Electronic Journal of Differential Equations, Vol. 2003(2003), No. 63, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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WEAK SOLUTIONS FOR A VISCOUS p-LAPLACIAN EQUATION

CHANGCHUN LIU

ABSTRACT. In this paper, we consider the pseudo-parabolic equation $u_t - k\Delta u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. By using the time-discrete method, we establish the existence of weak solutions, and also discuss the uniqueness.

1. Introduction

This paper concerns the study of the viscous p-Laplacian equation

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad x \in \Omega,\ p > 2, \tag{1.1}$$

with boundary condition

$$u\big|_{\partial\Omega} = 0, \tag{1.2}$$

and initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ and $k > 0$ is the viscosity coefficient. The term $k\frac{\partial\Delta u}{\partial t}$ in (1.1) is interpreted as due to viscous relaxation effects, or viscosity; hence, the equation (1.1) is called "viscous p-Laplacian equations". The well-known p-Laplacian equation is obtained by setting $k = 0$.

Equation (1.1) arises as a regularization of the pseudo-parabolic equation

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \Delta u, \tag{1.4}$$

which arises in various physical phenomena. (1.4) can be assumed as a model for diffusion of fluids in fractured porous media [1, 5, 4], or as a model for heat conduction involving a thermodynamic temperature $\theta = u - k\Delta u$ and a conductive temperature u [10, 3]. Equation (1.4) has been extensively studied, and there are many outstanding results concerning existence, uniqueness, regularity, and special properties of solutions, see for example [4, 5, 6, 7, 8, 9, 11].

To derive (1.4), B. D. Coleman, R. J. Duffin and V. J. Mizel considered a special kinematical situation, of nonsteady simple shearing flow [4]. In fact, when the influence of many factors, such as the molecular and ion effects, are considered, one has the nonlinear relation $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ in stead of Δu in right-hand side of (1.4). Hence, we obtain (1.1).

2000 *Mathematics Subject Classification.* 35G25, 35Q99, 35K55, 35K70.
Key words and phrases. Pseudo-parabolic equations, existence, uniqueness.

Submitted August 5, 2002. Published June 10, 2003.

Equation (1.1) is something like the p-Laplacian equation, but many methods which are useful for the p-Laplacian equation are no longer valid for this equation. Because of the degeneracy, problem (1.1)-(1.3) does not admit classical solutions in general. So, we study weak solutions in the sense of following

Definition A function u is said to be a weak solution of (1.1)-(1.3), if the following conditions are satisfied:

(1) $u \in L^\infty(0,T;W_0^{1,p}(\Omega)) \cap C(0,T;H^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,T;W^{-1,p'}(\Omega))$, where p' is conjugate exponent of p.

(2) For $\varphi \in C_0^\infty(Q_T)$ and $Q_T = \Omega \times (0,T)$,

$$\iint_{Q_T} u\frac{\partial\varphi}{\partial t}dx\,dt + k\iint_{Q_T} \nabla u\frac{\partial\nabla\varphi}{\partial t}dx\,dt - \iint_{Q_T} |\nabla u|^{p-2}\nabla u\nabla\varphi dx\,dt = 0\,.$$

(3) $u(x,0) = u_0(x)$.

In this paper, we discuss first the existence of weak solutions. Most proofs of existence for (1.4) are based on the Yoshida approximations [6], but these methods do not apply to (1.1). Our method for proving the existence of weak solutions is based on a time discrete method that constructs approximate solutions. Later on, we discuss the uniqueness of a solution. For simplicity we set $k = 1$ in this paper.

2. Existence of weak solutions

Theorem 2.1. *If $u_0 \in W_0^{1,p}(\Omega)$ with $p > 2$, then problem (1.1)-(1.3) has at least one solution.*

We use the a discrete method for constructing an approximate solution. First, divide the interval $(0,T)$ in N equal segments and set $h = \frac{T}{N}$. Then consider the problem

$$\frac{1}{h}(u_{k+1} - u_k) - \frac{1}{h}(\Delta u_{k+1} - \Delta u_k) = \operatorname{div}(|\nabla u_{k+1}|^{p-2}\nabla u_{k+1}), \tag{2.1}$$

$$u_{k+1}|_{\partial\Omega} = 0, \quad k = 0,1,\dots,N-1, \tag{2.2}$$

where u_0 is the initial value.

