

GLOBAL SOLUTION FOR THE KADOMTSEV-PETVIASHVILI EQUATION (KPII) IN ANISOTROPIC SOBOLEV SPACES OF NEGATIVE INDICES

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ABSTRACT. It is proved that the Cauchy problem for the Kadomtsev-Petviashvili equation (KPII) is globally well-posed for initial data in anisotropic Sobolev spaces $H^{s_0}(\mathbb{R}^2)$ with $s > -1/14$. The extension of a local solution to a solution in an arbitrary interval is carried out by means of an almost conservation property of the H^{s_0} norm of the solution.

1. INTRODUCTION

In this article, we consider the initial-value (IVP) problem for the Kadomtsev-Petviashvili Equation (KP-II):

$$\begin{aligned} \partial_t u + \partial_x^3 u + \partial_x^{-1} \partial_y^2 u + u \partial_x u &= 0 \\ u(x, y, 0) &= u_0(x, y), \end{aligned} \tag{1.1}$$

with initial data u_0 in anisotropic Sobolev spaces with negative indices. It is known that problem (1.1) is locally well-posed for initial data in the anisotropic Sobolev spaces $H^{s_1 s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{1}{3}$ and $s_2 \geq 0$ (see [4] and [9]).

For $s_1 \geq 0$, the local result and the conservation law for the L^2 norm show that (1.1) is globally well-posed. Using the high-low frequency technique, introduced by Bourgain [1], the global result for (1.1) was proved in [4] when $u_0 \in H^{s_0}(\mathbb{R}^2)$ with $s > -\frac{1}{64}$.

In this paper, we apply a modification of this technique, proposed in [2] for the Korteweg-de Vries Equation, KdV, to extend the global result for the KP-II equation mentioned above to indices s with $s \in (-\frac{1}{14}, 0)$. The solution in any time interval $[0, T]$ is obtained from the local solutions by means of an iterative process in a finite number of steps. Such process is possible because we have in hand a norm which allows us to control the size of the solution at any instant t in an adequate way. This norm, equivalent to the usual norm of H^{s_0} , is essentially the L^2 norm for frequencies below a chosen parameter N . Thus, when N is sufficiently large, we can take the advantages of the L^2 conservation norm.

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To establish our results in a precise manner, we present several definitions and introduce some notation. Our initial data will be in the anisotropic Sobolev space

$$H^{s0} = H^{s0}(\mathbb{R}^2) := \{u \in S'(\mathbb{R}^2) : \|u\|_s^2 := \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} |\widehat{u}(\zeta)|^2 d\zeta < \infty\},$$

where $s < 0$, $S'(\mathbb{R}^2)$ is the space of tempered distributions in \mathbb{R}^2 , \widehat{u} is the Fourier transform of u in the space variables, $\zeta = (\xi, \eta)$ is the variable in the frequency space, with ξ and η corresponding to the space variables x and y , respectively, and the symbol $\langle \cdot \rangle$ stands for $1 + |\cdot|$.

For $N \in \mathbb{N}$ we define in H^{s0} an equivalent norm $\|\cdot\|_{sN}$ by $\|u\|_{sN} := \|I_N u\|_{L^2}$, where $(I_N u)^\wedge(\zeta) := M(\xi)\widehat{u}(\zeta)$ and

$$M(\xi) := M_N(\xi) := \begin{cases} 1, & \text{if } |\xi| \leq N, \\ \frac{|\xi|^s}{N^s}, & \text{if } |\xi| > N. \end{cases}$$

It is easily seen that

$$\|u\|_s \leq \|u\|_{sN} \leq CN^{|s|} \|u\|_s.$$

The solutions of the problem will be in the space

$$X_{s\gamma\varepsilon} := \{u \in S'(\mathbb{R}^3) \mid \|u\|_{s\gamma\varepsilon}^2 := \int_{\mathbb{R}^3} \langle \xi \rangle^{2s} \langle \sigma \rangle^{2\gamma} \langle \theta \rangle^{2\varepsilon} |\widehat{u}(\lambda)|^2 d\lambda < \infty\},$$

where $\gamma \in \mathbb{R}$, $\varepsilon > 0$, \widehat{u} is the Fourier transform of u in the space-time variables, $\lambda = (\zeta, \tau) = (\xi, \eta, \tau)$ is the variable in the frequency space with ξ and η as before, and τ corresponding to the time variable t ; $\sigma := \tau - m(\zeta) \equiv \tau - \xi^3 + \frac{\eta^2}{\xi}$, and $\theta := \frac{\sigma}{1+|\xi|^3}$. We observe that $m(\zeta) = \xi^3 - \frac{\eta^2}{\xi}$ is the symbol associated to the linear part of the KP-II equation.

For $N \in \mathbb{N}$ we consider in $X_{s\gamma\varepsilon}$ the equivalent norm $\|\cdot\|_{s\gamma\varepsilon N}$ defined by

$$\|u\|_{s\gamma\varepsilon N} := \|I_N u\|_{0\gamma\varepsilon},$$

where $(I_N u)^\wedge(\lambda) = M(\xi)\widehat{u}(\lambda)$. If $\gamma > \frac{1}{2}$, then $X_{s\gamma\varepsilon}$ is continuously embedded in $C_b(\mathbb{R}_t; H^{s0})$, the space of continuous bounded functions from the variable t to H^{s0} .

When we use the norms $\|\cdot\|_{sN}$ and $\|\cdot\|_{s\gamma\varepsilon N}$ in H^{s0} and $X_{s\gamma\varepsilon}$, we will refer to these spaces as H_N^{s0} and $X_{s\gamma\varepsilon N}$, respectively.

