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NON-AUTONOMOUS RETARDED DIFFERENTIAL EQUATIONS: THE VARIATION OF CONSTANTS FORMULAS AND THE ASYMPTOTIC BEHAVIOUR

SAID BOULITE, LAHCEN MANIAR, & MOHAMMED MOUSSI

Abstract. This paper is devoted to show a variation of constants formula for the operator solution to the non-autonomous retarded differential equation

$$x'(t) = A(t)x(t) + L(t)x_t + f(t), \quad x_s = \varphi, \quad t \geq s \geq 0,$$

in terms of the inhomogeneous term f, which will allow us to study the asymptotic behaviour of this solution. We treat also the existence of fundamental solutions and the stability of semi-linear retarded differential equations.

1. Introduction

In this paper we study the retarded non-autonomous differential equation

$$\begin{aligned} x'(t) &= A(t)x(t) + L(t)x_t + f(t), \quad t \geq s \geq 0, \\ x_s &= \varphi \in \mathcal{C}_r := C([-r,0],E), \end{aligned} \tag{1.1}$$

where $(A(t), D(A(t)))_{t\geq 0}$ generates the strongly continuous evolution family $(V(t,s))_{t\geq s\geq 0}$ on a Banach space E, and $(L(t))_{t\geq 0}$ is a family of bounded linear operators from $\mathcal{C}_r$ into E.

In the autonomous case ($A(t) = A$, $L(t) = L$, $t \geq 0$), the retarded differential equation (1.1) has been studied by many authors using various techniques; see for example [4, 9, 14, 17, 27, 28, 29].

The case $A(t) = A$, $t \geq 0$, has been treated recently in [12], [19] and [22] using extrapolation theory. In [22], A. Rhandi showed recently that the solution of the homogeneous retarded differential equation ($f \equiv 0$) is given by a Dyson-Phillips series. Also in the same case, in [10, 12], the authors proved that the solution of the inhomogeneous retarded equation (1.1) is given in terms of the inhomogeneous term f by a variation of constants formula, which was used to study the asymptotic behaviour of the solutions of (1.1). Recently, in [13] the authors studied the asymptotic behaviour to (1.1) on $\mathbb{R}$ using the evolution semigroups and the characteristic equation. Also, R. Schnaubelt [26] showed the first (to our knowledge) variation of constants formula for the inhomogeneous retarded equation (1.1) by using the ideas of evolution semigroups.

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Our aim in this paper is to extend the results of [10, 12, 19, 22] to the fully non-autonomous equation (1.1). More precisely, after the preliminary section, we establish in Section 3 the existence of mild solutions of the homogeneous retarded differential equations, and that solutions are provided by some evolution families. Moreover, we show that these evolution families are given by Dyson-Phillips series and that the term retard do not affect the asymptotic behaviour, as boundedness and asymptotic almost periodicity, of the solutions to the non retarded Cauchy problem

$$\begin{aligned} x'(t) &= A(t)x(t), \quad t \geq s, \\ x(s) &= x \in E. \end{aligned} \tag{1.2}$$

Namely, we give conditions under which the evolution families solutions of the homogeneous retarded equations and the Cauchy problem (1.2) have the same asymptotic behaviour (see Theorem 3.3).

In section 4, we show that the solution x_t of (1.1) satisfies a variation of constants formula, different from the one of [26]. Using our variation of constant formula, we show that these solutions inherit the same asymptotic behaviour of the inhomogeneous term f. In the last section, we show the existence of fundamental solutions for the homogeneous retarded equations, which has been assumed in [10], and obtain the asymptotic behaviour of semi-linear retarded differential equations. Before ending this introduction, we shall mention that we can consider also the above differential retarded equations on $\mathbb{R}$ and we can obtain, using the same technics, the same results and all results of [12].

2. Preliminaries

In this section we recall some definitions and fix notations which will be used in the sequel.

Let X be a Banach space and denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X.

A family of operators $\mathcal{U} := (U(t,s))_{t\geq s\geq 0} \subset \mathcal{L}(X)$ is called *a strongly continuous evolution family* if

(1) $U(t,s) = U(t,r)U(r,s)$ and $U(s,s) = Id$ for all $t \geq r \geq s \geq 0$,
(2) the mapping $\{(t,s) : t \geq s \geq 0\} \ni (t,s) \mapsto U(t,s)$ is strongly continuous.

An evolution family $(U(t,s))_{t\geq s\geq 0}$ is said to have an *exponential dichotomy* if there exists a projection-valued function $P : \mathbb{R}_+ \to \mathcal{L}(X)$ such that the function $P(\cdot)x$ is continuous and bounded for each $x \in X$, and constants $\delta > 0, N = N(\delta) \geq 1$ such that

(i) $P(t)U(t,s) = U(t,s)P(s)$;
(ii) $U_Q(t,s)$, the restriction of $U(t,s)$ on $ImQ(s)$, is invertible as an operator from $ImQ(s)$ to $ImQ(t)$, with $Q(\cdot) := I - P(\cdot)$;
(iii) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$, and $\|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$

for $t \geq s$ and $t, s \in \mathbb{R}_+$. The family of operators $(\Gamma(t,s))_{t\geq s\geq 0} \subseteq \mathcal{L}(X)$ given by

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geq s, \\ -U_Q(t,s)Q(s), & t < s, \end{cases}$$

is called the corresponding *Green's* operator function.

For evolution families and well-posedness of the non-autonomous Cauchy problems we refer to [11, 21, 24, 25].

Let $BC(\mathbb{R}_+, X)$ be the Banach space of all bounded continuous functions from $\mathbb{R}_+$ to X, endowed with the uniform norm. The closed subspace of bounded uniformly continuous functions will be denoted by $BUC(\mathbb{R}_+, X)$.

If $f : \mathbb{R}_+ \to X$, the set of all translates, called the hull of f, is $H(f) := \{f(\cdot + t) : t \in \mathbb{R}_+\}$.

A function $f \in BC(\mathbb{R}_+, X)$ is said to be *asymptotically almost periodic* if $H(f)$ is relatively compact in $BC(\mathbb{R}_+, X)$.