Lemma 2.2. *For a fixed k, if $u_k \in H_0^1(\Omega)$, problem (2.1)-(2.2) admits a weak solution $u_{k+1} \in W_0^{1,p}(\Omega)$, such that for any $\varphi \in C_0^\infty(\Omega)$, have*

$$\frac{1}{h}\int_\Omega (u_{k+1} - u_k)\varphi dx + \frac{1}{h}\int_\Omega (\nabla u_{k+1} - \nabla u_k)\nabla\varphi dx + \int_\Omega |\nabla u_{k+1}|^{p-2}\nabla u_{k+1}\nabla\varphi dx = 0. \tag{2.3}$$

Proof. On the space $W_0^{1,p}(\Omega)$, we consider the functionals

$$\Phi_1[u] = \frac{1}{p}\int_\Omega |\nabla u|^p dx,$$

$$\Phi_2[u] = \frac{1}{2}\int_\Omega |u|^2 dx,$$

$$\Phi_3[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 dx,$$

$$\Psi[u] = \Phi_1[u] + \frac{1}{h}\Phi_2[u] + \frac{1}{h}\Phi_3[u] - \int_\Omega f u dx,$$

where $f \in H^{-1}(\Omega)$ is a known function. Using Young's inequality, there exist constants $C_1, C_2 > 0$, such that

$$\begin{aligned}\Psi[u] &= \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2h}\int_\Omega |u|^2 dx + \frac{1}{2h}\int_\Omega |\nabla u|^2 dx - \int_\Omega fu\,dx \\ &\geq C_1 \int_\Omega |\nabla u|^p dx - C_2\|f\|_{-1};\end{aligned}$$

hence $\Psi[u] \to \infty$, as $\|u\|_{1,p} \to +\infty$. Here $\|u\|_{1,p}$ denotes the norm of u in $W_0^{1,p}(\Omega)$.

Since the norm is lower semi-continuous and $\int_\Omega fudx$ is a continuous functional, $\Psi[u]$ is weakly lower semi-continuous on $W_0^{1,p}(\Omega)$ and satisfying the coercive condition. From [2] we conclude that there exists $u_* \in W_0^{1,p}(\Omega)$, such that

$$\Psi[u_*] = \inf \Psi[u],$$

and u_* is the weak solutions of the Euler equation corresponding to $\Psi[u]$,

$$\frac{1}{h}u - \frac{1}{h}\Delta u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f.$$

Taking $f = (u_k - \Delta u_k)/h$, we obtain a weak solutions u_{k+1} of (2.1)–(2.2). The proof is complete. □

Now, we need to establish a priori estimates, for the weak solutions u_{k+1} of (2.1)–(2.2). First, we define the weak solutions of (1.1)–(1.3) as follows:

$$\begin{gathered}u^h(x,t) = u_k(x), \quad kh < t \leq (k+1)h,\ k = 0, 1, \dots, N-1, \\ u^h(x,0) = u_0(x).\end{gathered}$$

Lemma 2.3. *The weak solutions u_k of (2.1)–(2.2) satisfy*

$$h\sum_{k=1}^{N}\int_\Omega |\nabla u_k|^p dx \leq C, \tag{2.4}$$

$$\sup_{0<t<T}\int_\Omega |\nabla u^h(x,t)|^p dx \leq C, \tag{2.5}$$

where C is a constant independent of h and k.

Proof. i) We take $\varphi = u_{k+1}$ in the integral equality (2.3) (we can easily prove that for $\varphi \in W_0^{1,p}(\Omega)$, (2.3) also holds).