For $T > 0$ and $\gamma > \frac{1}{2}$ we define $X_{s\gamma\varepsilon N}[0, T]$ as the set of all restrictions to $[0, T]$ of the elements of $X_{s\gamma\varepsilon}$ with norm defined by

$$\|u\|_{X_{s\gamma\varepsilon N}[0, T]} := \inf\{\|v\|_{s\gamma\varepsilon N} : v|_{[0, T]} = u\}.$$

In our exposition we will make use of the space $H^\infty(\mathbb{R}^2) := \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}^2)$, where $H^s(\mathbb{R}^2)$ is the classical Sobolev space of L^2 type defined by

$$H^s(\mathbb{R}^2) := \{u \in S'(\mathbb{R}^2) : \int_{\mathbb{R}^2} \langle \zeta \rangle^{2s} |\widehat{u}(\zeta)|^2 d\zeta < \infty\}.$$

Our concept of solution comes from Duhamel's formula for problem (1.1). Formally, u is a solution of (1.1) in $[0, T]$ if for $t \in [0, T]$

$$u(t) = W(t)u_0 - \frac{1}{2} \int_0^t W(t-t') \partial_x (u(t'))^2 dt', \quad (1.2)$$

where $\{W(t)\}$ is the group associated to the linear part of KP-II equation. This is:

$$[W(t)u_0]^\wedge(\zeta) := e^{itm(\zeta)} \widehat{u_0}(\zeta).$$

In order to stay in the context of the spaces $X_{s\gamma\varepsilon}$, we multiply the right side of (1.2) by $\Psi(T^{-1}t)$, where $\Psi \in C_0^\infty(\mathbb{R}_t)$, $\Psi \geq 0$, $\Psi \equiv 1$ in $[0, 1]$ and $\text{supp } \Psi \subset [-1, 2]$. In this way, we consider the integral equation

$$u(t) = \Psi(T^{-1}t)W(t)u_0 - \frac{1}{2}\Psi(T^{-1}t) \int_0^t W(t-t')\partial_x(u(t')^2) dt', \quad t \in \mathbb{R}. \quad (1.3)$$

We can see that, formally, for $t \in [0, T]$ expressions (1.3) and (1.2) coincide. By a direct calculation, it can be easily established that

$$\|\Psi(T^{-1}\cdot_t)W(\cdot_t)u_0\|_{s\gamma\varepsilon N} \leq C_T\|u_0\|_{sN}, \quad (1.4)$$

where C_T depends on T but not on N .

For $\gamma > \frac{1}{2}$ and $f \in S(\mathbb{R}^3) \cap X_{s(\gamma-1)\varepsilon}$, where $S(\mathbb{R}^3)$ is the space of Schwartz functions in \mathbb{R}^3 , we define:

$$G_T(f)(t) := \frac{1}{2}\Psi(T^{-1}t) \int_0^t W(t-t')f(t')dt',$$

where $f(t) := f(\cdot_x, \cdot_y, t)$. Following a procedure similar to that in the proof in [6, Lemma 3.3], it can be seen that

$$\|G_T(f)\|_{s\gamma\varepsilon N} \leq C_T\|f\|_{s(\gamma-1)\varepsilon N}. \quad (1.5)$$

Therefore, since $S(\mathbb{R}^3) \cap X_{s(\gamma-1)\varepsilon}$ is dense in $X_{s(\gamma-1)\varepsilon}$, G_T has a unique continuous extension, which we denote again by G_T , to the space $X_{s(\gamma-1)\varepsilon}$.

In the study of the nonlinear part of the equation, an important role is played by the bilinear form $\partial_x(uv)$; more precisely, we have the following result whose proof will be given in section 2.

Lemma 1.1. *Let $s \in (-\frac{1}{3}, 0)$. For $\gamma > \frac{1}{2}$ and $\varepsilon > \frac{1}{6}$ such that $(\frac{1}{3} + s) - (\gamma - \frac{1}{2}) - (\varepsilon - \frac{1}{6}) \geq 0$, $(\frac{1}{2} + s) - 3(\varepsilon - \frac{1}{6}) \geq 0$, and $\frac{1}{3}(\frac{1}{2} + s) - (\gamma - \frac{1}{2}) \geq 0$, it follows that*

$$\|\partial_x(uv)\|_{s(\gamma-1)\varepsilon N} \leq \bar{C}\|u\|_{s\gamma\varepsilon N}\|v\|_{s\gamma\varepsilon N} \quad \forall u, v \in X_{s\gamma\varepsilon}, \quad (1.6)$$

with \bar{C} independent of N .

Estimates (1.4), (1.5), and (1.6) allow us to define the concept of solution for (1.1):

Definition. For $u_0 \in H^{s_0}$, $T > 0$, s , γ , and ε as in Lemma 1.1, we say that $u \in X_{s\gamma\varepsilon N}[0, T]$ is a solution of the IVP (1.1) in the interval $[0, T]$ if there is an extension $v \in X_{s\gamma\varepsilon N}$ of u , such that

$$u(t) = W(t)u_0 - G_T(\partial_x v^2)(t) \quad \forall t \in [0, T].$$

It was proved in [4] that (1.1) is locally well-posed for initial data u_0 in H^{s_0} with $s > -\frac{1}{3}$. More precisely, the theorems of existence and uniqueness of local solutions were proved there. The proofs of continuous dependence on the initial data and of regularity follow the same procedure applied in [5] for the corresponding proofs in the case $s > 0$.

In this paper we will obtain global solution for initial data in H^{s_0} with $s > -\frac{1}{14}$. In this case, the extension from a local solution u to a solution in an arbitrary interval $[0, T]$ is carried out by keeping control of the norm $\|u(T)\|_{sN}$ with the use of the homogeneity properties of the KP equation and the aid of an almost conservation law which uses a cancellation effect expressed by the following estimate of a new bilinear form.

Lemma 1.2. For $s \in (-\frac{1}{4}, 0)$, let $\gamma > \frac{1}{2}$, and $\varepsilon > \frac{1}{6}$ be chosen to satisfy the hypotheses of Lemma 1.1 and the condition $(\frac{1}{2}-\gamma)+\frac{2}{3}(\frac{1}{4}+s) > 0$ (i.e., $2|s| < 2-3\gamma$), then, for $\alpha \in (2|s|, 2-3\gamma)$ it follows that

$$\|\partial_x[(I_N u)(I_N v) - I_N(uv)]\|_{0,\gamma-1,0} \leq CN^{-\alpha} \|I_N u\|_{0\gamma\varepsilon} \|I_N v\|_{0\gamma\varepsilon}. \quad (1.7)$$

The proof of this lemma will be given in section 3. Finally, in section 4 we will prove our main result, whose precise formulation is:

Theorem 1.3. For $s \in (-\frac{1}{14}, 0)$, $T > 0$, and $u_0 \in H^{s_0}$ with $\partial_x^{-1}u_0 \in S'(\mathbb{R}^2)$ (i.e., $\frac{\widehat{u_0}}{i\xi} \in S'(\mathbb{R}^2)$), there exists $N > 0$ such that problem (1.1) has a solution in $X_{s\gamma\varepsilon N}[0, T]$.

