-1) for* I_λ*.*

5. Existence results for critical exponents

In this section, we prove the existence of positive solutions of (1.1)-(1.2) in the cases where the exponents may be critical. The main theorem in this section is as follows.

Theorem 5.1. *Suppose that $k(x) > 0$ on M, $d(x) \geq 0$ on ∂M, $q < \gamma$, $p < \gamma \leq p^*$ and $p < q \leq p^{**}$ are satisfied. Assume that $F(\phi_1) < 0$ and $B(\phi_1) < 0$. Then for every λ in $(\lambda_1, \min\{\lambda^*(K), \lambda^*(D)\})$ there exists a weak positive solution u_1 of (1.1)-(1.2) such that $u_1 > 0$ on M and $u_1 \in W_p^1(M)$. Furthermore, it holds $I_\lambda(u_1) < 0$.*

Proof. For the cases $p < \gamma < p^*$ and $p < q < p^{**}$ the statement of this theorem follows from Theorem 4.5. For critical exponents $\gamma = p^*$ and $q = p^{**}$ the result will be obtained by limiting arguments from the subcritical cases. As an example, let us suppose that $\gamma = p^*$ and $p < q < p^{**}$. The other cases can be done analogously.

Let $p < \beta \leq p^*$. Then we define

$$F_\beta(u) = \int_M K(x)|u|^\beta d\mu_g, \ u \in W.$$

Analogously one defines $\lambda^*_\beta(K)$. We assume that $F_{p^*}(\phi_1) < 0$. Then it follows from Lemma 4.1 that $\lambda_1 < \min\{\lambda^*_{p^*}(K), \lambda^*(D)\})$. Furthermore, let $\lambda_0 \in (\lambda_1, \min\{\lambda^*_{p^*}(K), \lambda^*(D)\})$. Then it is easy to see that one can find a number $\varepsilon > 0$ such that $F_\beta(\phi_1) < 0$, $|p^* - \beta| < \varepsilon$ and $\lambda^*_\beta(K) \to \lambda^*_{p^*}(K)$ as $\beta \to p^*$. Hence we have $\lambda_0 \in (\lambda_1, \min\{\lambda^*_\beta(K), \lambda^*(D)\}$ if $|p^* - \beta| < \varepsilon_0$ for some $\varepsilon_0 > 0$. Applying now Theorem 4.5 we obtain the existence of a weak positive solution $u_{1,\beta}$ of (1.1)-(1.2) with $\gamma = \beta$ such that

$$\begin{aligned} &\int_M |\nabla u_{1,\beta}|^{p-2}(\nabla u_{1,\beta}, \nabla\psi) d\mu_g + \int_{\partial M} d(x)|u_{1,\beta}|^{p-2}u_{1,\beta}\psi d\nu_g \\ &-\lambda_0 \int_M k(x)|u_{1,\beta}|^{p-2}u_{1,\beta}\psi d\mu_g - \int_M K(x)|u_{1,\beta}|^{\beta-2}u_{1,\beta}\psi d\mu_g \\ &\qquad - \int_{\partial M} D(x)|u_{1,\beta}|^{q-2}u_{1,\beta}\psi d\nu_g = 0 \end{aligned} \tag{5.1}$$

holds for any $\psi \in C^\infty(\overline{M})$.

We show that the functions $u_{1,\beta}$ are uniformly bounded in the W-norm. Suppose to the contrary that $\|u_{1,\beta_i}\| \to \infty$ for some sequence β_i such that $\beta_i \to p^*$ as $i \to \infty$. Let $v_{1,\beta_i} = u_{1,\beta_i}/\|u_{1,\beta_i}\|$ for $i = 1, 2, ...,$. Then we have $u_{1,\beta_i} = t^1(v_{1,\beta_i})v_{1,\beta_i}$, where $\|v_{1,\beta_i}\| = 1$ and by assumption $t^1(v_{1,\beta_i}) \to \infty$. Since the functions v_{1,β_i} are uniformly bounded in the W-norm then, by weak compactness, we can find a weak convergent subsequence of $\{v_{1,\beta_i}\}$ (again denoted by $\{v_{1,\beta_i}\}$) which converges weakly to some point $w \in W$.