$$\begin{aligned}\frac{1}{h}\int_\Omega (u_{k+1} - u_k)u_{k+1}dx &+ \frac{1}{h}\int_\Omega (\nabla u_{k+1} - \nabla u_k)\nabla u_{k+1}dx \\ &+ \int_\Omega |\nabla u_{k+1}|^{p-2}\nabla u_{k+1}\nabla u_{k+1}dx = 0,\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{h}\int_\Omega |u_{k+1}|^2 dx &+ \frac{1}{h}\int_\Omega |\nabla u_{k+1}|^2 dx - \frac{1}{h}\int_\Omega u_k u_{k+1}dx \\ &- \frac{1}{h}\int_\Omega \nabla u_{k+1}\nabla u_k dx + \int_\Omega |\nabla u_{k+1}|^p dx = 0.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{h}\int_\Omega |u_{k+1}|^2dx+\frac{1}{h}\int_\Omega |\nabla u_{k+1}|^2dx+\int_\Omega |\nabla u_{k+1}|^pdx\\&=\frac{1}{h}\int_\Omega u_ku_{k+1}dx+\frac{1}{h}\int_\Omega \nabla u_{k+1}\nabla u_kdx.\end{aligned}$$

By Young's inequality,

$$\begin{aligned}&\frac{1}{h}\int_\Omega |u_{k+1}|^2dx+\frac{1}{h}\int_\Omega |\nabla u_{k+1}|^2dx+\int_\Omega |\nabla u_{k+1}|^pdx\\&\le\frac{1}{2h}\int_\Omega |u_k|^2dx+\frac{1}{2h}\int_\Omega |u_{k+1}|^2dx+\frac{1}{2h}\int_\Omega |\nabla u_k|^2dx+\frac{1}{2h}\int_\Omega |\nabla u_{k+1}|^2dx;\end{aligned}$$

that is,

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u_{k+1}|^2dx+\frac{1}{2}\int_\Omega |\nabla u_{k+1}|^2dx+h\int_\Omega |\nabla u_{k+1}|^pdx\\&\le\frac{1}{2}\int_\Omega |u_k|^2dx+\frac{1}{2}\int_\Omega |\nabla u_k|^2dx.\end{aligned}\tag{2.6}$$

Adding these inequalities for k from 0 to $N-1$, we have

$$h\sum_{k=1}^N\int_\Omega |\nabla u_k|^pdx\le\frac{1}{2}\int_\Omega |u_0|^2dx+\frac{1}{2}\int_\Omega |\nabla u_0|^2dx.$$

Therefore, (2.4) holds.
ii) We take $\varphi=u_{k+1}-u_k$ in the integral equality (2.3) and have

$$\begin{aligned}&\frac{1}{h}\int_\Omega (u_{k+1}-u_k)(u_{k+1}-u_k)dx+\frac{1}{h}\int_\Omega (\nabla u_{k+1}-\nabla u_k)\nabla(u_{k+1}-u_k)dx\\&\quad+\int_\Omega |\nabla u_{k+1}|^{p-2}\nabla u_{k+1}\nabla(u_{k+1}-u_k)dx=0\,.\end{aligned}$$

Since the first term and the second term of the left hand side of the above equality is nonnegative, it follows that

$$\begin{aligned}\int_\Omega |\nabla u_{k+1}|^pdx&\le\int_\Omega |\nabla u_{k+1}|^{p-2}\nabla u_{k+1}\nabla u_k\,dx\\&\le\frac{p-1}{p}\int_\Omega |\nabla u_{k+1}|^pdx+\frac{1}{p}\int_\Omega |\nabla u_k|^pdx;\end{aligned}$$

thus,

$$\int_\Omega |\nabla u_{k+1}|^pdx\le\int_\Omega |\nabla u_k|^pdx.$$

For any m, with $1\le m\le N-1$, adding the above inequality for k from 0 to $m-1$, we have

$$\int_\Omega |\nabla u_m|^pdx\le\int_\Omega |\nabla u_0|^pdx.$$

Therefore, (2.5) holds. □

Lemma 2.4. *For a weak solutions u_{k+1} of (2.1)–(2.2), we have*

$$-Ch\le\int_\Omega |u_{k+1}|^2dx+\int_\Omega |\nabla u_{k+1}|^2dx-\int_\Omega |u_k|^2dx-\int_\Omega |\nabla u_k|^2dx\le 0,\tag{2.7}$$

where C is a constant independently of h.

