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ENERGY DECAY ESTIMATES FOR LIENARD'S EQUATION WITH QUADRATIC VISCOUS FEEDBACK

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Abstract. This article concerns the stabilization for a well-known Lienard's system of ordinary differential equations modelling oscillatory phenomena. It is known that such a system is asymptotically stable when a linear viscous (motion-activated) damping with constant gain is engaged. However, in many applications it seems more realistic that the aforementioned gain is not constant and does depend on the deviation from equilibrium. In this article, we consider a (nonlinear) gain, introduced in [2], which is proportional to the square of such deviation and derive an explicit energy decay estimate for solutions of the corresponding "damped" Lienard's system. We also discuss the place of our result in the framework of stabilization of so-called critical bilinear systems.

1. Introduction

Motivation and main results. This article concerns the stabilization of a single oscillatory motion (or, more generally, of an oscillatory phenomenon) by means of suitable damping. Without loss of generality (that is, one might need to perform a routine change of variables first) such motion is described by a two dimensional system of ordinary differential equations of Lienard's type like

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x + uBx, \tag{1.1}$$

where $x = (x_1, x_2)$ and uBx models a damping device of structure described by matrix B with gain $u = u(x_1, x_2)$.

A system like (1.1) is widely used to model various kinds of periodic oscillatory phenomena, for example, in electrical or civil engineering (see, e.g., [4, 7] and the references therein). In particular, if we assume that (1.1) models a motion of a particle, then $x_1(t)$ would describe the deviation (position) of this particle with respect to equilibrium $x_1 = 0$ at time t, while $-x_2(t)$ would describe its velocity.

One of the most typical damping devices used in applications is so-called the "*viscous (or motion-activated)" damping* which, in the framework of model (1.1),

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is associated with the matrix

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and a *negative* gain u, in which case system (1.1) looks as follows:

$$\begin{gathered} \dot{x}_1 = -x_2, \quad t > 0, \\ \dot{x}_2 = x_1 + u x_2, \end{gathered} \tag{1.2}$$

$$x_1(0) = x_{10}, x_2(0) = x_{20}. \tag{1.3}$$

Note that the damping term in (1.2) is active only when the "velocity" $x_2(t)$ is not zero and it "acts" in the direction which is opposite of the "motion of the particle" at any given time t.

It is classical result that system (1.2)–(1.3) is asymptotically stable when a linear viscous damping with a negative constant gain is engaged, that is, when u in (1.2) is a negative number. (Note that in this case one can obtain the explicit formula for solutions.) However, it seems more realistic in many applications that the aforementioned gain u should depend on deviation from equilibrium. In this respect we refer the reader to Jurdjevic and Quinn [4] who considered the *quadratic* gain

$$u(x_1, x_2) = -x_1^2, \tag{1.4}$$

in which case the gain is proportional to the square of the magnitude of deviation. It was shown in [4] that system (1.2)–(1.4) is asymptotically stable. On the other hand, the method of [4] does not provide any explicit estimates for the energy decay of the corresponding solutions, which evaluate the effectiveness of nonlinear feedback (1.4) and thus are of principal importance in applications.

The main goal of this paper is to derive such an *explicit energy decay estimate* for system (1.2)–(1.4). It is clear from the start that it cannot be of exponential type, because system (1.2)–(1.4) is critical (see the discussion in the next section for details). We have the following result.

Theorem 1.1. *There exist positive constants β, γ, and $\bar{t} = \bar{t}(\| (x_{10}, x_{20}) \|_{R^2})$ such that the energy*

$$E(t) = \frac{x_1^2(t) + x_2^2(t)}{2}$$

of system (1.2)–(1.4) *satisfies*

$$E(t) \leq \frac{E(0)}{\beta + \gamma E(0) t}, \quad t > \bar{t}. \tag{1.5}$$

The values of the constants β and γ are given in the proof of this theorem (see (2.69) below).

Problem background: Critical bilinear systems. System (1.2)–(1.4) (or (1.1)) is an important principal case of planar, so-called critical bilinear systems (BLS), which are of traditional interest in the context of asymptotic stabilization (see, e.g., [2] for more detail). Let us remind the reader that a planar BLS of the form

$$\dot{x} = Ax + u(x_1, x_2) Bx, \ x(0) = x_0, \tag{1.6}$$

is called *critical* if the given matrices A and B satisfy the following two conditions:

- For each $u^\star \in R$ at least one eigenvalue of $A + u^\star B$ has nonnegative real part.