Note that the role played by N is merely technical and also that the obtained solution is in

$$X_{s\gamma\varepsilon}[0, T] := \{v|_{[0,T]} \mid v \in X_{s\gamma\varepsilon}\}.$$

Through this article, the letter C will denote diverse constants and the notation $x \sim y$, for two variables x and y , will mean the existence of positive constants C_1 and C_2 such that $C_1|x| \leq |y| \leq C_2|x|$.

2. PROOF OF LEMMA 1.1

It was proved in [4] that, under the hypotheses of Lemma 1.1, the following estimate takes place:

$$\|\partial_x(uv)\|_{s(\gamma-1)\varepsilon} \leq C \|u\|_{s\gamma\varepsilon} \|v\|_{s\gamma\varepsilon}. \quad (2.1)$$

If $\lambda_1 = (\zeta_1, \tau_1)$, $\lambda_2 = (\zeta_2, \tau_2) = (\zeta - \zeta_1, \tau - \tau_1) = \lambda - \lambda_1$; $\sigma_1 = \sigma(\zeta_1, \tau_1)$, $\sigma_2 = \sigma(\zeta_2, \tau_2)$; $\theta_1 = \theta(\zeta_1, \tau_1)$, $\theta_2 = \theta(\zeta_2, \tau_2)$, estimate (2.1) is equivalent to the estimate

$$\begin{aligned} & \int_{\mathbb{R}^6} \widetilde{K}(\lambda, \lambda_1) f(\lambda_1) g(\lambda_2) h(\lambda) d\lambda_1 d\lambda \\ & := \int_{\mathbb{R}^6} \frac{|\xi| \langle \theta \rangle^\varepsilon}{\langle \sigma \rangle^{1-\gamma} \langle \sigma_1 \rangle^\gamma \langle \sigma_2 \rangle^\gamma \langle \theta_1 \rangle^\varepsilon \langle \theta_2 \rangle^\varepsilon} \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s} f(\lambda_1) g(\lambda_2) h(\lambda) d\lambda_1 d\lambda \\ & \leq C \|h\| \|f\| \|g\|, \end{aligned} \quad (2.2)$$

where $f, g, h \geq 0$ and $\|\cdot\| := \|\cdot\|_{L^2_\lambda}$. Using duality, estimate (1.6) is equivalent to:

$$\begin{aligned} & \int_{\mathbb{R}^6} \frac{|\xi| \langle \theta \rangle^\varepsilon}{\langle \sigma \rangle^{1-\gamma} \langle \sigma_1 \rangle^\gamma \langle \sigma_2 \rangle^\gamma \langle \theta_1 \rangle^\varepsilon \langle \theta_2 \rangle^\varepsilon} \frac{M(\xi)}{M(\xi_1)M(\xi_2)} f(\lambda_1) g(\lambda_2) h(\lambda) d\lambda_1 d\lambda \\ & \leq \overline{C} \|h\| \|f\| \|g\|. \end{aligned} \quad (2.3)$$

In this way, to establish (2.3) it suffices to prove that

$$\frac{M(\xi)}{M(\xi_1)M(\xi_2)} \leq C \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s}, \quad (2.4)$$

with C independent of N . In the proof of (2.4) we will take into account that

$$1 \leq 2^{|s|} \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s},$$

and besides that, for $s < 0$, $M(\xi) \leq 1$.

By a symmetry argument it is sufficient to analyze the following cases:

(i) $|\xi_1| \leq \frac{N}{2} \wedge |\xi_2| \leq \frac{N}{2}$. Therefore,

$$\frac{M(\xi)}{M(\xi_1)M(\xi_2)} = M(\xi) \leq 1 \leq 2^{|\xi|} \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s}.$$

(ii) $|\xi_1| \geq \frac{N}{2} \wedge |\xi_2| \leq 1$. Then, $|\xi| \sim |\xi_1| \sim \langle \xi \rangle \sim \langle \xi_1 \rangle$ and $\langle \xi_2 \rangle \sim 1$. Hence,

$$\frac{M(\xi)}{M(\xi_1)M(\xi_2)} \leq C \frac{\frac{|\xi|^s}{N^s}}{\frac{|\xi_1|^s}{N^s} 1} \leq C \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s}.$$

(iii) $\frac{N}{2} \leq |\xi_1| \leq N \wedge 1 \leq |\xi_2| \leq \frac{N}{2}$. Thus,

$$\frac{M(\xi)}{M(\xi_1)M(\xi_2)} = M(\xi) \leq 1 \leq 2^{|\xi|} \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s}.$$

(iv) $N \leq |\xi_1| \wedge 1 \leq |\xi_2| \leq \frac{N}{2}$. Then, $|\xi| \geq \frac{N}{2}$, $|\xi| \sim \langle \xi \rangle$, and $|\xi_1| \sim \langle \xi_1 \rangle$. Therefore,

$$\frac{M(\xi)}{M(\xi_1)M(\xi_2)} \leq C \frac{\frac{|\xi|^s}{N^s}}{\frac{|\xi_1|^s}{N^s} 1} \leq C \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s} \leq C \frac{\langle \xi \rangle^s \langle \xi_2 \rangle^{|\xi|}}{\langle \xi_1 \rangle^s} = C \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s}.$$

(v) $|\xi_1| \geq \frac{N}{2} \wedge |\xi_2| \geq \frac{N}{2}$. Thus, $|\xi_1| \sim \langle \xi_1 \rangle$ and $|\xi_2| \sim \langle \xi_2 \rangle$.

If $|\xi| \leq 1$, then $\langle \xi \rangle \sim 1$ and

$$\frac{M(\xi)}{M(\xi_1)M(\xi_2)} \leq C \frac{1}{N^{2|\xi|} |\xi_1|^s |\xi_2|^s} \leq C \frac{\langle \xi \rangle^s}{N^{2|\xi|} \langle \xi_1 \rangle^s \langle \xi_2 \rangle^s} \leq C \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s}.$$

If $1 \leq |\xi| \leq N$, then $\frac{|\xi|}{N} \leq 1$, $\frac{|\xi|^s}{N^s} \geq 1$, and $|\xi| \sim \langle \xi \rangle$. Therefore,

$$\begin{aligned} \frac{M(\xi)}{M(\xi_1)M(\xi_2)} &\leq C \frac{1}{N^{2|\xi|} |\xi_1|^s |\xi_2|^s} \leq C \frac{\frac{|\xi|^s}{N^s}}{N^{2|\xi|} |\xi_1|^s |\xi_2|^s} \\ &\leq C \frac{\langle \xi \rangle^s}{N^{|\xi|} \langle \xi_1 \rangle^s \langle \xi_2 \rangle^s} \leq C \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s}. \end{aligned}$$

If $N \leq |\xi|$, then

$$\frac{M(\xi)}{M(\xi_1)M(\xi_2)} \leq C \frac{\frac{|\xi|^s}{N^s}}{N^{2|\xi|} |\xi_1|^s |\xi_2|^s} \leq C \frac{\langle \xi \rangle^s}{N^{|\xi|} \langle \xi_1 \rangle^s \langle \xi_2 \rangle^s} \leq C \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s}.$$

Which completes the proof.