Proof. The second inequality in (2.7) is an immediate consequence of (2.6). To prove the first inequality, we choose $\varphi = u_k$ in (2.3) and obtain

$$\frac{1}{h}\int_\Omega (u_{k+1}-u_k)u_k dx + \frac{1}{h}\int_\Omega (\nabla u_{k+1}-\nabla u_k)\nabla u_k\, dx + \int_\Omega |\nabla u_{k+1}|^{p-2}\nabla u_{k+1}\nabla u_k\, dx = 0\,.$$

Therefore,

$$\begin{aligned}
&\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^2 dx - \int_\Omega u_{k+1}u_k dx - \int_\Omega \nabla u_{k+1}\nabla u_k dx\\
&= h\int_\Omega |\nabla u_{k+1}|^{p-2}\nabla u_{k+1}\nabla u_k dx\\
&\le h\Big(\int_\Omega |\nabla u_{k+1}|^p dx\Big)^{(p-1)/p}\Big(\int_\Omega |\nabla u_k|^p dx\Big)^{1/p}.
\end{aligned}$$

Here we have used Hölder inequality. By (2.5) again, we obtain

$$\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^2 dx - \int_\Omega u_{k+1}u_k dx - \int_\Omega \nabla u_{k+1}\nabla u_k dx \le Ch.$$

Therefore,

$$\begin{aligned}
&\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^2 dx\\
&\le Ch + \int_\Omega u_{k+1}u_k dx + \int_\Omega \nabla u_{k+1}\nabla u_k dx\\
&\le Ch + \frac{1}{2}\int_\Omega |u_{k+1}|^2 dx + \frac{1}{2}\int_\Omega |u_k|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_{k+1}|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_k|^2 dx.
\end{aligned}$$

i.e.,

$$\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^2 dx - \int_\Omega |u_{k+1}|^2 dx - \int_\Omega |\nabla u_{k+1}|^2 dx \le Ch$$

which completes the proof. □

Lemma 2.5.

$$\sup_{0<t<T}\Big(\int_\Omega |u^h|^2 dx + \int_\Omega |\nabla u^h|^2 dx\Big) \le \int_\Omega |u_0|^2 dx + \int_\Omega |\nabla u_0|^2 dx. \tag{2.8}$$

The proof follows by adding (2.4), for m with $1 \le m \le N-1$, for k from 0 to $m-1$.

Proof of Theorem 2.1. First, we define the operator A^t, $A^t(\nabla u^h) = |\nabla u_k|^{p-2}\nabla u_k$, $\Delta^h u^h = u_{k+1} - u_k$, where $kh < t \le (k+1)h, k = 0, 1, \dots, N-1$. By the dispersion equation (2.1) and the (2.4) in Lemma 2.2, we know that

$$\frac{1}{h}(u_{k+1}-u_k) \quad \text{in } L^\infty(0,T;W^{-1,p'}(\Omega)) \quad \text{is bounded.} \tag{2.9}$$

By (2.5), (2.7), (2.9) and (2.4) we known that exists a subsequence of $\{u^h\}$ (which we denote as the original sequence) such that

$$\begin{gathered}
u^h \to u \quad \text{in } L^\infty(0,T;W^{1,p}(\Omega)) \quad \text{weak-}\star, \\
\nabla u^h \to \nabla u \quad \text{in } L^\infty(0,T;L^2(\Omega)) \quad \text{weak-}\star, \\
\frac{1}{h}(u_{k+1}-u_k) \to \frac{\partial u}{\partial t} \quad \text{in } L^\infty(0,T;W^{-1,p'}(\Omega) \quad \text{weak-}\star, \\
|\nabla u^h|^{p-2}\nabla u^h \to w \quad \text{in } L^\infty(0,T;L^{p'}(\Omega)) \quad \text{weak-}\star,
\end{gathered}$$

where p' is conjugate exponent of p. From (2.3), we known, for any $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} (\frac{1}{h}\Delta^h u^h \varphi + \frac{1}{h}\Delta^h \nabla u^h \nabla\varphi + |\nabla u^h|^{p-2}\nabla u^h \nabla\varphi) dx\,dt = 0,$$

i.e.,

$$\iint_{Q_T} (\frac{1}{h}\Delta^h u^h \varphi - \frac{1}{h}\Delta^h u^h \Delta\varphi + |\nabla u^h|^{p-2}\nabla u^h \nabla\varphi) dx\,dt = 0.$$