Suppose that $w = 0$. Since W is compactly embedded in $L_p(M)$ and compactly trace - embedded in $L_p(\partial M)$ we may assume that $\int_M k(x)|v_{1,\beta_i}|^p d\mu_g \to 0$ as $i \to \infty$. This implies $H_{\lambda_0}(v_{1,\beta_i}) > 0$ for β_i near p^*. Therefore we get a contradiction to the fact that $H_{\lambda_0}(v_{1,\beta_i}) < 0$ for $v_{1,\beta_i} \in \Theta^o_{1,\beta_i}$. Thus $w \neq 0$ and therefore we can find $\psi_0 \in C^\infty(\overline{M})$ such that

$$\int_M K(x)|w|^{p^*-2}w\psi_0 d\mu_g \neq 0. \tag{5.2}$$

It follows from (5.1) that

$$\begin{aligned}\int_M |\nabla v_{1,\beta_i}|^{p-2}(\nabla v_{1,\beta_i}, \nabla\psi_0)d\mu_g &+ \int_{\partial M} d(x)|u_{1,\beta_i}|^{p-2}u_{1,\beta}\psi d\nu_g \\ -\lambda_0 \int_M k(x)|v_{1,\beta_i}|^{p-2}v_{1,\beta_i}\psi_0 d\mu_g &= t^1(v_{1,\beta_i})^{\beta_i-2}\int_M K|v_{1,\beta_i}|^{\beta_i-2}v_{1,\beta_i}\psi_0 d\mu_g \\ &+ t^1(v_{1,\beta_i})^{q-2}\int_{\partial M} D|v_{1,\beta_i}|^{q-2}v_{1,\beta_i}\psi_0 d\nu_g.\end{aligned} \tag{5.3}$$

Since W_p^1 is compactly embedded in $L_s(M)$ for $p < s < p^*$ and trace-embedded in $L_q(\partial M)$ for $p < q < p^{**}$, it follows that $v_{1,\beta_i} \to w$ in $L_s(M)$, $p < s < p^*$ and in $L_q(\partial M)$, $p < q < p^{**}$. Hence and by (5.2) it follows that the right hand side of (5.3) converges to infinity as $i \to \infty$ in contrast to the fact that the left hand side of this equality is bounded. Thus we get a contradiction and the functions $u_{1,\beta}$ are uniformly bounded in the W-norm.

Therefore, by weak compactness, we can find a weak convergent subsequence of $\{u_{1,\beta}\}$ (again denoted by $\{u_{1,\beta}\}$). Since W_p^1 is compactly embedded in $L_s(M)$ for $p < s < p^*$ and trace-embedded in $L_q(\partial M)$ for $p < q < p^{**}$, it follows easily that the weak W_p^1-limit u_{1,p^*} of the sequence $u_{1,\beta}$ satisfies also (5.1). To prove our theorem it remains to show that u_{1,p^*} is nonzero. Suppose to the contrary that $u_{1,p^*} = 0$. Let $v_{1,\beta} = u_{1,\beta}/\|u_{1,\beta}\|$. Then $u_{1,\beta} = t^1(v_{1,\beta})v_{1,\beta}$ where $\|v_{1,\beta}\| = 1$. Hence $t^1(v_{1,\beta}) \to 0$ and/or $v_{1,\beta} \rightharpoonup 0$ weakly with respect to W as $\beta \to p^*$.

Suppose the second case holds: $v_{1,\beta} \rightharpoonup 0$ weakly as $\beta \to p^*$. Since W_p^1 is compactly embedded in $L_s(M)$ for $p < s < p^*$, we may assume $f(v_{1,\beta}) \to 0$ as $\beta \to p^*$. This implies $H_{\lambda_0}(v_{1,\beta}) > 0$ for β near p^*. Therefore we have a contradiction to the fact that $H_{\lambda_0}(v_{1,\beta})) < 0$ for $v_{1,\beta} \in \Theta^0_{1,\beta}$.