3. PROOF OF LEMMA 1.2

We begin this section by describing the procedure that leads to an almost conservation law, according to which, for a solution v of problem (1.1) in the interval $[0, 1]$, the size of $\|v(1)\|_{sN}$ is controlled by the size of $\|v(0)\|_{sN}$. In the description of this procedure, the necessity of estimate (1.7) comes out in a natural way.

Let $v_0 \in H^\infty(\mathbb{R}^2)$ with $\partial_x^{-1}v_0 \in H^\infty(\mathbb{R}^2)$. If $w \in X_{s\gamma\epsilon N}$ is such that $v := w|_{[0,1]}$ is a solution of the IVP (1.1) with initial datum v_0 , then, from a regularity theorem for problem (1.1) (see [5, Theorem IV]) and from the application of standard techniques of the theory of semigroups, we can conclude that $v(t)$ and $\partial_x^{-1}v(t)$ are in $H^\infty(\mathbb{R}^2)$, and

$$v'(t) + \partial_x^3 v(t) + \partial_x^{-1} \partial_y^2 v(t) + v(t) \partial_x v(t) = 0 \quad \text{in } H^\infty(\mathbb{R}^2), \quad \forall t \in [0, 1]. \quad (3.1)$$

When we apply the operator I_N to the equation above and take the inner product in $L^2(\mathbb{R}^2)$ with $I_N v(t)$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|I_N v(t)\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x I_N(v(t)^2)) I_N v(t) dx dy = 0.$$

Taking into account that $\frac{1}{2} \int_{\mathbb{R}^2} (\partial_x (I_N v(t)^2)) I_N v(t) dx dy = 0$, and denoting by χ the characteristic function of the interval $[0, 1]$, an integration with respect to t in $[0, 1]$ yields

$$\begin{aligned} \|I_N v(1)\|_{L^2}^2 &= \|I_N v_0\|_{L^2}^2 - \int_0^1 \langle \partial_x [I_N(v(t)^2) - (I_N v(t))^2], I_N v(t) \rangle dt \\ &\leq \|I_N v_0\|_{L^2}^2 + \left| \int_{-\infty}^{+\infty} \langle \chi(t) \partial_x [I_N(w(t)^2) - (I_N w(t))^2], \chi(t) I_N w(t) \rangle dt \right| \\ &\leq \|I_N v_0\|_{L^2}^2 + C \|\chi(\cdot) \partial_x [I_N(w^2) - (I_N w)^2]\|_{0, \gamma^- - 1, 0} \|\chi(\cdot) I_N w\|_{0, 1 - \gamma^-, 0} \\ &\leq \|I_N v_0\|_{L^2}^2 + C \|\partial_x [I_N(w^2) - (I_N w)^2]\|_{0, \gamma^- - 1, 0} \|I_N w\|_{0, (1 - \gamma^-)^+, 0}, \end{aligned} \quad (3.2)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^2)$, $\frac{1}{2} < \gamma^- < \gamma$, and, in the last inequality we have used the following lemma of technical character.

Lemma 3.1. *If $\bar{\gamma} \in (0, \frac{1}{2})$ and $\bar{\gamma} < \bar{\gamma}^+ < \frac{1}{2}$, then*

$$\|\chi(\cdot) u\|_{0, \bar{\gamma}^+, 0} \leq C \|u\|_{0, \bar{\gamma}^+, 0}, \quad (3.3)$$

$$\|\chi(\cdot) u\|_{0, (-\bar{\gamma}^+), 0} \leq C \|u\|_{0, -\bar{\gamma}^+, 0}. \quad (3.4)$$

Proof. We prove (3.3) only, since (3.4) follows from (3.3) by duality. For $u \in S(\mathbb{R}^3) \cap X_{0, \bar{\gamma}^+, 0}$,

$$\begin{aligned} \|\chi(\cdot) u\|_{0, \bar{\gamma}^+, 0}^2 &= \iint \langle \sigma \rangle^{2\bar{\gamma}^+} |\widehat{[\chi *_{\tau} \widehat{u}(\zeta, \cdot)]}(\tau)|^2 d\tau d\zeta \\ &\leq C \iint |\widehat{[\chi(\cdot) u(\cdot)]}(\zeta)|^2 d\tau d\zeta \\ &\quad + C \iint |\tau|^{2\bar{\gamma}^+} |\widehat{[\chi *_{\tau} \widehat{u}(\zeta, \cdot)]}(\tau + m(\zeta))|^2 d\tau d\zeta =: I + II. \end{aligned} \quad (3.5)$$

Using Plancherel identity in the variable t we obtain

$$\begin{aligned} I &\leq C \iint |\chi(t) \widehat{u(t)}(\zeta)|^2 dt d\zeta \leq C \iint |\widehat{u(t)}(\zeta)|^2 dt d\zeta \\ &= C \iint |\widehat{u}(\zeta, \tau)|^2 d\zeta d\tau = \|u\|_{000}^2 \leq \|u\|_{0, \bar{\gamma}^+, 0}^2. \end{aligned} \quad (3.6)$$

To estimate II we use the following Leibniz formula for fractional derivatives proved in [7]: If $\beta \in (0, 1)$ and $1 < q < \infty$, then

$$\|D^\beta(fg) - fD^\beta g\|_{L^q(\mathbb{R})} \leq C \|g\|_{L^\infty(\mathbb{R})} \|D^\beta f\|_{L^q(\mathbb{R})}. \quad (3.7)$$