Letting $h \to 0$, we obtain, in the sense of distributions,

$$\frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t} - \operatorname{div}(w) = 0. \tag{2.10}$$

Next, we prove that $w = |\nabla u|^{p-2}\nabla u$ a.e. in Q_T. Define

$$\begin{aligned}
f_h(t) = \frac{t-kh}{2h}\Big(&\int_\Omega |u_{k+1}|^2 dx + \int_\Omega |\nabla u_{k+1}|^2 dx - \int_\Omega |u_k|^2 dx - \int_\Omega |\nabla u_k|^2 dx\Big) \\
&+ \frac{1}{2}\int_\Omega |u_k|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_k|^2 dx,
\end{aligned}$$

where $kh < t \le (k+1)h$. by (2.7) we have

$$\begin{gathered}
\frac{1}{2}\int_\Omega |\nabla u_k|^2 dx + \frac{1}{2}\int_\Omega |u_k|^2 dx - Ch \le f_h(t) \le \frac{1}{2}\int_\Omega |u_k|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_k|^2 dx, \\
-C \le f_h'(t) \le 0.
\end{gathered}$$

By Ascoli–Arzela theorem, there exists a function $f(t) \in C([0,T])$, such that

$$\lim_{h\to 0} f_h(t) = f(t) \quad \text{for } t \in [0,T] \text{ uniformly}.$$

Using (2.7), we have

$$\lim_{h\to 0}\Big(\frac{1}{2}\int_\Omega |\nabla u^h|^2 dx + \frac{1}{2}\int_\Omega |u^h|^2 dx\Big) = f(t) \quad \text{for } t \in [0,T] \text{ uniformly}. \tag{2.11}$$

By (2.6) again, we obtain

$$\frac{1}{2}\int_\Omega |u_N|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_N|^2 dx + \iint_{Q_T} |\nabla u^h|^p dx\,dt \le \frac{1}{2}\int_\Omega |u_0|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx.$$

In the above inequality letting $h \to 0$, and using (2.10) we have

$$
\begin{aligned}
\lim_{h\to 0}\iint_{Q_T} |\nabla u^h|^p dx\,dt &\le f(0) - f(T)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{T-\varepsilon}(f(t)-f(t+\varepsilon))dt\\
&= \lim_{\varepsilon\to 0}\lim_{h\to 0}\Big[\frac{1}{2\varepsilon}\int_0^{T-\varepsilon}\int_\Omega(|u^h(x,t)|^2-|u^h(x,t+\varepsilon)|^2)dx\,dt\\
&\quad+\frac{1}{2\varepsilon}\int_0^{T-\varepsilon}\int_\Omega(|\nabla u^h(x,t)|^2-|\nabla u^h(x,t+\varepsilon)|^2)dx\,dt\Big].
\end{aligned}
$$

Since $\Phi_2[u] = \frac{1}{2}\int_\Omega |u|^2 dx$ and $\Phi_3[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$ are convex functionals, and

$$
\frac{\delta\Phi_2[u]}{\delta u} = u, \quad \frac{\delta\Phi_3[u]}{\delta u} = -\Delta u,
$$

we have

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |u^h(x,t)|^2 dx - \frac{1}{2}\int_\Omega |u^h(x,t+\varepsilon)|^2 dx\\
&+\frac{1}{2}\int_\Omega |\nabla u^h(x,t)|^2 dx - \frac{1}{2}\int_\Omega |\nabla u^h(x,t+\varepsilon)|^2 dx\\
&\le \int_\Omega u^h(x,t)(u^h(x,t)-u^h(x,t+\varepsilon))dx\\
&\quad+\int_\Omega \nabla u^h(x,t)(\nabla u^h(x,t)-\nabla u^h(x,t+\varepsilon))dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lim_{h\to 0}\frac{1}{2\varepsilon}\Big[\int_0^{T-\varepsilon}\int_\Omega |u^h(x,t)|^2-|u^h(x,t+\varepsilon)|^2)dx\,dt\\
&+\int_0^{T-\varepsilon}\int_\Omega(|\nabla u^h(x,t)|^2-|\nabla u^h(x,t+\varepsilon)|^2)dx\,dt\Big]\\
&\le \frac{1}{\varepsilon}\int_0^{T-\varepsilon}\int_\Omega(u(x,t)-u(x,t+\varepsilon))u\,dx\,dt\\
&\quad+\frac{1}{\varepsilon}\int_0^{T-\varepsilon}\int_\Omega(\nabla u(x,t)-\nabla u(x,t+\varepsilon))\nabla u\,dx\,dt\,.
\end{aligned}
$$