- There exists a $u_0 \in R$ for which all the real parts of the eigenvalues of $A + u_0 B$ are non positive.

Note that the latter condition implies that system (1.6) is stable when the constant feedback $u(x_1, x_2) = u_0$ is engaged, while the former condition implies that (1.6) is not asymptotically stabilizable by any constant feedback.

It is well-known (e.g., [2]) that, by the standard linear change of coordinates and time-rescaling, all the matrices A satisfying the above-cited two conditions can be transformed either into the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as our matrix A in (1.1) or in (1.2)–(1.4), associated with a periodic oscillatory motion, or into one of the following three "simpler" matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We refer the reader to [1, 2, 3, 5] for various positive and negative results on stabilization of planar critical BLS by means of constant, linear, quadratic and piecewise constant feedback laws. However, it seems that all the available results do not provide any explicit energy decay rates.

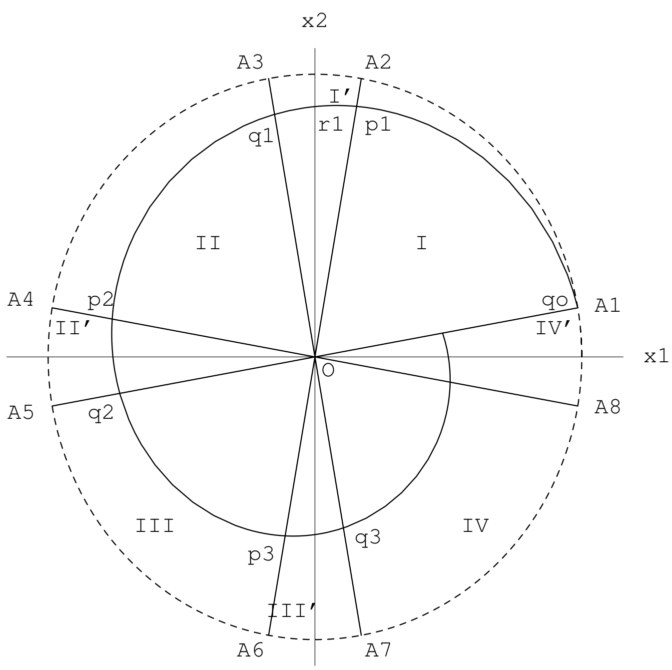

FIGURE 1. Trajectory for system (1.2)–(1.4)

2. Proof of Theorem 1.1

Our plan for this section as follows: First of all, as we are concerned with the large time behavior of (1.2)–(1.4), without loss of generality (see Step 8 for the general case), we can assume that

$$\{(x_{10}, x_{20})\} \subset B = \{(\xi, \eta) | \xi^2 + \eta^2 \le \frac{1}{2}\}. \tag{2.1}$$

Select any angle $\theta_0 \in (0, \frac{\pi}{4})$ and split the ball B into eight sectors I, $I,'$ II, $II,'$ III, III' and IV,IV' as shown on Figure 1, so that the central angles $\angle A_1OA_8$, $\angle A_2OA_3$, $\angle A_4OA_5$ and $\angle A_6OA_7$ are all equal to $2\theta_0$. Again, without loss of generality, we may assume that the initial point (x_{10}, x_{20}) is located at point A_1, also denoted by q_0 on Figure 1 (see Step 8 otherwise), where respectively the solid line starting from q_0 describes the trajectory $(x_1(t), x_2(t))$ of system (1.2)–(1.4) as it crosses all the eight aforementioned sectors.

Our plan is to derive the energy decay estimate for the solution to (1.2)–(1.4) starting from q_0 by evaluating its energy subsequently in each of the above mentioned sectors $I - IV'$. Then we extend this result to the general case.

Remark 2.1. Below we use the notation $x_1 = x_1(t)$, $x_2 = x_2(t)$, keeping in mind that $(x_1, x_2) = (x_1(t), x_2(t))$ represent the solution to (1.2)–(1.4) at hand.