If $H(f)$ is weakly relatively compact in $BC(\mathbb{R}_+, X)$, the bounded continuous function $f : \mathbb{R}_+ \to X$ is called *Eberlein weakly asymptotically almost periodic.*

We recall also that a closed subspace $\mathcal{E}$ of $BUC(\mathbb{R}_+, X)$ is said to be *translation bi-invariant* if for all $t \geq 0$

$$f \in \mathcal{E} \Longleftrightarrow f(\cdot + t) \in \mathcal{E},$$

and *operator invariant* if $M \circ f \in \mathcal{E}$ for every $f \in \mathcal{E}$ and $M \in \mathcal{L}(X)$, where $M \circ f$ is defined by $(M \circ f)(t) = M(f(t))$, $t \geq 0$. A closed subspace $\mathcal{E}$ of $BUC(\mathbb{R}_+, X)$ is said to be *homogeneous* if it is translation bi-invariant and operator invariant.

From [5] the following classes of X-valued functions are homogeneous closed subspaces of $BUC(\mathbb{R}_+, X)$:

- The space $C_0(\mathbb{R}_+, X)$ of all continuous functions vanishing at infinity;
- The space $AAP(\mathbb{R}_+, X)$ of asymptotically almost periodic functions;
- The space $W(\mathbb{R}_+, X)$ of Eberlein weakly asymptotically almost periodic functions.

For more details on almost periodic functions, we refer to [1, 15]. For the almost periodicity of solutions of Cauchy problems, see, e.g., [2, 3, 6, 16, 23].

For the sequel, we need also the following fundamental lemma, see [18] for more details and [5] for an autonomous version.

Lemma 2.1. *Let $(U(t,s))_{t\geq s\geq 0}$ be a bounded evolution family on X. Let $\mathcal{E}$ be a homogeneous closed subspace of $BUC(\mathbb{R}_+, X)$. Assume that $\mathbb{R}_+ \ni t \mapsto U(t+s, s)x$ belongs to $\mathcal{E}$ for every $x \in X$ and $s \geq 0$. If $h \in L^1(\mathbb{R}_+, X)$, then $\mathbb{R}_+ \ni t \mapsto \int_0^t U(t+s, s+\sigma)h(\sigma)d\sigma$ belongs to $\mathcal{E}$ for all $s \geq 0$.*

3. The homogeneous case: Dyson-Phillips series and asymptotic behaviour

Let $(A(t), D(A(t))_{t\geq 0}$ be a stable family and generate an evolution family $(V(t,s))_{t\geq s\geq 0}$, on a Banach space E, such that $\|V(t,s)\| \leq Me^{\omega(t-s)}$, for some constants $\omega \in \mathbb{R}$ and $M \geq 1$. Consider also the family $(L(t))_{t\geq 0}$ of bounded linear operators from $\mathcal{C}_r$ into E, with $L(\cdot) \in BC(\mathbb{R}_+, \mathcal{L}_s(\mathcal{C}_r, E))$, i.e., $t \mapsto L(t)$ is a bounded and strongly continuous function.

The well-posedness of the homogeneous non-autonomous retarded differential equation

$$\begin{aligned} x'(t) &= A(t)x(t) + L(t)x_t, \quad t \geq s, \\ x_s &= \varphi \in \mathcal{C}_r, \end{aligned} \tag{3.1}$$

was treated recently, e.g., in [10, 13, 20, 26]. In these papers, the authors showed the existence of a unique mild solution to (3.1), i.e., a continuous function x :

$[s-r,\infty) \to E$ satisfying

$$x(t) = \begin{cases} V(t,s)\varphi(0) + \int_s^t V(t,\sigma)L(\sigma)x_\sigma d\sigma, & t \geq s, \\ \varphi(t-s), & s-r \leq t \leq s, \end{cases} \tag{3.2}$$

and the operator solutions (x_t) are evolution families on $\mathcal{C}_r$. In this section, we show, by a different way, the existence of mild solutions to (3.1). Our aim exactly here is to show that the evolution family solution of the equation (3.1) is given by a Dyson-Phillips series, and satisfies a variation of constants formula. These formulas are used to show that the mild solutions of the retarded equations have the same asymptotic behaviour of the trajectories $\mathbb{R}_+ \ni t \mapsto V(t+s,s)x$, $s \geq 0$, $x \in E$.

It is known that the evolution family solution to the non-retarded equation $(L(t) \equiv 0)$ is given by

$$U(t,s)\varphi(\tau) = \begin{cases} V(t+\tau,s)\varphi(0), & t+\tau \geq s, \\ \varphi(t+\tau-s), & s-r \leq t+\tau \leq s\,; \end{cases}$$

see for example [12, 26]. To obtain our aim, we need the following fundamental result.

Lemma 3.1. *Let* $g \in C(\mathbb{R}_+, E)$. *Then*

$$\lim_{\lambda\to+\infty} \int_s^t U(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda, A(0))g(\sigma)d\sigma$$

exists in $\mathcal{C}_r$ *uniformly in compact sets of* $\{(t,s) : t \geq s \geq 0\}$.

Proof. Set, for $\lambda \geq \lambda_0 > \max(\omega, 0)$ and $0 \leq s \leq t \leq T$ (for some $T > 0$),

$$\mathcal{W}_\lambda(t,s) := \int_s^t U(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda, A(0))g(\sigma)d\sigma.$$

For $\tau \in [-r, 0]$ and $t+\tau \geq s$, we have

$$\begin{aligned} &\mathcal{W}_\lambda(t,s)(\tau) \\ &= \int_s^{t+\tau} U(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda, A(0))g(\sigma)(\tau)d\sigma + \int_{t+\tau}^t U(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda, A(0))g(\sigma)(\tau)d\sigma \\ &= \int_s^{t+\tau} V(t+\theta,\sigma)\lambda R(\lambda, A(0))g(\sigma)d\sigma + \int_{t+\tau}^t \lambda e^{\lambda(t+\tau-\sigma)}R(\lambda, A(0))g(\sigma)d\sigma. \end{aligned}$$