Hence, we obtain

$$
\lim_{h\to 0}\iint_{Q_T}|\nabla u^h|^p dx\,dt \le -\int_0^T\langle\frac{\partial u}{\partial t},u\rangle dt + \int_0^T\langle\frac{\partial u}{\partial t},\Delta u\rangle dt,
$$

where $\langle,\rangle$ denotes the inner product. Form (2.10), we obtain

$$
\lim_{h\to 0}\iint_{Q_T}|\nabla u^h|^p dx\,dt \le \iint_{Q_T} w\nabla u dx\,dt\,. \tag{2.12}
$$

Again by $\frac{\delta\Phi_1[u]}{\delta u} = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and the convexity of $\Phi_1[u]$, for any $g \in L^\infty(0,T;W_0^{1,p}(\Omega))$ we have

$$\begin{aligned}
&-\frac{1}{p}\iint_{Q_T}|\nabla g|^p dx\,dt + \frac{1}{p}\iint_{Q_T}|\nabla u^h|^p dx\,dt\\
&\le \iint_{Q_T} -\operatorname{div}(|\nabla u^h|^{p-2}\nabla u^h)(u^h-g)dx\,dt,
\end{aligned}$$

that is

$$\begin{aligned}
\frac{1}{p}\iint_{Q_T}|\nabla g|^p dx\,dt - \frac{1}{p}\iint_{Q_T}|\nabla u^h|^p dx\,dt &\ge \iint_{Q_T}\operatorname{div}(|\nabla u^h|^{p-2}\nabla u^h)(u^h-g)dx\,dt\\
&= \iint_{Q_T}(|\nabla u^h|^{p-2}\nabla u^h)\nabla(g-u^h)dx\,dt.
\end{aligned}$$

By (2.11) and $F(u)$ is weakly lower semicontinuous, in above equality letting $h\to 0$, we obtain

$$\frac{1}{p}\iint_{Q_T}|\nabla g|^p dx\,dt - \frac{1}{p}\iint_{Q_T}|\nabla u|^p dx\,dt \le \iint_{Q_T} w\nabla(g-u)dx\,dt. \tag{2.13}$$

In (2.13), we take $g=\varepsilon g+u$ to obtain

$$\frac{1}{\varepsilon}\Big[\frac{1}{p}\iint_{Q_T}|\nabla(\varepsilon g+u)|^p dx\,dt - \frac{1}{p}\iint_{Q_T}|\nabla u|^p dx\,dt\Big] \ge \iint_{Q_T} w\nabla g\,dx\,dt.$$

Letting $\varepsilon\to 0$,

$$\iint_{Q_T}\frac{\delta\Phi_1[u]}{\delta u} g\,dx\,dt = \iint_{Q_T}|\nabla u|^{p-2}\nabla u\nabla g\,dx\,dt \ge \iint_{Q_T} w\nabla g\,dx\,dt\,.$$

Since g is arbitrary, taking $g=-g$, we get the opposite inequality above; hence

$$w = |\nabla u|^{p-2}\nabla u.$$

The strong convergence of u^h in $C(0,T;H^1(\Omega))$ and the fact that $u^h(x,0)=u_0(x)$ completes the proof. □

3. Uniqueness of solutions

In this section, we prove that the weak solution is unique. To this end we need the following lemma.