Thus $v_{1,p^*} \neq 0$. Suppose now that $t^1(v_{1,\beta}) \to 0$ as $\beta \to p^*$. By virtue of (5.1) we have

$$\begin{aligned}\int_M |\nabla v_{1,\beta}|^{p-2}(\nabla v_{1,\beta}, \nabla\psi)d\mu_g &+ \int_{\partial M} d(x)|v_{1,\beta}|^{p-2}v_{1,\beta}\psi d\nu_g - \\ -\lambda_0 \int_M k(x)|v_{1,\beta}|^{p-2}v_{1,\beta}\psi d\mu_g &= t^1(v_{1,\beta})^{\beta-1}\int_M K(x)|v_{1,\beta}|^{\beta-2}v_{1,\beta}\psi d\mu_g + \\ &+ t^1(v_{1,\beta})^{q-1}\int_{\partial M} D(x)|v_{1,\beta}|^{q}v_{1,\beta}\psi d\nu_g.\end{aligned} \tag{5.4}$$

Passing to the limit in (5.4) as $\beta \to p^*$ we get

$$\begin{aligned}\int_M |\nabla v_{1,p^*}|^{p-2}\nabla v_{1,p^*}\nabla\psi d\mu_g &+ \int_{\partial M} d(x)|v_{1,p^*}|^{p-2}v_{1,p^*}\psi d\nu_g - \\ &-\lambda_0 \int_M k(x)|v_{1,p^*}|^{p-2}v_{1,p^*}\psi d\mu_g = 0.\end{aligned} \tag{5.5}$$

Observe that $v_{1,p^*} \geq 0$. Hence by the maximum principle and the Hopf lemma, since $v_{1,p^*} \not\equiv 0$, we see that $v_{1,p^*} > 0$ in $\overline{M}$. But $\lambda_1 < \lambda_0$ and λ_1 is a simple and isolated eigenvalue. Hence we get a contradiction.

Thus there exists a weak solutions $u_{1,p^*} \geq 0$ of problem (1.1)-(1.2) with $\gamma = p^*$ and $p < q < p^{**}$. Since the functional H_λ is weakly lower semi-continuous on W_p^1, $H_\lambda(u_{1,p^*}) \leq \liminf_{\beta\to p^*} H_\lambda(u_{1,\beta}) < 0$. Then for $\lambda \in (\lambda_1, \min\{\lambda^*_{p^*}(K), \lambda^*(D)\})$ by Proposition 4.4 it follows $F(u_{1,p^*}) < 0$, $B(u_{1,p^*}) < 0$. It implies $I_\lambda(u_{1,p^*}) < 0$. By the maximum principle and the Hopf lemma, since $u_{1,p^*} \not\equiv 0$, we see that $u_{1,p^*} > 0$ in $\overline{M}$. This completes the proof of the Theorem 5.1. □

The following result shows existence of ground state in critical cases.

Theorem 5.2. *Suppose that* (1.5), $k(x) \geq 0$ *on* M, $d(x) \geq 0$ *on* ∂M, $p < \gamma \leq p^*$, $p < q \leq p^{**}$ *and* $q < \gamma$ *are satisfied. Furthermore, we assume*

(1) $F(\phi_1) < 0$
(2) $D(x) \leq 0$ *on* ∂M

Then for every $\lambda \in (\lambda_1, \lambda^*(K))$ *there exists a ground state* $u_1 \in W^1_p(M)$ *of* I_λ. *Furthermore,* $u_1 > 0$, $I_\lambda(u_1) < 0$.

Proof. The existence of ground state in subcritical cases of exponents $p < \gamma < p^*$, $p < q < p^{**}$ follows from Lemma 4.9. As an example, let us prove the assertion of the theorem for the following critical case $p < q < p^{**}$, $\gamma = p^*$. The other cases can be done analogously.