Let $\lambda, \mu \geq \lambda_0$. We have then,

$$\begin{aligned} &\mathcal{W}_\lambda(t,s)(\tau) - \mathcal{W}_\mu(t,s)(\tau) \\ &= \int_s^{t+\tau} V(t+\theta,\sigma)\left[\lambda R(\lambda, A(0)) - \mu R(\mu, A(0))\right]g(\sigma)d\sigma \\ &\quad + \int_{t+\tau}^t \left[\lambda e^{\lambda(t+\tau-\sigma)}R(\lambda, A(0)) - \mu e^{\mu(t+\tau-\sigma)}R(\mu, A(0))\right]g(\sigma)d\sigma, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{W}_\lambda(t,s)(\tau) - \mathcal{W}_\mu(t,s)(\tau) \\ &= \int_s^t \left[\lambda e^{\lambda(t+\tau-\sigma)}R(\lambda, A(0)) - \mu e^{\mu(t+\tau-\sigma)}R(\mu, A(0))\right]g(\sigma)d\sigma \end{aligned}$$

for $t+\tau \le s$. Thus,

$$\begin{aligned}&\|\mathcal{W}_\lambda(t,s)(\tau)-\mathcal{W}_\mu(t,s)(\tau)\|\\&\le \tilde{M}(T)\int_0^T \|\left[\lambda R(\lambda,A(0))-\mu R(\mu,A(0))\right]g(\sigma)\|d\sigma+\tilde{M}(\frac{1}{\lambda}+\frac{1}{\mu})\sup_{0\le\sigma\le T}\|g(\sigma)\|.\end{aligned}$$

Hence, as

$$\lim_{\lambda,\mu\to+\infty}\|\left[\lambda R(\lambda,A(0))-\mu R(\mu,A(0))\right]g(\sigma)\|=0 \quad \text{for all } \sigma\in[0,T],$$

by Lebesgue dominated convergence theorem, we have

$$\lim_{\lambda,\mu\to+\infty}\int_0^T \|\left[\lambda R(\lambda,A(0))-\mu R(\mu,A(0))\right]g(\sigma)\|d\sigma=0.$$

Consequently,

$$\sup_{\tau\in[-r,0]}\|\mathcal{W}_\lambda(t,s)(\tau)-\mathcal{W}_\mu(t,s)(\tau)\|\to 0 \quad \text{as } \lambda,\mu\to+\infty$$

uniformly for $0\le s\le t\le T$. This completes the proof. □

From this lemma, we can define the operators $U_n(t,s)$ as

$$U_0(t,s)\varphi=U(t,s)\varphi,$$

$$U_n(t,s)\varphi=\lim_{\lambda\to+\infty}\int_s^t U(t,\sigma)e^{\lambda\cdot}\lambda R(\lambda,A(0))L(\sigma)U_{n-1}(\sigma,s)\varphi\,d\sigma$$

for all $\varphi\in\mathcal{C}_r$, $n\ge 1$, and $0\le s\le t$.

The first main result of this section can now be stated.

Theorem 3.2. (i) *The expansion* $U_L(t,s):=\sum_{n\ge0}U_n(t,s)$, $t\ge s\ge 0$, *converges in* $\mathcal{L}(\mathcal{C}_r)$ *uniformly on* $\{(t,s):0\le s\le t\le T\}$ *(for all* $T>0$*), and* $(U_L(t,s))_{t\le s\le 0}$ *is an evolution family on* $\mathcal{C}_r$. *Further, this variation of constants formula*

$$U_L(t,s)\varphi=U(t,s)\varphi+\lim_{\lambda\to\infty}\int_s^t U(t,\sigma)e^{\lambda\cdot}\lambda R(\lambda,A(0))L(\sigma)U_L(\sigma,s)\varphi\,d\sigma \tag{3.3}$$

holds for all $\varphi\in\mathcal{C}_r$ *and* $t\ge s\ge 0$.

(ii) *For every* $\varphi\in\mathcal{C}_r$ *and* $s\ge 0$ *the function defined by*

$$x(t,s,\varphi):=\begin{cases}U_L(t,s)\varphi(0), & t\ge s,\\ \varphi(t-s), & s-r\le t\le s,\end{cases} \tag{3.4}$$

is the unique mild solution of (3.1)*, and*

$$x_t=U_L(t,s)\varphi,\quad t\ge s\ge 0.$$

Proof. For $n=0$, we have

$$\|U_0(t,s)\|\le Me^{\omega(t-s)},\qquad t\ge s\ge 0.$$

For $n=1$, $t\ge s\ge 0$ and $\varphi\in\mathcal{C}_r$

$$U_1(t,s)\varphi=\lim_{\lambda\to+\infty}\int_s^t U(t,\sigma)e^{\lambda\cdot}\lambda R(\lambda,A(0))L(\sigma)U_0(\sigma,s)\varphi\,d\sigma.$$

For all $\tau \in [-r, 0]$, we have

$$\begin{aligned}\|U_1(t,s)\varphi\| &= \lim_{\lambda\to+\infty} \|\int_s^t U(t,\sigma)e^{\lambda\cdot}\lambda R(\lambda, A(0))L(\sigma)U_0(\sigma,s)\varphi\, d\sigma\| \\ &\le M^2\int_s^t e^{\omega(t-\sigma)}\|L(\sigma)\|Me^{\omega(\sigma-s)}\|\varphi\|\, d\sigma \\ &\le M^2\|L(\cdot)\|_\infty Me^{\omega(t-s)}(t-s)\|\varphi\|.\end{aligned}$$

Hence

$$\|U_1(t,s)\| \le M^2\|L(\cdot)\|_\infty Me^{\omega(t-s)}(t-s)\|\varphi\|.$$

By induction, one can see

$$\|U_n(t,s)\| \le \frac{(M^2\|L(\cdot)\|_\infty)^n}{n!}Me^{\omega(t-s)}, \quad t \ge s \ge 0.$$

Therefore, the expansion $\sum_{n\ge0} U_n(t,s)$ converges in $\mathcal{L}(\mathcal{C}_r)$ uniformly for $0 \le s \le t \le T$. The strong continuity of $(U_L(t,s))_{t\ge s\ge0}$ can be obtained from Lemma (3.1) and the uniform convergence of the series. The rest of (i) is easy to see.
For (ii), from the variation of constants formula (3.3), we have

$$U_L(t,s)\varphi(\tau) = \begin{cases} V(t+\tau,s)\varphi(0) & \\ + \int_s^{t+\tau} V(t+\tau,\sigma)L(\sigma)U_L(\sigma,s)\varphi\, d\sigma, & t+\tau \ge s, \\ \varphi(t+\tau-s), & s-r \le t+\tau \le s. \end{cases} \tag{3.5}$$