Therefore, an application of Plancherel identity and of estimate (3.7) with $q = 2$ gives us

$$\begin{aligned}
 II &= C \int \|D_t^{\bar{\gamma}}[e^{-i(\cdot)t}m(\zeta)\chi(\cdot_t)\widehat{u(\cdot_t)}(\zeta)]\|_{L_t^2}^2 d\zeta \\
 &\leq C \int \|D_t^{\bar{\gamma}}[e^{-i(\cdot)t}m(\zeta)\widehat{u(\cdot_t)}(\zeta)]\|_{L_t^2}^2 \|\chi(\cdot_t)\|_{L_t^\infty}^2 d\zeta \\
 &+ C \int \|e^{-i(\cdot)t}m(\zeta)\widehat{u(\cdot_t)}(\zeta)D_t^{\bar{\gamma}}\chi(\cdot_t)\|_{L_t^2}^2 d\zeta \\
 &\leq C\|u\|_{0\bar{\gamma}0}^2 + C \int \|e^{-i(\cdot)t}m(\zeta)\widehat{u(\cdot_t)}(\zeta)\|_{L^{2p}}^2 \|D_t^{\bar{\gamma}}\chi(\cdot_t)\|_{L^{2p'}}^2 d\zeta,
 \end{aligned}
 \tag{3.8}$$

where p and p' are conjugate exponents. If we choose p in such a way that $\frac{1}{2} - \frac{1}{2p} = \bar{\gamma}^+$, then $H^{\bar{\gamma}^+}(\mathbb{R}_t) \hookrightarrow L^{2p}(\mathbb{R}_t)$. Bearing in mind that the inverse Fourier transform operator is bounded from $L^{\frac{2p'}{2p'-1}}(\mathbb{R})$ to $L^{2p'}(\mathbb{R})$; it follows from (3.8) that

$$II \leq C\|u\|_{0\bar{\gamma}0}^2 + C \int \|e^{-i(\cdot)t}m(\zeta)\widehat{u(\cdot_t)}(\zeta)\|_{H_t^{\bar{\gamma}^+}}^2 d\zeta \cdot \|(D_t^{\bar{\gamma}}\chi)^{\wedge t}\|_{L^{\frac{2p'}{2p'-1}}(\mathbb{R}_\tau)}^2.$$

Since $\frac{2p'}{2p'-1} = \frac{1}{1-\bar{\gamma}^+}$ and $|\widehat{\chi}(\tau)| \leq \frac{C}{\langle \tau \rangle}$, we have

$$\|(D_t^{\bar{\gamma}}\chi)^{\wedge t}\|_{L^{\frac{2p'}{2p'-1}}} \leq C \left(\int_{-\infty}^{+\infty} \frac{d\tau}{\langle \tau \rangle^{\frac{1-\bar{\gamma}^+}{1-\bar{\gamma}^+}}} \right)^{1-\bar{\gamma}^+} < \infty.$$

Therefore,

$$II \leq C\|u\|_{0\bar{\gamma}0}^2 + C\|u\|_{0\bar{\gamma}+0}^2.
 \tag{3.9}$$

From (3.5), (3.6), and (3.9) we obtain (3.3) for $u \in S(\mathbb{R}^3) \cap X_{0\bar{\gamma}+0}$. The result of the lemma follows from a density argument. \square

Proof of Lemma 1.2. We use the notation introduced in the proof of Lemma 1.1. Reasoning by duality, estimate (1.7) is equivalent to the estimate

$$\begin{aligned}
 &\iint K(\lambda, \lambda_1)f(\lambda_1)g(\lambda_2)h(\lambda) d\lambda_1 d\lambda \\
 &:= \int_{\mathbb{R}_\lambda^3} \int_{\mathbb{R}_{\lambda_1}^3} \frac{|\xi|}{\langle \sigma \rangle^{1-\gamma} \langle \sigma_1 \rangle^\gamma \langle \theta_1 \rangle^\varepsilon \langle \sigma_2 \rangle^\gamma \langle \theta_2 \rangle^\varepsilon} \frac{(M(\xi_1)M(\xi_2) - M(\xi))}{M(\xi_1)M(\xi_2)} f(\lambda_1)g(\lambda_2)h(\lambda) d\lambda_1 d\lambda \\
 &\leq CN^{-\alpha} \|h\| \|f\| \|g\|.
 \end{aligned}
 \tag{3.10}$$

For $A \subseteq \mathbb{R}^6$, we will denote by J_A the integral over the set A of the former integrand. By a symmetry argument, it suffices to show that

$$J := \sum_{i=1}^4 J_{A_i} \leq CN^{-\alpha} \|h\| \|f\| \|g\|,
 \tag{3.11}$$

where

$$\begin{aligned} A_1 &:= \{(\lambda, \lambda_1) \mid |\xi_1| \leq \frac{N}{2} \wedge |\xi_2| \leq \frac{N}{2}\}, \\ A_2 &:= \{(\lambda, \lambda_1) \mid |\xi_1| \geq \frac{N}{2} \wedge |\xi_2| \leq 1\}, \\ A_3 &:= \{(\lambda, \lambda_1) \mid |\xi_1| \geq \frac{N}{2} \wedge 1 < |\xi_2| < N\}, \text{ and} \\ A_4 &:= \{(\lambda, \lambda_1) \mid |\xi_1| \geq \frac{N}{2} \wedge |\xi_2| \geq N\}. \end{aligned}$$

For the rest of this article, we will use the notation

$$L(\xi_1, \xi_2) := \frac{M(\xi_1)M(\xi_2) - M(\xi)}{M(\xi_1)M(\xi_2)}. \quad (3.12)$$

Estimate of J_{A_1} : For $(\lambda, \lambda_1) \in A_1$ we have that $L(\xi_1, \xi_2) = 0$. Thus

$$J_{A_1} = 0. \quad (3.13)$$

Estimate of J_{A_2} : For $(\lambda, \lambda_1) \in A_2$, an application of the mean value theorem leads to

$$\begin{aligned} |L(\xi_1, \xi_2)| &= \left| \frac{M(\xi_1) - M(\xi_1 + \xi_2)}{M(\xi_1)} \right| \leq \frac{N^{|s|}}{M(\xi_1)} \left| \frac{1}{|\xi_1|^{|s|}} - \frac{1}{|\xi_1 + \xi_2|^{|s|}} \right| \\ &\leq C \frac{N^{|s|}|\xi_2|}{M(\xi_1)|\xi_1|^{|s|+1}} \leq C \frac{N^{|s|}|\xi_2|}{\frac{N^{|s|}}{|\xi_1|^{|s|}}|\xi_1|^{|s|+1}} \\ &\leq \frac{C}{|\xi_1|} \leq \frac{C}{N} \leq \frac{C}{N} \frac{\langle \xi_1 \rangle^{|s|} \langle \xi_2 \rangle^{|s|}}{\langle \xi \rangle^{|s|}}. \end{aligned}$$