Suppose $\lambda \in (\lambda_1, \lambda^*(K))$ and let $u_{1,\beta}$ be a ground state of $I_{\lambda,\beta}$ when $p < \beta < p^*$. Using the same arguments as in proving of Theorem 5.1 it can be shown the existence of weak convergent subsequence $u_{1,\beta_i} \rightharpoonup u_{1,p^*}$ with respect to W as $\beta_i \to p^*$ where u_{1,p^*} is a positive solution of (1.1)-(1.2). Let us show that u_{1,p^*} is a ground state.

First note that the functional $J^1_{\lambda,\beta}(\cdot)$ defined on $\Theta^o_{1,\lambda,\beta}$ is bounded below, i.e.,

$$-\infty < \hat{I}^1_{\lambda,\beta} = \inf\{J^1_{\lambda,\beta}(w) : w \in \Theta_{1,\lambda}\}.$$

for $\lambda \in (\lambda_1, \lambda^*(K))$ and $p < \beta \leq p^*$ (see Lemma 4.7). Next we remark that for every $w \in \Theta_{1,\lambda}$ the function $J^1_{\lambda,\beta}(w)$ is continuous with respect to $\beta \in (p, p^*]$. Hence it follows that $\hat{I}^1_{\lambda,\beta}$ is also continuous with respect to $\beta \in (p, p^*]$ and

$$\hat{I}^1_{\lambda,\beta} \to \hat{I}^1_{\lambda,p^*}, \ \text{as}\beta \to p^*. \tag{5.6}$$

Thus to prove the claim it is sufficient to show that

$$J^1_{\lambda,p^*}(v_{1,p^*}) \leq \hat{I}^1_{\lambda,p^*} = \lim_{\beta\to p^*} J^1_{\lambda,\beta}(v_{1,\beta}). \tag{5.7}$$

where $v_{1,\beta} = u_{1,\beta}/\|u_{1,\beta}\|$. Observe from the convergence $u_{1,\beta_i} \rightharpoonup u_{1,p^*}$ it follows that

$$B(u_{1,\beta_i}) \to \bar{B},\ F_{\beta_i}(u_{1,\beta_i}) \to \bar{F} \text{ as } i \to +\infty, \tag{5.8}$$

and

$$H_\lambda(u_{1,\beta_i}) \to \bar{H} \text{ as } i \to +\infty, \tag{5.9}$$

where $\bar{H}$, $\bar{F}$, $\bar{B}$ are finite. Since $H_\lambda(\cdot)$ is weakly lower semi-continuous with respect to W we have

$$H_\lambda(u_{1,p^*}) \leq \bar{H}. \tag{5.10}$$

Let us show that

$$F_{p^*}(u_{1,p^*}) \leq \bar{F}. \tag{5.11}$$

Consider a finite partition of unity for M: $\psi_j : M \to \mathbb{R}$, $Supp\psi_j \subset M$, $0 \leq \psi_j \leq 1$, $\sum_j \psi_j(x) \equiv 1$ on M. Let $p < \beta < p^*$ then testing (1.1) by $\psi_j u_{1,\beta_i}$ we obtain

$$\begin{aligned} &\int_M |\nabla u_{1,\beta_i}|^p \psi_j d\mu_g + \int_M |\nabla u_{1,\beta_i}|^{p-2}(\nabla u_{1,\beta_i}, \nabla\psi_j) d\mu_g \\ &-\lambda \int_M k(x)|u_{1,\beta_i}|^p \psi_j d\mu_g - \int_M K(x)|u_{1,\beta_i}|^{\beta_i}\psi_j d\mu_g = 0. \end{aligned} \tag{5.12}$$