Hence, the translation property

$$U_L(t,s)\varphi(\tau) = \begin{cases} U_L(t+\tau,s)\varphi(0), & t+\tau \ge s, \\ \varphi(t+\tau-s), & s-r \le t+\tau \le s, \end{cases}$$

holds, and then the function defined by (3.4) satisfies $x_t = U_L(t,s)\varphi$. Consequently, it is now clear that x is a mild solution of (3.1). □

Now we deal with the question of the robustness of some asymptotic behaviour of the non-retarded equation (1.2) under the introduction of the term retard. More precisely if we assume that the trajectories $t \mapsto V(t+s,s)x$, $s \ge 0$, $x \in E$ belong to some homogeneous closed subspace $\mathcal{E}$ of $BUC(\mathbb{R}_+, E)$, and there exist constants $0 < q < \frac{1}{M}$ and $s_0 \ge 0$ such that

$$\int_0^{+\infty} \|L(\tau+s)U(\tau+s,s)\varphi\|d\tau \le q\|\varphi\| \tag{3.6}$$

for all $\varphi \in \mathcal{C}_r$ and $s \ge s_0$, we can obtain the following main result.

Theorem 3.3. *Assume* (3.6) *and that the trajectories* $t \mapsto V(t+s,s)x$, $s \ge 0$, $x \in E$ *belong to some homogeneous closed subspace* $\mathcal{E}$ *of* $BUC(\mathbb{R}_+, E)$*. Then, the solution* $t \mapsto x(t+s,s,\varphi)$ *of* (3.1) *belongs to* $\mathcal{E}$ *for all* $\varphi \in \mathcal{C}_r$ *and* $s \ge 0$.

Proof. Take $t \ge 0, s \ge s_0$ and $\varphi \in \mathcal{C}_r$. By the above results, we have that

$$\begin{aligned} x(t+s,s,\varphi) &= U_L(t+s,s)\varphi(0) \\ &= V(t+s,s)\varphi(0) + \int_0^t V(t+s,\sigma+s)L(\sigma+s)U_L(\sigma+s,s)\varphi\, d\sigma\,. \end{aligned}$$

According to Lemma 2.1, we have to show only that $L(\cdot+s)U_L(\cdot+s)\varphi$ is in $L^1(\mathbb{R}_+,E)$. For this, let $t\geq 0$. The variation of constants formula (3.3) leads to

$$\begin{aligned}
&L(t+s)U_L(t+s,s)\varphi\\
&= L(t+s)U(t+s,s)\varphi\\
&\quad + \lim_{\lambda\to+\infty}\int_s^{t+s} L(t+s)U(t+s,\sigma)e^{\lambda\cdot}\lambda R(\lambda,A(0))L(\sigma)U_L(\sigma,s)\varphi\,d\sigma.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^t L(\tau+s)U_L(\tau+s,s)\varphi d\tau\\
&= \int_0^t L(\tau+s)U(\tau+s,s)\varphi d\tau\\
&\quad + \lim_{\lambda\to+\infty}\int_0^t\int_s^{\tau+s} L(\tau+s)U(\tau+s,\sigma)e^{\lambda\cdot}\lambda R(\lambda,A(0))L(\sigma)U_L(\sigma,s)\varphi\,d\sigma d\tau\\
&= \int_0^t L(\tau+s)U(\tau+s,s)\varphi d\tau\\
&\quad + \lim_{\lambda\to+\infty}\int_s^{t+s}\int_0^{t+s-\sigma} L(\tau+\sigma)U(\tau+\sigma,\sigma)e^{\lambda\cdot}\lambda R(\lambda,A(0))L(\sigma)U_L(\sigma,s)\varphi\,d\tau d\sigma,
\end{aligned}$$

and the estimate (3.6) implies

$$\begin{aligned}
&\int_0^t \|L(\tau+s)U_L(\tau+s,s)\varphi\| d\tau\\
&\leq \int_0^t \|L(\tau+s)U(\tau+s,s)\varphi\| d\tau\\
&\quad + \lim_{\lambda\to+\infty}\int_s^{t+s}\int_0^{t+s-\sigma} \|L(\tau+\sigma)U(\tau+\sigma,\sigma)e^{\lambda\cdot}\lambda R(\lambda,A(0))L(\sigma)U_L(\sigma,s)\varphi\|\,d\tau d\sigma\\
&\leq q\|\varphi\| + q\lim_{\lambda\to+\infty}\int_s^{t+s}\|e^{\lambda\cdot}\lambda R(\lambda,A(0))L(\sigma)U_L(\sigma,s)\varphi\|\,d\sigma\\
&\leq q\|\varphi\| + qM\int_0^t \|L(\sigma+s)U_L(\sigma+s,s)\varphi\|\,d\sigma.
\end{aligned}$$

This proves our claim. For $0\leq s\leq s_0$ and $t\geq 0$, one can write

$$U_L(t+s_0+s,s)\varphi(0) = U_L(t+s+s_0,s+s_0)U_L(s+s_0,s)\varphi(0).$$

As $s+s_0\geq s_0$, then as shown above $t\mapsto U_L(t+s_0+s,s)\varphi(0)$ belongs to $\mathcal{E}$ and by the translation bi-invariance of $\mathcal{E}$, $t\mapsto U_L(t+s,s)x$ belongs to $\mathcal{E}$. This completes the proof. □

4. The inhomogeneous case: the variation of constants formula and the asymptotic behaviour

Consider again the inhomogeneous retarded differential equation

$$\begin{aligned}
x'(t) &= A(t)x(t) + L(t)x_t + f(t), \quad t\geq s\geq 0,\\
x_s &= \varphi\in\mathcal{C}_r,
\end{aligned} \tag{4.1}$$

where $(A(t), D(A(t)))_{t\geq 0}$ is a stable family and generates a strongly continuous evolution family $(V(t,s))_{t\geq s\geq 0}$, on a Banach space E, such that $\|V(t,s)\| \leq Me^{\omega(t-s)}$, with $\omega \in \mathbb{R}$ and $M \geq 1$, $L(\cdot) \in BC(\mathbb{R}_+, \mathcal{L}_s(\mathcal{C}_r, E))$, and $f \in L^1_{\rm loc}(\mathbb{R}_+, E)$.

The following definition is standard in the literature [10, 13, 26].