Step 1: Denote by t_{p_1} the time required for the trajectory of (1.2)–(1.4) starting from $q_0 = (x_1(0), x_2(0)) = (x_{10}, x_{20})$ to reach point p_1. We have (see Figure 1):

$$x_1(t) \ge \sqrt{2E(t_{p_1})}\sin(\theta_0), \quad x_2(t) \ge \sqrt{2E(0)}\sin(\theta_0) \tag{2.2}$$

and hence

$$x_1^2(t)x_2^2(t) \ge 4E(0)E(t_{p_1})\sin^4(\theta_0). \tag{2.3}$$

Multiplying equations (1.2)–(1.4) respectively by x_1 and x_2 and adding them, we obtain:

$$x_1(t)\dot{x_1}(t) + x_2(t)\dot{x_2}(t) = \frac{dE(t)}{dt} = \frac{1}{2}\frac{d \parallel x(t) \parallel^2_{R^2}}{dt} = -x_1^2(t)x_2^2(t). \tag{2.4}$$

Integrating this equation over $(0, t_{p_1})$ yields:

$$E(t_{p_1}) = \frac{x_1^2(t_{p_1}) + x_2^2(t_{p_1})}{2} = -\int_0^{t_{p_1}} x_1^2x_2^2 dt + E(0). \tag{2.5}$$

Applying (2.3) (2.5), we obtain

$$E(t_{p_1}) \le E(0) - 4E(0)E(t_{p_1})\sin^4(\theta_0)t_{p_1}, \tag{2.6}$$

which yields:

$$E(t_{p_1}) \le \frac{E(0)}{1 + 4E(0)\delta^2 t_{p_1}}, \tag{2.7}$$

where $\delta = \sin^2(\theta_0)$. Thus, to estimate $E(t_{p_1})$, we need to evaluate t_{p_1}. This evaluation is accomplished in steps 2-4 below, based on the "visualization technique," [5] which makes use of the phase-portrait of system (1.2)–(1.4). Namely, to evaluate t_{p_1}, we intend to derive first explicit estimates both (a) for the length of arc of the trajectory of (1.2)–(1.4) lying in sector I and (b) for the speed at which the point describing the position of system (1.2)–(1.4) at time t moves along this arc. These estimates, obtained in terms of the energy function, will provide us with the bounds for the duration of the time-interval required by the system to traverse sector I, that is, for t_{p_1}. It turns out that the derived bounds for t_{p_1}, do not depend on the energy of the system in sector I.

Remark 2.2. From (2.4) we conclude that the distance $\| x(t) \|_{R^2}$ monotonically decreases as $t \to \infty$. This in turn implies that the trajectory initiated at q_0 remains in B for all $t > 0$. In other words, system (1.2)–(1.4) is stable (see also [4]).

Step 2: Estimate for $\sqrt{\dot{x_1}(t)^2 + \dot{x_2}(t)^2}$ in sector I. In this step we obtain an estimate for the value of the speed $\sqrt{\dot{x_1}^2(t) + \dot{x_2}^2(t)}$ at which the point $(x_1(t), x_2(t))$ moves in sector I. Since $(x_1, x_2) \in B$, in sector I we have

$$0 < x_1 \le \frac{1}{\sqrt{2}}, \quad 0 < x_2 \le \frac{1}{\sqrt{2}} \quad \text{and} \quad x_1 x_2 \le \frac{1}{2}. \tag{2.8}$$

Hence, using (1.2) and applying (2.8),

$$\dot{x_1} = -x_2, \quad \dot{x_2} = x_1 - x_1^2 x_2 = x_1(1 - x_1 x_2) \ge \frac{x_1}{2}. \tag{2.9}$$

Therefore, from (2.9) we derive:

$$\dot{x_1}^2 + \dot{x_2}^2 \ge \frac{x_1^2}{4} + x_2^2 \ge \frac{x_1^2 + x_2^2}{4} = \frac{E(t)}{2}. \tag{2.10}$$

Since $\dot{E}(t) \le 0$, in view of (2.10), we have the following estimates:

$$\sqrt{\dot{x_1}^2(t) + \dot{x_2}^2(t)} \ge \sqrt{\frac{E(t)}{2}} \ge \frac{\sqrt{E(t_{p_1})}}{\sqrt{2}}, \quad t \in [0, t_{p_1}]. \tag{2.11}$$

Applying the triangular inequality to the second equation in (1.2) and using (2.8), we obtain:

$$|\dot{x_2}(t)| = |x_1(t) - x_1^2(t) x_2(t)| \le |x_1(t)| + x_1^2(t)|x_2(t)| \le |x_1(t)| + \frac{|x_1(t)|}{2} = \frac{3|x_1(t)|}{2}. \tag{2.12}$$

Remark 2.3. Note that inequalities (2.10)–(2.12) hold everywhere in B.