In this way, taking into account the definition of the kernel \tilde{K} in (2.2), it follows that $K(\lambda, \lambda_1) \leq \frac{C}{N} \tilde{K}(\lambda, \lambda_1)$ and therefore, according to estimate (2.2),

$$J_{A_2} \leq \frac{C}{N} \|h\| \|f\| \|g\| \leq CN^{-\alpha} \|h\| \|f\| \|g\|. \quad (3.14)$$

Estimate of J_{A_3} : For $(\lambda, \lambda_1) \in A_3$,

$$|L(\xi_1, \xi_2)| = \left| \frac{M(\xi_1) - M(\xi_1 + \xi_2)}{M(\xi_1)} \right|.$$

When $(\frac{N}{2} \leq |\xi_1| \leq N \wedge |\xi| \geq N)$ or $(N \leq |\xi_1| \leq \frac{3N}{2} \wedge 1 \leq |\xi_2| \leq \frac{N}{2})$ or $|\xi_1| \geq \frac{3N}{2}$, from an application of the mean value theorem we have

$$|L(\xi_1, \xi_2)| \leq C \frac{N^{|s|}}{M(\xi_1)} \frac{|\xi_2|}{|\xi_1|^{|s|+1}} \leq C \frac{|\xi_2|}{|\xi_1|}.$$

If $\frac{N}{2} \leq |\xi_1| \leq N$ and $|\xi| \leq N$, then $L(\xi_1, \xi_2) = 0$. If $N \leq |\xi_1| \leq \frac{3N}{2}$ and $\frac{N}{2} \leq |\xi_2| \leq N$, then

$$|L(\xi_1, \xi_2)| \leq 1 + \frac{M(\xi)}{M(\xi_1)} \leq C \leq C \frac{|\xi_2|}{|\xi_1|},$$

since $|\xi_2| \sim |\xi_1|$.

Thus, for $(\lambda, \lambda_1) \in A_3$:

$$|K(\lambda, \lambda_1)| \leq C \frac{|\xi|}{\langle \sigma \rangle^{1-\gamma}} \frac{|\xi_2|}{|\xi_1|} \frac{1}{\langle \sigma_1 \rangle^\gamma \langle \sigma_2 \rangle^\gamma}. \quad (3.15)$$

According to our definitions we have

$$\sigma_1 + \sigma_2 - \sigma = 3\xi\xi_1\xi_2 + \frac{(\eta\xi_1 - \eta_1\xi)^2}{\xi\xi_1\xi_2}.$$

Therefore, $|\xi\xi_1\xi_2| \leq \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$. If $|\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$, then

$$\begin{aligned} \frac{|\xi|}{\langle\sigma\rangle^{1-\gamma}} \frac{|\xi_2|}{|\xi_1|} &\leq \frac{|\xi|^\gamma |\xi_2|^\gamma |\xi\xi_1\xi_2|^{1-\gamma}}{\langle\sigma\rangle^{1-\gamma} |\xi_1|^{2-\gamma}} \\ &\leq C \frac{|\xi|^\gamma |\xi_2|^\gamma}{|\xi_1|^{2-\gamma}} \leq C \frac{|\xi_1|^\gamma |\xi_2|^\gamma}{|\xi_1|^{2-\gamma}} + C \frac{|\xi_2|^\gamma |\xi_2|^\gamma}{|\xi_1|^{2-\gamma}} \\ &\leq C \frac{N^\gamma}{N^{2-2\gamma}} + C \frac{N^{2\gamma}}{N^{2-\gamma}} \leq CN^{-(2-3\gamma)}, \end{aligned}$$

and from (3.15),

$$|K(\lambda, \lambda_1)| \leq CN^{-(2-3\gamma)} \frac{1}{\langle\sigma_1\rangle^\gamma \langle\sigma_2\rangle^\gamma}. \tag{3.16}$$

If $|\sigma_1| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$, then

$$\begin{aligned} \frac{|\xi|}{\langle\sigma_1\rangle^\gamma} \frac{|\xi_2|}{|\xi_1|} \frac{1}{\langle\sigma\rangle^{1-\gamma}} &\leq \frac{|\xi||\xi_2|}{\langle\sigma_1\rangle^{1-\gamma} |\xi_1| \langle\sigma\rangle^\gamma} \leq \frac{|\xi|^\gamma |\xi_2|^\gamma |\xi\xi_1\xi_2|^{1-\gamma}}{\langle\sigma_1\rangle^{1-\gamma} |\xi_1|^{2-\gamma} \langle\sigma\rangle^\gamma} \\ &\leq C \frac{|\xi|^\gamma |\xi_2|^\gamma}{|\xi_1|^{2-\gamma} \langle\sigma\rangle^\gamma} \leq C \frac{N^{-(2-3\gamma)}}{\langle\sigma\rangle^\gamma}, \end{aligned}$$

and from (3.15)

$$|K(\lambda, \lambda_1)| \leq CN^{-(2-3\gamma)} \frac{1}{\langle\sigma\rangle^\gamma \langle\sigma_2\rangle^\gamma}. \tag{3.17}$$

If $|\sigma_2| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$, then, as in the former case,

$$\frac{|\xi|}{\langle\sigma_2\rangle^\gamma} \frac{|\xi_2|}{|\xi_1|} \frac{1}{\langle\sigma\rangle^{1-\gamma}} \leq C \frac{N^{-(2-3\gamma)}}{\langle\sigma\rangle^\gamma}$$

and from (3.15),

$$|K(\lambda, \lambda_1)| \leq CN^{-(2-3\gamma)} \frac{1}{\langle\sigma\rangle^\gamma \langle\sigma_1\rangle^\gamma}. \tag{3.18}$$

Let us denote by A_{30} , A_{31} , and A_{32} the subsets of A_3 corresponding to each one of the former cases respectively. Then, from (3.16) we have

$$\begin{aligned} J_{A_{30}} &\leq CN^{-(2-3\gamma)} \int_{\mathbb{R}^3} h(\lambda) \int_{\mathbb{R}^3} \frac{f(\lambda_1) g(\lambda_2)}{\langle\sigma_1\rangle^\gamma \langle\sigma_2\rangle^\gamma} d\lambda_1 d\lambda_2 \\ &\leq CN^{-(2-3\gamma)} \|h\| \|FG\| \\ &\leq CN^{-(2-3\gamma)} \|h\| \|F\|_{L^4_{xyt}} \|G\|_{L^4_{xyt}}, \end{aligned}$$

where $\widehat{F}(\lambda) := \frac{f(\lambda)}{\langle\sigma\rangle^\gamma}$ and $\widehat{G}(\lambda) := \frac{g(\lambda)}{\langle\sigma\rangle^\gamma}$.