Definition 4.1. A continuous function $x := x(\cdot, s, \varphi) : [-r, \infty) \to E$ is called a mild solution of (4.1) if

$$x(t) = \begin{cases} V(t,s)\varphi(0) + \int_s^t V(t,\sigma)\,[L(\sigma)x_\sigma + f(\sigma)]\; d\sigma, & t \geq s, \\ \varphi(t-s), & s-r \leq t \leq s. \end{cases} \tag{4.2}$$

The existence of mild solutions of (4.1) has been treated recently by many authors, e.g., [13, 26]. Our aim, in this section, is to show that the mild solutions of these equations are given by variation of constants formulas in terms of the inhomogeneous term f. To this purpose, we need the fundamental lemma.

Lemma 4.2. *For every $f \in L^1_{\rm loc}(\mathbb{R}_+, E)$, the limit*

$$\lim_{\lambda\to+\infty} \int_s^t U_L(t,\sigma) e^{\lambda\cdot} \lambda R(\lambda, A(0)) f(\sigma)\, d\sigma$$

exists in $\mathcal{C}_r$ uniformly in compact sets of $\{(t,s) : 0 \leq s \leq t\}$.

Proof. Let $T > 0$, $0 \leq s \leq t \leq T$, and $\lambda \geq \max(0, \omega)$. Assume first that $f \in C([0,T], E)$, and set

$$\mathcal{Z}_\lambda(t,s) = \int_s^t U_L(t,\sigma)\lambda e^{\lambda\cdot} R(\lambda, A(0)) f(\sigma) d\sigma.$$

For $\tau \in [-r, 0]$ such that $t + \tau \geq s$, from the formula (3.5), we have

$$\begin{aligned} &\mathcal{Z}_\lambda(t,s)(\tau) \\ &= \int_s^{t+\tau} U_L(t,\sigma)\lambda e^{\lambda\cdot} R(\lambda, A(0)) f(\sigma)(\tau) d\sigma + \int_{t+\tau}^t U_L(t,\sigma)\lambda e^{\lambda\cdot} R(\lambda, A(0)) f(\sigma)(\tau) d\sigma \\ &= \int_s^{t+\tau} V(t+\tau,\sigma)\lambda R(\lambda, A(0)) f(\sigma) d\sigma + \int_{t+\tau}^t \lambda e^{\lambda(t+\tau-\sigma)} R(\lambda, A(0)) f(\sigma) d\sigma \\ &\quad + \int_s^{t+\tau}\int_\sigma^{t+\tau} V(t+\tau,\delta) L(\delta) U_L(\delta,\sigma)\lambda e^{\lambda\cdot} R(\lambda, A(0)) f(\sigma)\, d\delta\, d\sigma. \end{aligned} \tag{4.3}$$

The last term in the right-hand side of this equality leads to

$$\begin{aligned} &\int_s^{t+\tau}\int_\sigma^{t+\tau} V(t+\tau,\delta) L(\delta) U_L(\delta,\sigma)\lambda e^{\lambda\cdot} R(\lambda, A(0)) f(\sigma)\, d\delta\, d\sigma \\ &= \int_s^{t+\tau}\int_s^{\delta} V(t+\tau,\delta) L(\delta) U_L(\delta,\sigma)\lambda e^{\lambda\cdot} R(\lambda, A(0)) f(\sigma) d\sigma\, d\delta \\ &= \int_s^{t+\tau} V(t+\tau,\delta) L(\delta) \mathcal{Z}_\lambda(\delta,s) d\delta. \end{aligned}$$

Then, if $t+\tau \geq s$ we obtain

$$\begin{aligned}
&\mathcal{Z}_\lambda(t,s)(\tau)\\
&= \int_s^{t+\tau} V(t+\tau,\sigma)\lambda R(\lambda,A(0))f(\sigma)d\sigma + \int_{t+\tau}^t \lambda e^{\lambda(t+\tau-\sigma)} R(\lambda,A(0))f(\sigma)d\sigma\\
&\quad + \int_s^{t+\tau} V(t+\tau,\sigma)L(\sigma)\mathcal{Z}_\lambda(\sigma,s)d\sigma.
\end{aligned}$$

If $t+\tau \leq s$, we have

$$\mathcal{Z}_\lambda(t,s)(\tau) = \int_s^t \lambda e^{\lambda(t+\tau-\sigma)} R(\lambda,A(0))f(\sigma)d\sigma.$$

Now, let $\lambda,\mu \geq \lambda_0 > \max(\omega,0)$,

$$\begin{aligned}
&\mathcal{Z}_\lambda(t,s)(\tau) - \mathcal{Z}_\mu(t,s)(\tau)\\
&= \begin{cases}
\int_s^{t+\tau} V(t+\tau,\sigma)\big[\lambda R(\lambda,A(0)) - \mu R(\mu,A(0))\big]f(\sigma)d\sigma \\
+\int_{t+\tau}^t \big[\lambda e^{\lambda(t+\tau-\sigma)}R(\lambda,A(0)) - \mu e^{\mu(t+\tau-\sigma)}R(\mu,A(0))\big]f(\sigma)d\sigma \\
+\int_s^{t+\tau} V(t+\tau,\sigma)L(\sigma)\big[\mathcal{Z}_\lambda(\sigma,s) - \mathcal{Z}_\mu(\sigma,s)\big]d\sigma, & t+\tau \geq s\\
\int_s^t [\lambda e^{\lambda(t+\tau-\sigma)}R(\lambda,A(0)) - \mu e^{\mu(t+\tau-\sigma)}R(\mu,A(0))]f(\sigma)d\sigma, & t+\tau \leq s.
\end{cases}
\end{aligned}$$

For $t+\tau \geq s$, we get easily

$$\begin{aligned}
&\|\mathcal{Z}_\lambda(t,s)(\tau) - \mathcal{Z}_\mu(t,s)(\tau)\|\\
&\leq \tilde{M}(T)\int_0^T \|(\lambda R(\lambda,A(0)) - \mu R(\mu,A(0)))f(\sigma)\|d\sigma\\
&\quad + M(\frac{1}{\lambda}+\frac{1}{\mu})\sup_{0\leq\sigma\leq T}\|f(\sigma)\| + Me^{\omega T}\|L(\cdot)\|_\infty \int_0^T \|\mathcal{Z}_\lambda(\sigma,s) - \mathcal{Z}_\mu(\sigma,s)\|d\sigma,
\end{aligned}$$

and for $t+\tau \leq s$

$$\begin{aligned}
&\|\mathcal{Z}_\lambda(t,s)(\tau) - \mathcal{Z}_\mu(t,s)(\tau)\|\\
&\leq \tilde{M}(T)\int_0^T \|(\lambda R(\lambda,A(0)) - \mu R(\mu,A(0)))f(\sigma)\|d\sigma + M(\frac{1}{\lambda}+\frac{1}{\mu})\sup_{0\leq\sigma\leq T}\|f(\sigma)\|.
\end{aligned}$$