Using (2.12) and the first equation in (1.2), we further derive that for $t \in [0, t_{p_1}]$,

$$\sqrt{\dot{x_1}^2(t) + \dot{x_2}^2(t)} \le \sqrt{\frac{9x_1^2(t)}{4} + x_2^2(t)} \le \sqrt{\frac{9(x_1^2(t) + x_2^2(t))}{4}} \le \frac{3\sqrt{E(0)}}{\sqrt{2}}. \tag{2.13}$$

Combining equations (2.11) and (2.13) yields

$$\frac{\sqrt{E(t_{p_1})}}{\sqrt{2}} \le \sqrt{\dot{x_1}^2(t) + \dot{x_2}^2(t)} \le \frac{3\sqrt{E(0)}}{\sqrt{2}}, \quad t \in [0, t_{p_1}]. \tag{2.14}$$

These estimates will be used to evaluate t_{p_1} in step 4 below.

Step 3: Estimate for $|q_0 p_1|$. In this step we obtain an estimate for the length of the curve $q_0 p_1$ connecting the points q_0 and p_1 (see Figure 1). It follows from (2.8) and (2.9) that the curve $q_0 p_1$ is rising as t increases in $[0, t_{p_1}]$. Hence, we can describe it geometrically as a graph of some monotone decreasing function $x_2 = x_2(x_1)$, $x_1 \in [\sqrt{2E(t_{p_1})} \sin(\theta_0), \sqrt{2E(0)} \cos(\theta_0)]$ and

$$|q_0 p_1| = \int_{\sqrt{2E(t_{p_1})} \sin(\theta_0)}^{\sqrt{2E(0)} \cos(\theta_0)} \sqrt{1 + (\frac{dx_2}{dx_1})^2} dx_1. \tag{2.15}$$

Using the first equation of (1.2) and the fact that $\dot{x_2}(t)$ is strictly positive on the interval $t \in [0, t_{p_1}]$, we have from (2.12):

$$\dot{x_1} = -x_2(t), \dot{x_2} \le \frac{3x_1(t)}{2}. \tag{2.16}$$

Note that in sector I, $x_1(t)$ and $x_2(t)$ are both positive, while $\dot{x_1}(t)$ is strictly negative. Hence,

$$\frac{\dot{x_2}}{-\dot{x_1}} \leq \frac{3x_1(t)}{2x_2(t)}, t \in [0, t_{p_1}]. \tag{2.17}$$

Therefore, [8, pp.105], we conclude that

$$|\frac{dx_2}{dx_1}| < \frac{3x_1(t)}{2x_2(t)}, t \in [0, t_{p_1}]. \tag{2.18}$$

Thus,

$$1 + (\frac{dx_2}{dx_1})^2 \leq \frac{9(x_1^2(t) + x_2^2(t))}{4x_2^2(t)} = \frac{9E(t)}{2x_2^2(t)}, \quad t \in [0, t_{p_1}]. \tag{2.19}$$

Using (2.19) and the fact that $E(t)$ is a decreasing function of t, we further obtain that

$$1 + (\frac{dx_2}{dx_1})^2 \leq \frac{9E(0)}{2x_2(t)^2} \quad \text{or} \quad \sqrt{1 + (\frac{dx_2}{dx_1})^2} \leq \frac{3\sqrt{E(0)}}{\sqrt{2}x_2(t)}, \quad t \in [0, t_{p_1}]. \tag{2.20}$$

Since $\dot{x_2}(t)$ is strictly positive,

$$\sqrt{2E(0)}\sin(\theta_0) \leq x_2(t) \leq \sqrt{2E(t_{p_1})}\cos(\theta_0) \leq \sqrt{2E(t_{p_1})}, \quad t \in [0, t_{p_1}]. \tag{2.21}$$

Applying (2.21) to (2.20) gives

$$\sqrt{1 + (\frac{dx_2}{dx_1})^2} \leq \frac{3}{2\sin(\theta_0)}, \quad t \in [0, t_{p_1}]\,. \tag{2.22}$$

Remark 2.4. Note that, in view of Remark 2.1, inequality (2.22) also holds in sector I', namely in the portion that lies in the first quadrant.