From the weak convergence $u_{1,\beta_i} \rightharpoonup u_{1,p^*}$ with respect to W and strong convergence $u_{1,\beta_i} \to u_{1,p^*}$ in $L_s(M)$, $p < s < p^*$ it follows that

$$\begin{aligned}&\int_M |\nabla u_{1,\beta_i}|^p \psi_j d\mu_g - \lambda \int_M k(x)|u_{1,\beta_i}|^p \psi_j d\mu_g \to \bar{H}_j,\\ &F_{\beta_i}(u_{1,\beta_i}(\psi_j)^{1/\beta_i}) \to \bar{F}_j \quad \text{as } i \to +\infty,\end{aligned} \tag{5.13}$$

and

$$\int_M |\nabla u_{1,\beta_i}|^{p-2}(\nabla u_{1,\beta_i}, \nabla \psi_j) d\mu_g \to \int_M |\nabla u_{1,p^*}|^{p-2}(\nabla u_{1,p^*}, \nabla \psi_j) d\mu_g$$

as $i \to +\infty$. Hence passing to the limit in (5.12) we deduce

$$\bar{H}_j + \int_M |\nabla u_{1,p^*}|^{p-2}(\nabla u_{1,p^*}, \nabla \psi_j) d\mu_g = \bar{F}_j. \tag{5.14}$$

On the other hand from (1.1) in critical case $\gamma = p^*$ we have

$$\begin{aligned}&\int_M |\nabla u_{1,p^*}|^p \psi_j d\mu_g - \lambda \int_M k(x)|u_{1,p^*}|^p \psi_j d\mu_g +\\ &+ \int_M |\nabla u_{1,p^*}|^{p-2}(\nabla u_{1,p^*}, \nabla \psi_j) d\mu_g = F_{p^*}(u_{1,p^*}(\psi_j)^{1/p^*}).\end{aligned}$$

Since $\bar{H}_j \ge H_\lambda(u_{1,p^*}\psi_j)$, it follows that $F_{p^*}(u_{1,p^*}\psi_j) \le \bar{F}_j$. Thus by summing these inequalities we obtain (5.11). Observe that from the equation $Q_\lambda(t^1(v_{1,\beta}), v_{1,\beta}) = 0$, $\beta \in (p, p^*]$ it follows

$$J^1_{\lambda,p^*}(v_{1,\beta}) = \frac{q-p}{pq}(t^1(v_{1,\beta}))^p H_\lambda(v_{1,\beta}) + \frac{\gamma - q}{\gamma q}(t^1(v_{1,\beta}))^\gamma F_\beta(v_{1,\beta}).$$

Hence from (5.8), (5.9), (5.10), (5.11) we deduce

$$J^1_{\lambda,p^*}(v_{1,p^*}) \le \frac{q-p}{pq}\bar{H} + \frac{\gamma-q}{\gamma q}\bar{F} = \lim_{\beta \to p^*} J^1_{\lambda,\beta}(v_{1,\beta}).$$

Thus we obtain (5.7) and the proof of theorem is complete. □

Acknowledgements. We would like to thank S. I. Pohozaev for his helpful discussions on the subject. The first author was supported by Deutsche Forschungsgemeinschaft, grant DFG-9510728 and by the grants RFBR 02-01-00069, INTAS 00-136. The second author was supported by grant DFG-Tr 374/1-2.

References

[1] Alama S., Tarantello G., *On semilinear elliptic problems with indefinite nonlinearities.* Calculus Var. and Partial Differential Equations **1** (1993), 439-475.
[2] Berestycki, H., Capuzzo–Dolcetta, I., Nirenberg, L., *Variational methods for indefinite superlinear homogeneous elliptic problems.* NoDEA **2** (1995), 553-572.
[3] Brezis, H., Nirenberg L., *Positive solutions of nonlinear elliptic equations involving critical exponents.* Comm. Pure Appl. Math. **36** (1983), 437-477.
[4] Carslow. H, Jaeger J., *Conduction of Heat in Solid.* 2nd edition, Oxford Press, (1959), 17-23.
[5] Cherrier P., *Probl'emes de Neumann nonlinéaires sur les varétés Riemanniennes.* J. Funct. Anal. **57** (1984), 154-207.
[6] Coleman, S., Glazer, V., Martin, A., *Action minima among solution to a class of Euclidean scalar field equations.* Comm. Math. Phys.**58** , 2 ,211-221 (1978)
[7] Diaz, J.I., *Nonlinear Partial Differential Equation and Free Boundaries vol. I: Elliptic Equations.* Vol. 106 of Research Notes in Mathematics (Pitman Advanced Publishing Program, London), (1985).