Hence, as

$$\lim_{\lambda,\mu\to+\infty} \|\,[\lambda R(\lambda,A(0)) - \mu R(\mu,A(0))]\,f(\sigma)\| = 0 \quad \text{for all } \sigma\in[0,T],$$

by the Lebesgue dominated convergence theorem, for $\varepsilon > 0$ and λ,μ sufficiently large we have

$$\|\mathcal{Z}_\lambda(t,s)(\tau) - \mathcal{Z}_\mu(t,s)(\tau)\| \leq \varepsilon + Me^{\omega T}\int_s^t \|L(\cdot)\|_\infty \|\mathcal{Z}_\lambda(\sigma,s) - \mathcal{Z}_\mu(\sigma,s)\|d\sigma.$$

An application of Gronwall's inequality yields

$$\|\mathcal{Z}_\lambda(t,s)(\tau) - \mathcal{Z}_\mu(t,s)(\tau)\| \leq \varepsilon e^{Me^{\omega T}(t-s)\|L(\cdot)\|_\infty}.$$

Then, we can conclude that

$$\sup \tau\in[-r,0]\|\mathcal{Z}_\lambda(t,s)(\tau) - \mathcal{Z}_\mu(t,s)(\tau)\| \to 0 \quad \text{as } \lambda,\mu\to+\infty$$

uniformly on $\{(t,s) : 0\leq s\leq t\leq T\}$.

Let f_n in $C([0,T],E)$ be a subsequence converging to f in $L^1(0,T;E)$. We have

$$\|\int_0^t U_L(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))[f_n(\sigma)-f(\sigma)]d\sigma\| \le M(T,s)\|f_n-f\|_{L^1}.$$

Hence, this provides the existence of the limit for $f\in L^1_{\rm loc}(\mathbb{R}_+,E)$, and this completes the proof. □

Theorem 4.3. *Let $\varphi\in\mathcal{C}_r$ and $s\ge 0$, then the function x defined by*

$$x(t):=\begin{cases} u(t)(0), & t\ge s,\\ \varphi(t-s), & s-r\le t\le s,\end{cases}$$

with

$$u(t):=U_L(t,s)\varphi+\lim_{\lambda\to+\infty}\int_s^t U_L(t,\sigma)e^{\lambda\cdot}\lambda R(\lambda,A(0))f(\sigma)\,d\sigma,\quad t\ge s,$$

is a mild solution of (4.1)*. Conversely, if x is a mild solution of* (4.1)*, then*

$$x_t=U_L(t,s)\varphi+\lim_{\lambda\to+\infty}\int_s^t U_L(t,\sigma)e^{\lambda\cdot}\lambda R(\lambda,A(0))f(\sigma)\,d\sigma,\quad t\ge s. \tag{4.4}$$

Proof. Let $\tau\in[-r,0]$, then for $t+\tau\ge s$, we have

$$\begin{aligned}
&u(t)(\tau)\\
&=V(t+\tau,s)\varphi(0)+\int_s^{t+\tau}V(t+\tau,\sigma)L(\sigma)U_L(\sigma,s)\varphi\,d\sigma\\
&\quad+\lim_{\lambda\to+\infty}\int_s^{t+\tau}V(t+\tau,\sigma)\lambda R(\lambda,A(0))f(\sigma)\,d\sigma\\
&\quad+\lim_{\lambda\to+\infty}\int_{t+\tau}^{t}\lambda e^{\lambda(t+\tau-\sigma)}R(\lambda,A(0))f(\sigma)\,d\sigma\\
&\quad+\lim_{\lambda\to+\infty}\int_s^{t+\tau}\int_\sigma^{t+\tau}V(t+\tau,\delta)L(\delta)U_L(\delta,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)\,d\delta d\sigma\\
&=V(t+\tau,s)\varphi(0)+\int_s^{t+\tau}V(t+\tau,\sigma)L(\sigma)U_L(\sigma,s)\varphi\,d\sigma+\int_s^{t+\tau}V(t+\tau,\sigma)f(\sigma)\,d\sigma\\
&\quad+\lim_{\lambda\to+\infty}\int_s^{t+\tau}V(t+\tau,\delta)L(\delta)\int_s^\delta U_L(\delta,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)\,d\sigma d\delta\\
&=V(t+\tau,s)\varphi(0)+\int_s^{t+\tau}V(t+\tau,\sigma)L(\sigma)U_L(\sigma,s)\varphi\,d\sigma+\int_s^{t+\tau}V(t+\tau,\sigma)f(\sigma)\,d\sigma\\
&\quad+\int_s^{t+\tau}V(t+\tau,\delta)L(\delta)\Big[\lim_{\lambda\to+\infty}\int_s^\delta U_L(\delta,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)\,d\sigma\Big]d\delta\\
&=V(t+\tau,s)\varphi(0)+\int_s^{t+\tau}V(t+\tau,\sigma)f(\sigma)\,d\sigma\\
&\quad+\lim_{\lambda\to+\infty}\int_s^{t+\tau}V(t+\tau,\delta)L(\delta)[U_L(\delta,s)\varphi\\
&\quad+\lim_{\lambda\to+\infty}\int_s^\delta U_L(\delta,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)\,d\sigma]d\delta.
\end{aligned}$$

Then,

$$u(t)(\tau) = V(t+\tau, s)\varphi(0) + \int_s^{t+\tau} V(t+\tau,\delta)L(\delta)\left[u(\delta)+f(\delta)\right]\, d\delta. \tag{4.5}$$

For $t+\tau \le s$, we obtain

$$u(t)(\tau) = \varphi(t+\tau-s) + \lim_{\lambda\to+\infty}\int_s^t e^{\lambda(t+\tau-\sigma)}\lambda R(\lambda, A(0))f(\sigma)\, d\sigma.$$

Therefore,

$$u(t)(\tau) = \varphi(t+\tau-s). \tag{4.6}$$

This implies that the function x satisfies $x_t = u(t), \quad t \ge 0$, and (4.2). Thus x is a mild solution of (4.1).