Using the Strichartz inequality (see [8], Proposition 2.1 and [3, Lemma 3.3])

$$\|F\|_{L^4_{xyt}} \leq C \|f\| \quad \text{if } \gamma > \frac{1}{2}, \tag{3.19}$$

it follows that

$$J_{A_{30}} \leq CN^{-(2-3\gamma)} \|h\| \|f\| \|g\|.$$

In a similar way, from (3.17) and (3.18),

$$J_{A_{31}} + J_{A_{32}} \leq CN^{-(2-3\gamma)} \|h\| \|f\| \|g\|.$$

In this manner,

$$J_{A_3} \leq CN^{-\alpha} \|h\| \|f\| \|g\|. \tag{3.20}$$

Estimate of J_{A_4} : To estimate J_{A_4} we require the following lemma.

Lemma 3.2. *If $\bar{s} \in (-\frac{1}{4}, 0)$, $\gamma > \frac{1}{2}$ satisfy $(\frac{1}{2} - \gamma) + \frac{2}{3}(\frac{1}{4} + \bar{s}) > 0$, and $\varepsilon > \frac{1}{6}$, then, for u and v such that $\text{supp } \hat{u}, \text{supp } \hat{v} \subseteq \{\lambda \mid |\xi| \geq 1\}$, the following inequality holds*

$$\|\partial_x(uv)\|_{0(\gamma-1)0} \leq C \|u\|_{\bar{s}\gamma\varepsilon} \|v\|_{\bar{s}\gamma\varepsilon}. \tag{3.21}$$

The proof of this lemma is a direct adaptation of [4, lemma 3.1], with the exception that in this case it is not necessary to consider the set Ω_6 used there for the study of low frequencies and that demands the condition $\varepsilon > \frac{1}{6} + \frac{2|\bar{s}|}{3}$, which we have weakened here.

Inequality (3.21) is equivalent to the estimate

$$\begin{aligned} & \iint_{\{(\lambda, \lambda_1) \mid |\xi_1|, |\xi_2| \geq 1\}} |\xi| \langle \sigma \rangle^{\gamma-1} \langle \xi_1 \rangle^{|\bar{s}|} \langle \xi_2 \rangle^{|\bar{s}|} h(\lambda) \frac{\bar{f}(\lambda_1)}{\langle \sigma_1 \rangle^\gamma \langle \theta_1 \rangle^\varepsilon} \frac{\bar{g}(\lambda_2)}{\langle \sigma_2 \rangle^\gamma \langle \theta_2 \rangle^\varepsilon} d\lambda_1 d\lambda \\ & \leq C \|h\| \|\bar{f}\| \|\bar{g}\|, \end{aligned} \tag{3.22}$$

which we will use now to estimate J_{A_4} . For $(\lambda, \lambda_1) \in A_4$,

$$\begin{aligned} |L(\xi_1, \xi_2)| & \leq 1 + \frac{M(\xi)}{M(\xi_1)M(\xi_2)} \leq C \frac{\langle \xi_1 \rangle^{\frac{\alpha}{2}} \langle \xi_2 \rangle^{\frac{\alpha}{2}}}{N^{\frac{\alpha}{2}} N^{\frac{\alpha}{2}}} + \frac{C}{\frac{N^{|\bar{s}|}}{|\xi_1|^{|\bar{s}|}} \frac{N^{|\bar{s}|}}{|\xi_2|^{|\bar{s}|}}} \\ & \leq C \frac{\langle \xi_1 \rangle^{\frac{\alpha}{2}} \langle \xi_2 \rangle^{\frac{\alpha}{2}}}{N^\alpha} + C \frac{\langle \xi_1 \rangle^{|\bar{s}|} \langle \xi_2 \rangle^{|\bar{s}|}}{N^{|\bar{s}|} N^{|\bar{s}|}} \cdot \frac{\langle \xi_1 \rangle^{\frac{\alpha}{2}-|\bar{s}|} \langle \xi_2 \rangle^{\frac{\alpha}{2}-|\bar{s}|}}{N^{\frac{\alpha}{2}-|\bar{s}|} N^{\frac{\alpha}{2}-|\bar{s}|}} \\ & \leq C \frac{\langle \xi_1 \rangle^{\frac{\alpha}{2}} \langle \xi_2 \rangle^{\frac{\alpha}{2}}}{N^\alpha}, \end{aligned}$$

since $\frac{\alpha}{2} - |\bar{s}| \geq 0$.

Therefore, using (3.22) with $\bar{s} = -\frac{\alpha}{2}$ (which is possible since if $2|\bar{s}| < \alpha < 2 - 3\gamma$, then $|\bar{s}| < \frac{\alpha}{2} < 1 - \frac{3}{2}\gamma$ and thus, $-\frac{\alpha}{2} \in (-\frac{1}{4}, 0)$ and $(\frac{1}{2} - \gamma) + \frac{2}{3}(\frac{1}{4} - \frac{\alpha}{2}) = \frac{1}{3}(2 - 3\gamma - \alpha) > 0$), it follows that

$$J_{A_4} \leq CN^{-\alpha} \|h\| \|f\| \|g\|. \tag{3.23}$$

The statement of Lemma 1.2 follows now from (3.11), (3.13), (3.14), (3.20), and (3.23). \square

4. PROOF OF THEOREM 1.3

In our proof we will make use of the homogeneity properties of the KP-II equation which we now describe as: For $\delta > 0$ we define

$$\begin{aligned} u_{0\delta}(x, y) & := \delta^{\frac{2}{3}} u_0(\delta^{\frac{1}{3}} x, \delta^{\frac{2}{3}} y), \\ u_\delta(x, y, t) & := \delta^{\frac{2}{3}} u(\delta^{\frac{1}{3}} x, \delta^{\frac{2}{3}} y, \delta t). \end{aligned}$$