[8] Drábek, P., Pohozaev, S.I., *Positive solution for the p-Laplacian: application of the fibering method.* Proc. Roy. Soc. Edinb. Sect. A **127** (1997), 703-726.

[9] Escobar, J., *The Yamabe problem on manifolds with boundary.* J. Diff. Geom. **35** (1992), 21-84.

[10] Escobar, J., *Conformal deformation of Riemannian metric to a scalar flat metric with constant mean curvature on the boundary.* Ann. Math. **136** (1992), 1-50.

[11] Escobar, J., *Conformal metric with prescribed mean curvature on the boundary.* Calculus Var. and Partial Differential Equations **4** (1996), 559-592.

[12] Il'yasov, Ya.Sh., *The existence of conformally equivalent metrics for Riemannian manifolds with boundary*, Differ. Equations **35**, No.3, (1999), 335-340.

[13] Il'yasov, Y., Runst, T., *Existence and multiplicity results for a class of non-linear Neumann boundary value problems.* Jenaer Schriften zur Mathematik und Informatik, Univ. Jena, Math/Inf/99/14, 1-23.

[14] Il'yasov, Y., *On positive solutions of indefinite elliptic equations.* C. R. Acad. Sci., Paris **333** (2001), 533-538.

[15] Michalek R., *Existence of positive solution of a general quasilinear elliptic equation with a nonlinear boundary condition of mixed type.* Nonl. An., Th. Meth. Appl. **15** (1990), 871-882.

[16] Pierotti D., Terracini S., *On a Neumann problem with critical exponent and critical nonlinearity on the boundary.* Comm. Part. Diff. Equations **20** (1995), 1155-1187.

[17] Pohozaev, S.I., *On an approach to Nonlinear equations.* Doklady Acad. Sci. USSR **247** (1979), 1327-1331.

[18] Pohozaev, S.I., Véron L., *Multiple positive solutions of some quasilinear Neumann problems.* Applicable. Analysis. **74** (2000), 363-391.

[19] Ouyang, T.C., *On the positive solutions of semilinear equations* $\Delta u + \lambda u + hu^p = 0$ *on the compact manifolds.* Indiana Univ. Math. J. **40** (1991), 1083-1141.

[20] Struwe, M., *Variational Methods, Application to Nonlinear Partial Differential Equations and Hamiltonian Systems.* Springer- Verlag, Berlin, Heidelberg, New-York, 1996.

[21] Taira, K., *The Yamabe problem and nonlinear boundary value problems.* J. Differential Equations **122** (1995), 316-372.

[22] Tehrani, H.T., *On indefinite superlinear elliptic equations.* Calculus Var. and Partial Differential Equations **4** (1996), 139-153.

[23] Vazquez J. L. & Véron L., *Solutions positives d' éuations elliptiques semi-linéaires sur des variétés riemanniennes compactes*, C. R. Acad. Sci., Paris **312**, (1991), 811-815.

[24] Véron L., *Première valeur propre non nulle du p-laplacien et éuations quasilin éaires elliptiques sur une vari ét é riemannienne compacte*, C. R. Acad. Sci., Paris **314**, (1992), 271-276.

Yavdat Il'yasov

Universite de La Rochelle, 1700 La Rochelle, France

E-mail address: `IlyasovYS@ic.bashedu.ru`

Thomas Runst

Mathematisches Institut, Friedrich-Schiller-Universität Jena, D-07740 Jena, Germany

E-mail address: `runst@minet.uni-jena.de`