Lemma 3.1. *For $\varphi \in L^\infty(t_1,t_2;W_0^{1,p}(\Omega))$ with $\varphi_t \in L^2(t_1,t_2;H^1(\Omega))$, the weak solutions u of the problem (1.1)-(1.3) on Q_T satisfies*

$$\begin{aligned}
&\int_\Omega u(x,t_1)\varphi(x,t_1)dx + \int_\Omega \nabla u(x,t_1)\nabla\varphi(x,t_1)dx\\
&+\int_{t_1}^{t_2}\int_\Omega\Big(u\frac{\partial\varphi}{\partial t} + \nabla u\frac{\partial\nabla\varphi}{\partial t} - |\nabla u|^{p-2}\nabla u\nabla\varphi\Big)dx\,dt\\
&= \int_\Omega u(x,t_2)\varphi(x,t_2)dx + \int_\Omega \nabla u(x,t_2)\nabla\varphi(x,t_2)dx.
\end{aligned}$$

In particular, for $\varphi \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}\int_\Omega (u(x,t_1)-u(x,t_2))\varphi dx + \int_\Omega \nabla(u(x,t_1)-u(x,t_2))\nabla\varphi dx \\ - \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi dx\, dt = 0\,. \end{aligned} \tag{3.1}$$

Proof. From $\varphi \in L^\infty(t_1,t_2;W_0^{1,p}(\Omega))$ and $\varphi_t \in L^2(t_1,t_2;H^1(\Omega))$, it follows that there exists a sequence of functions $\{\varphi_k\}$, for fixed $t \in (t_1,t_2), \varphi_k(\cdot,t) \in C_0^\infty(\Omega)$, and as $k \to \infty$

$$\|\varphi_{kt}-\varphi_t\|_{L^2(t_1,t_2;H^1(\Omega))} \to 0, \quad \|\varphi_k-\varphi\|_{L^\infty(t_1,t_2;W_0^{1,p}(\Omega))} \to 0.$$

Choose a function $j(s) \in C_0^\infty(R)$ such that $j(s) \geq 0$, for $s \in R$; $j(s)=0$, for $\forall |s|>1$; $\int_R j(s)ds = 1$. For $h>0$, define $j_h(s) = \frac{1}{h}j(\frac{s}{h})$ and

$$\eta_h(t) = \int_{t-t_2+2h}^{t-t_1-2h} j_h(s)ds.$$

Clearly $\eta_h(t) \in C_0^\infty(t_1,t_2)$, $\lim_{h\to 0^+}\eta_h(t)=1$, for all $t \in (t_1,t_2)$. In the definition of weak solutions, choose $\varphi = \varphi_k(x,t)\eta_h(t)$. We have

$$\begin{aligned}&\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_1-2h)dx\,dt - \int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_2+2h)dx\,dt \\ &+ \int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_k j_h(t-t_1-2h)dx\,dt - \int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_k j_h(t-t_2+2h)\,dx\,dt \\ &+ \int_{t_1}^{t_2}\int_\Omega u\varphi_{kt}\eta_h dx\,dt + \int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_{kt}\eta_h\,dx\,dt \\ &- \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi_k\eta_h\,dx\,dt = 0.\end{aligned}$$

Observe that

$$\begin{aligned}&|\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_1-2h)dx\,dt - \int_\Omega (u\varphi_k)|_{t=t_1}dx\big| \\ &= |\int_{t_1+h}^{t_1+3h}\int_\Omega u\varphi_k j_h(t-t_1-2h)dx\,dt - \int_{t_1+h}^{t_1+3h}\int_\Omega (u\varphi_k)|_{t=t_2} j_h(t-t_1-2h)dx\,dt\big| \\ &\leq \sup_{t_1+h<t<t_1+3h}\int_\Omega \big|(u\varphi_k)|_t - (u\varphi_k)|_{t_1}\big|dx,\end{aligned}$$

and $u \in C(0,T;L^2(\Omega))$. We see that the right hand side tends to zero as $h \to 0$. Similarly,

$$\Big|\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_2+2h)dx\,dt - \int_\Omega (u\varphi_k)|_{t=t_2}dx\Big| \to 0, \quad \text{as } h \to 0,$$

$$\Big|\int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_k j_h(t-t_1-2h)dx\,dt - \int_\Omega (\nabla u\nabla\varphi_k)|_{t=t_1}dx\Big| \to 0, \quad \text{as } h \to 0,$$

$$\Big|\int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_k j_h(t-t_2+2h)dx\,dt - \int_\Omega (\nabla u\nabla\varphi_k)|_{t=t_2}dx\Big| \to 0, \quad \text{as } h \to 0.$$