Then $u \in X_{s\gamma\varepsilon N}[0, T]$ is a solution of the IVP (1.1) in $[0, T]$ with initial datum u_0 if and only if $u_\delta \in X_{s\gamma\varepsilon N}[0, T/\delta]$ is a solution of (1.1) in $[0, T/\delta]$ with initial datum $u_{0\delta}$. We also observe that,

$$\|u_{0\delta}\|_{sN} \leq CN^{|\bar{s}|} \delta^{(\frac{1}{6} - \frac{|\bar{s}|}{3})} \|u_0\|_s. \tag{4.1}$$

Let $C := \max\{C_1, \overline{C}\}$, where C_1 is the constant in estimates (1.4) and (1.5) corresponding to $T = 1$, and \overline{C} is the constant in (1.6). Let $R := \frac{1}{8C^2}$. If $v_0 \in H^{s_0}$ with $\|v_0\|_{s_N} \leq R$, then from (1.4), (1.5), and (1.6) we can conclude that the operator Φ_{v_0} defined by

$$\Phi_{v_0}(w) := \Psi(\cdot_t)W(\cdot_t)v_0 - G_1(\partial_x w^2)$$

maps the closed ball $\overline{B(0, 2CR)}$ of $X_{s\gamma\epsilon N}$ into itself and is a contraction. Therefore, Φ_{v_0} has a unique fixed point w in this ball. Thus, $w|_{[0,1]} \in X_{s\gamma\epsilon N}[0, 1]$ is a solution in $[0, 1]$ of (1.1) with initial datum v_0 .

Let $u_0 \in H^{s_0}$ with $\partial_x^{-1}u_0 \in S'(\mathbb{R}^2)$ and suppose that N and $\delta > 0$ are such that

$$\|u_{0\delta}\|_{s_N} \leq CN^{|s|}\delta^{\left(\frac{1}{6}-\frac{|s|}{3}\right)}\|u_0\|_s = \frac{R}{4}. \tag{4.2}$$

(We are supposing without loss of generality that $\|u_0\|_s \neq 0$). Since $\|u_{0\delta}\|_{s_N} \leq R$, the operator $\Phi_{u_{0\delta}}$ has a fixed point $w \in X_{s\gamma\epsilon N}$ in $\overline{B(0, 2CR)}$. Let $v := w|_{[0,1]}$. We can take a sequence $\{v_{0n}\}$ in $H^\infty(\mathbb{R}^2)$ with $\partial_x^{-1}v_{0n} \in H^\infty(\mathbb{R}^2)$, such that $v_{0n} \rightarrow u_{0\delta}$ in H^{s_0} and such that $\|v_{0n}\|_{s_N} \leq \frac{R}{2}$.

For each n , let $w_n \in X_{s\gamma\epsilon N}$ be the fixed point in $\overline{B(0, 2CR)}$ of $\Phi_{v_{0n}}$ and set $v_n := w_n|_{[0,1]}$.

Since the local problem is locally well-posed in $X_{s\gamma\epsilon}$ for initial data in H^{s_0} and the norms $\|\cdot\|_s$ and $\|\cdot\|_{s\gamma\epsilon}$ are equivalent to $\|\cdot\|_{s_N}$ and $\|\cdot\|_{s\gamma\epsilon N}$, respectively, we have that

$$v_n \rightarrow v \quad \text{in } X_{s\gamma\epsilon N}[0, 1]. \tag{4.3}$$

Since $v_{0n} \in H^\infty(\mathbb{R}^2)$ and $\partial_x^{-1}v_{0n} \in H^\infty(\mathbb{R}^2)$, from (3.1) and (3.2) it follows that

$$\|I_N v_n(1)\|_{L^2}^2 \leq \|I_N v_{0n}\|_{L^2}^2 + C\|\partial_x[I_N(w_n^2) - (I_N w_n)^2]\|_{0(\gamma-1)0}\|I_N w_n\|_{0\gamma 0},$$

and from Lemma 1.2 we obtain that

$$\begin{aligned} \|I_N v_n(1)\|_{L^2}^2 &\leq \frac{R^2}{4} + CN^{-\alpha}\|I_N w_n\|_{0\gamma\epsilon}^2\|I_N w_n\|_{0\gamma\epsilon} \\ &\leq \frac{R^2}{4} + CN^{-\alpha}\|w_n\|_{s\gamma\epsilon N}^3 \\ &\leq \frac{R^2}{4} + CN^{-\alpha}8C^3R^3. \end{aligned}$$

From (4.3) and the immersion of $X_{s\gamma\epsilon N}[0, 1]$ in $C([0, 1]; H_N^{s_0})$, we conclude that

$$\|I_N v(1)\|_{L^2}^2 \leq \frac{R^2}{4} + CN^{-\alpha}, \tag{4.4}$$

where C is independent of N and δ .

In virtue of (4.4), for $k \in \mathbb{N}$, we can obtain a solution of problem (1.1) with initial datum $u_{0\delta}$ in the interval $[0, k]$ whenever $(k-1)CN^{-\alpha} \leq \frac{3}{4}R^2$.

The largest k with this property is the integer \bar{k} for which $(\bar{k}-1)CN^{-\alpha} \leq \frac{3}{4}R^2 < \bar{k}CN^{-\alpha}$. If we wish to have a solution with initial datum u_0 in the interval $[0, T]$, it suffices to have $\bar{k} > \frac{T}{\delta}$. We know from (4.2) that $\delta^{\frac{1}{6}-\frac{|s|}{3}} = CN^{-|s|}$; or, which is the same, $\delta = CN^{\frac{-6|s|}{1-2|s|}}$ and $T/\delta = CTN^{\frac{6|s|}{1-2|s|}}$. Now, since $\bar{k} > CN^\alpha$, if we choose N in such a way that $CN^\alpha > CTN^{\frac{6|s|}{1-2|s|}}$, then we will have a solution of the problem with initial datum $u_{0\delta}$ in the interval $[0, \frac{T}{\delta}]$. This is possible if

$$\alpha > \frac{6|s|}{1-2|s|}. \tag{4.5}$$

If s is such that $\frac{6|s|}{1-2|s|} < 2 - 3\gamma$, then we can find α which satisfies (4.5) and the hypotheses of Lemma 1.2. This last inequality is satisfied by an allowed value of $\gamma > \frac{1}{2}$ if $\frac{6|s|}{1-2|s|} < \frac{1}{2}$; i.e. for $s \in (-\frac{1}{14}, 0)$.

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