Conversely, let $x(t)$ be a mild solution of (4.1), we have to show that $x_t = u(t)$ for all $t \ge s$. Let $t \ge s$ and $\tau \in [-r, 0]$, from (4.2), (4.5) and (4.6), we obtain

$$x_t(\tau) - u(t)(\tau) = \begin{cases} \int_s^{t+\tau} V(t+\tau,\sigma)L(\sigma)(x_\sigma - u(\sigma))d\sigma, & t+\tau \ge s, \\ 0, & s-r \le t+\tau \le s, \end{cases}$$

and by the Gronwall's inequality, we conclude that $x_t = u(t)$. $\square$

The variation of constants formula (4.4) will play a crucial role in the study of the asymptotic behaviour of the solutions to the retarded differential equation (4.1). For this purpose, we begin by the following lemma.

Lemma 4.4. *Assume that* $(U_L(t,s))_{t\ge s\ge 0}$ *has an exponential dichotomy. Let* $f \in BC(\mathbb{R}_+, X)$*, then the limit*

$$\lim_{\lambda\to+\infty}\int_0^\infty \Gamma(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda, A(0))f(\sigma)d\sigma \tag{4.7}$$

exists uniformly for t *in compact intervals of* $\mathbb{R}_+$*.*

Proof. Let $t \in [0,T]$, for some $T > 0$. We have

$$\begin{aligned} &\int_0^{+\infty} \Gamma(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda, A(0))f(\sigma)d\sigma \\ &= \int_0^t \Gamma(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda, A(0))f(\sigma)d\sigma + \int_t^{+\infty} \Gamma(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda, A(0))f(\sigma)d\sigma. \end{aligned}$$

By the definition of the Green's function, the two members of the above sum are well defined. by Lemma 4.2, the limit of the first member exists. Hence, it remains to show that the second one

$$\mathcal{E}_\lambda(t) := \int_t^{+\infty} \Gamma(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda, A(0))f(\sigma)d\sigma$$

is a Cauchy sequence.

First, one can verify for all $t \ge s \ge 0$

$$\mathcal{E}_\lambda(t) = U_L(t,s)\omega_\lambda(s) + \int_s^t U_L(t,\sigma)Q(\sigma)\lambda e^{\lambda\cdot}R(\lambda, A(0))f(\sigma)d\sigma.$$

Then, for $r > 0$ we have

$$\begin{aligned} Q(t+r)\mathcal{E}_\lambda(t+r) &= U(t+r,t)Q(t)\mathcal{E}_\lambda(t) \\ &\quad + Q(t+r)\int_t^{t+r} U_L(t+r,s)Q(\sigma)\lambda e^{\lambda\cdot}R(\lambda, A(0))f(\sigma)d\sigma, \end{aligned}$$

and as $\mathcal{E}_\lambda(t) \in \operatorname{Im} Q(t)$, we obtain

$$\begin{aligned}\mathcal{E}_\lambda(t) &= [U_Q(t+r,t)]^{-1}Q(t+r)\mathcal{E}_\lambda(t+r)\\ &\quad - Q(t+r)\int_t^{t+r} U_L(t+r,s)Q(\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma.\end{aligned}$$

Now, for $\lambda,\mu \geq \lambda_0 > \max(\omega,0)$

$$\begin{aligned}\mathcal{E}_\lambda(t)-\mathcal{E}_\mu(t) &= [U_Q(t+r,t)]^{-1}Q(t+r)[\mathcal{E}_\lambda(t+r)-\mathcal{E}_\mu(t+r)]\\ &\quad - [U_Q(t+r,t)]^{-1}Q(t+r)\int_t^{t+r} U_L(t+r,\sigma)[\lambda e^{\lambda\cdot}R(\lambda,A(0))\\ &\quad - \mu e^{\mu\cdot}R(\mu,A(0))]f(\sigma)d\sigma.\end{aligned}$$

Passing to the norm, it follows

$$\begin{aligned}\|\mathcal{E}_\lambda(t)-\mathcal{E}_\mu(t)\| &\leq \|[U_Q(t+r,t)]^{-1}Q(t+r)[\mathcal{E}_\lambda(t+r)-\mathcal{E}_\mu(t+r)]\|\\ &\quad + \Big\|[U_Q(t+r,t)]^{-1}Q(t+r)\int_t^{t+r} U_L(t+r,\sigma)[\lambda e^{\lambda\cdot}R(\lambda,A(0))\\ &\quad - \mu e^{\mu\cdot}R(\mu,A(0))]f(\sigma)d\sigma\Big\|\\ &\leq C_1e^{-\alpha r}\Big[C_2\|f\| + \|Q(t+r)\int_t^{t+r} U_L(t+r,\sigma)[\lambda e^{\lambda\cdot}R(\lambda,A(0))\\ &\quad - \mu e^{\mu\cdot}R(\mu,A(0))]f(\sigma)d\sigma\|\Big].\end{aligned}$$

For r sufficiently large and in view of Lemma 4.2, we can conclude that $\mathcal{E}_\lambda$ is a Cauchy sequence. □

Proposition 4.5. *Assume that $(U_L(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy, then for $f \in BC(\mathbb{R}_+,E)$, the function $v:[0,\infty)\to E$ defined by*

$$v(t) := \begin{cases}[U_L(t,0)P(0)\varphi\\ +\lim_{\lambda\to+\infty}\int_0^\infty \Gamma(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma](0), & t\geq 0\\ \varphi(t), & -r\leq t\leq 0\end{cases} \tag{4.8}$$

is the unique bounded mild solution of (4.1) *with φ is an initial condition which satisfies*

$$Q(0)\varphi = \lim_{\lambda\to+\infty}\int_0^\infty \Gamma(0,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma. \tag{4.9}$$

Moreover, if $f\in C_0(\mathbb{R}_+,E)$ then v also belongs to $C_0(\mathbb{R}_+,E)$.