Letting $h \to 0$ and $k \to \infty$, we obtain

$$\begin{aligned}
&\int_\Omega u(x,t_1)\varphi(x,t_1)dx + \int_\Omega \nabla u(x,t_1)\nabla\varphi(x,t_1)dx \\
&+ \int_{t_1}^{t_2}\int_\Omega \Big(u\frac{\partial\varphi}{\partial t} + \nabla u\frac{\partial\nabla\varphi}{\partial t} - |\nabla u|^{p-2}\nabla u\nabla\varphi\Big)dx\,dt \\
&= \int_\Omega u(x,t_2)\varphi(x,t_2)dx + \int_\Omega \nabla u(x,t_2)\nabla\varphi(x,t_2)dx.
\end{aligned}$$

In particular for $\varphi \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega (u(x,t_1) - u(x,t_2))\varphi dx + \int_\Omega (\nabla u(x,t_1) - \nabla u(x,t_2))\nabla\varphi dx \\
- \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx\,dt = 0
\end{aligned}$$

which completes the proof. □

For a fixed $\tau \in (0,T)$, set h satisfying $0 < \tau < \tau + h < T$. Letting $t_1 = \tau$, $t_2 = \tau + h$, then multiply (3.1) by $\frac{1}{h}$, for $\varphi \in W_0^{1,p}(\Omega)$, we obtain

$$\int_\Omega (u_h(x,\tau))_\tau\varphi(x)dx + \int_\Omega ((\nabla u)_h(x,\tau))_\tau\varphi(x)dx + \int_\Omega (|\nabla u|^{p-2}\nabla u)_h(x,\tau)\nabla\varphi dx = 0, \tag{3.2}$$

where

$$u_h(x,t) = \begin{cases} \frac{1}{h}\int_t^{t+h} u(\cdot,\tau)d\tau, & t \in (0,T-h), \\ 0, & t > T-h. \end{cases}$$

Theorem 3.2. *Problem (1.1)-(1.3) admits only one weak solution.*

Proof. Suppose u_1, u_2 are two solutions of (1.1)-(1.3), then

$$\begin{aligned}
\int_\Omega (u_1(x,\tau) - u_2(x,\tau))_{h\tau}\varphi(x)dx + \int_\Omega ((\nabla u_1 - \nabla u_2)_h(x,\tau))_\tau\varphi(x)dx \\
+ \int_\Omega (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)_h(x,\tau)\nabla\varphi dx = 0.
\end{aligned}$$

For a fixed τ, we take $\varphi(x) = [u_1 - u_2]_h \in W_0^{1,p}(\Omega)$, and hence

$$\begin{aligned}
&\int_\Omega (u_1(x,\tau) - u_2(x,\tau))_{h\tau}(u_1 - u_2)_h dx \\
&+ \int_\Omega \nabla(u_1(x,\tau) - u_2(x,\tau))_{h\tau}\nabla(u_1 - u_2)_h dx \\
&= -\int_\Omega [(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)_h](x,\tau)\nabla(u_1 - u_2)_h dx,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_\Omega (u_1(x,\tau) - u_2(x,\tau))_{h\tau}(u_1 - u_2)_h dx \\
&+ \int_\Omega \nabla(u_1(x,\tau) - u_2(x,\tau))_{h\tau}\nabla(u_1 - u_2)_h dx \\
&= -\int_\Omega [(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)_h](x,\tau)\nabla(u_1 - u_2)_h dx.
\end{aligned}$$

Integrating the above equality with respect to τ over $(0,t)$,

$$\int_\Omega |(u_1-u_2)_h|^2(x,t)dx + \int_\Omega |\nabla(u_1-u_2)_h|^2(x,t)dx \le 0,$$

we have $\int_\Omega |(u_1-u_2)_h|^2 dx = 0$; therefore, $u_1 = u_2$. □

Acknowledgment. The author would like to thank referee for his/her valuable suggestions and for providing the references E. Di Benedtto & M. Pierre [5] and E. Di Benedetto & R. E. Showalter [6].

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Changchun Liu
Department of Mathematics, Nanjing Normal University, Nanjing 210097, China
Department of Mathematics, Jilin University, Changchun 130012, China
E-mail address: mathlcc@21cn.com