Proof. Let v be a mild solution of (4.1). From Theorem 4.3, one can verify that

$$\begin{aligned}v(t) &= U_L(t,0)(\varphi + \lim_{\lambda\to+\infty}\int_0^\infty \Gamma(0,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma)\\ &\quad + \lim_{\lambda\to+\infty}\int_0^\infty \Gamma(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma.\end{aligned}$$

As $t\mapsto \lim_{\lambda\to+\infty}\int_0^\infty \Gamma(t,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma$ is a bounded function, then v is bounded if and only if

$$t\mapsto U_L(t,0)(\varphi + \lim_{\lambda\to+\infty}\int_0^\infty \Gamma(0,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma)$$

is bounded, and this is equivalent to $[\varphi+\lim_{\lambda\to+\infty}\int_0^\infty \Gamma(0,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma]$ belongs to $\operatorname{Im}P(0)$, i.e.,

$$Q(0)\varphi = -\lim_{\lambda\to+\infty}\int_0^\infty \Gamma(0,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma.$$

Consequently, v is given by (4.8).

Now, let us take $f\in C_0(\mathbb{R}_+,E)$ and show that

$$C_0(\mathbb{R}_+,E)\ni\omega(\cdot):=\lim_{\lambda\to+\infty}\int_0^\infty \Gamma(\cdot,\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma.$$

Let

$$\begin{aligned} I(t) &:= \lim_{\lambda\to+\infty}\int_0^t U_L(t,\sigma)P(\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma,\\ J(t) &:= \lim_{\lambda\to+\infty}\int_t^\infty U_Q(t,\sigma)Q(\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(\sigma)d\sigma. \end{aligned}$$

Then, $\omega(t)=I(t)-J(t)$.

Since $f(t)\to 0$ as $t\to\infty$, there exists $t_0\geq 0$ such that $\|f(t)\|\leq\frac{\varepsilon}{2MN\delta}$ for all $t\geq t_0$. Hence, $\|I\|\leq\frac{2MN}{\delta}\|f\|_\infty e^{-\delta(t-t_0)}+\frac{\varepsilon}{2}$ and this implies that $I(t)\to 0$ as $t\to+\infty$.

For $J(\cdot)$, we have

$$J(t)=\lim_{\lambda\to+\infty}\int_0^\infty U_Q(t,t+\sigma)Q(t+\sigma)\lambda e^{\lambda\cdot}R(\lambda,A(0))f(t+\sigma)d\sigma.$$

Then,

$$\|J(t)\|\leq MN\int_0^\infty e^{-\delta\sigma}\|f(t+\sigma)\|d\sigma.$$

Therefore, letting t approach infinity we obtain the result. □

5. Fundamental Solutions and Stability Results

This section is devoted to use the notion of fundamental solutions to study the stability of the semi-linear retarded equation

$$\begin{gathered} x'(t)=A(t)x(t)+L(t)x_t+F(t,x_t),\quad t\geq s\geq 0,\\ x_s=\varphi\in\mathcal{C}_r, \end{gathered}\tag{5.1}$$

where $(A(t),D(A(t)))_{t\geq 0}$ and $(L(t))_{t\geq 0}$ are defined as above and F a nonlinear function from $\mathbb{R}_+\times\mathcal{C}_r$ to E.

In [10], the authors showed the existence of fundamental solutions of (3.1), when $A(t)=A$. But to study the stability of the non-autonomous retarded equation (5.1) they assumed the existence of these fundamental solutions. Here, using our previous variation of constants formulas, we are able to show this assumed result, and then obtain the same stability results for (5.1) as in [10, Theorems 4.6, 4.7, 4.10].

Definition 5.1. A family $(\Phi(s,t))_{t\geq s\geq 0}$ of bounded linear operators on E is called a fundamental solution of (3.1) if

(i) $\Phi(t,t)=Id,\quad t\geq 0$.
(ii) $(t,s)\mapsto\Phi(t,s)$ is strongly continuous for $t\geq s\geq 0$.

(iii) For every $g \in L^1_{\mathrm{loc}}(\mathbb{R}_+, E)$ and $t \in [s-r, \infty)$, the map

$$t \mapsto \begin{cases} \int_s^t \Phi(t,\sigma)g(\sigma)d\sigma, & t \geq s \geq 0, \\ \varphi(t-s), & s-r \leq t \leq s, \end{cases}$$

is the unique mild solution of

$$\begin{gathered} v'(t) = A(t)x(t) + L(t)x_t + g(t), \quad t \geq s \geq 0, \\ v_s = \varphi \in \mathcal{C}_r. \end{gathered}$$

Let us define on $\mathcal{C}_r$ the function

$$[R_{\lambda,x}(t,s)](\tau) = \begin{cases} \lambda R(\lambda, A(0))x, & t+\tau \geq s, \\ \lambda e^{\lambda(t+\tau-s)} R(\lambda, A(0))x, & t+\tau < s, \end{cases} \tag{5.2}$$

with $x \in E$, $\tau \in [-r, 0]$ and $\lambda > \max(\omega, 0)$.

Theorem 5.2. *Suppose that, for all $x \in E$, the limit*

$$\lim_{\lambda \to +\infty} \int_s^t V(t,\sigma)L(\sigma)R_{\lambda,x}(\sigma,s)d\sigma \tag{5.3}$$

exists uniformly on compact sets of $\{(t,s) : t \geq s \geq 0\}$. Then, there exists a fundamental solution $(\Phi(t,s))_{t \geq s \geq 0}$ of (3.1) given by

$$\Phi(t,s)x = \lim_{\lambda \to +\infty} \lambda [U_L(t,s)e^{\lambda \cdot} R(\lambda, A(0))x](0) \tag{5.4}$$

for $t \geq s \geq 0$ and $x \in E$.

This theorem follows fom the variation of constants formula in Theorem 4.3, and the proof is similar to the one given in [10, Theorem 3.2].

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Said Boulite

Department of Mathematics, University Cadi Ayyad, Faculty of Sciences Semlalia, B.P. 2390, 40000 Marrakesh, Morocco

E-mail address: `sboulite@ucam.ac.ma`

Lahcen Maniar

Department of Mathematics, University Cadi Ayyad, Faculty of Sciences Semlalia, B.P. 2390, 40000 Marrakesh, Morocco

E-mail address: `maniar@ucam.ac.ma`

Mohammed Moussi

Department of Mathematics, Faculty of Sciences, University Mohammed I, 60000 Oujda, Morocco

E-mail address: `moussi@sciences.univ-oujda.ac.ma`