Combining (2.15) with (2.22) yields:

$$\begin{aligned} |q_0 p_1| &= \int_{\sqrt{2E(t_{p_1})}\sin(\theta_0)}^{\sqrt{2E(0)}\cos(\theta_0)} \sqrt{1 + (\frac{dx_2}{dx_1})^2} dx_1 \\ &\leq \frac{3(\sqrt{2E(0)}\cos(\theta_0) - \sqrt{2E(t_{p_1})}\sin(\theta_0))}{2\sin(\theta_0)}. \end{aligned} \tag{2.23}$$

Moreover, since $E(0) > E(t_{p_1})$ and $\cos(\theta_0) \geq \sin(\theta_0)$ for any $\theta_0 \in [0, \frac{\pi}{4}]$, further from (2.23), we obtain

$$|q_0 p_1| = \int_{\sqrt{2E(t_{p_1})}\sin(\theta_0)}^{\sqrt{2E(0)}\cos(\theta_0)} \sqrt{1 + (\frac{dx_2}{dx_1})^2} dx_1 \ \leq \frac{3\sqrt{E(0)}}{\sqrt{2}\sin(\theta_0)}. \tag{2.24}$$

We shall now evaluate the length of the curve $q_0 p_1$ from below. Once again, (2.9) yields

$$\frac{\dot{x_2}}{-\dot{x_1}} = |\frac{dx_2}{dx_1}| \geq \frac{x_1(t)}{2x_2(t)}, t \in [0, t_{p_1}]. \tag{2.25}$$

Therefore,

$$1 + (\frac{dx_2}{dx_1})^2 \geq \frac{4x_1^2(t) + x_2^2(t)}{4x_2(t)^2} \geq \frac{x_1^2(t) + x_2^2(t)}{4x_2(t)^2}, \quad t \in [0, t_{p_1}].$$

Applying (2.24) to this inequality, we obtain

$$\sqrt{1 + (\frac{dx_2}{dx_1})^2} \geq \frac{\sqrt{E(t_{p_1})}}{\sqrt{2}x_2(t)}, \quad t \in [0, t_{p_1}]. \tag{2.26}$$

Combining (2.21) with (2.26) yields

$$\sqrt{1+(\frac{dx_2}{dx_1})^2} \geq \frac{1}{2}, \quad t \in [0, t_{p_1}]. \tag{2.27}$$

Applying (2.27) to (2.15), we obtain the following estimate for the length of the curve q_0p_1 from below:

$$\begin{aligned} |q_0p_1| &= \int_{\sqrt{2E(t_{p_1})}\sin(\theta_0)}^{\sqrt{2E(0)}\cos(\theta_0)} \sqrt{1+(\frac{dx_2}{dx_1})^2}dx_1 \\ &\geq \frac{(\sqrt{2E(0)}\cos(\theta_0) - \sqrt{2E(t_{p_1})}\sin(\theta_0))}{2} \geq \frac{K}{\sqrt{2}}\sqrt{E(0)}, \end{aligned} \tag{2.28}$$

where $K = \cos(\theta_0) - \sin(\theta_0)$. Combining (2.24) and (2.28), we finally obtain:

$$\frac{3\sqrt{E(0)}}{\sqrt{2}\sin(\theta_0)} \geq |q_0p_1| \geq \frac{K}{\sqrt{2}}\sqrt{E(0)}. \tag{2.29}$$

Step 4: Estimate for t_{p_1}. In this step we shall derive an estimate for t_{p_1}, i.e., the time required for the trajectory of (1.2)–(1.4), initiated at q_0, to reach the point p_1. We begin by computing the lower bound estimate for t_{p_1}. Making use of (2.14)and (2.29), we obtain that

$$\frac{K}{\sqrt{2}}\sqrt{E(0)} \leq |q_0p_1| = \int_0^{t_{p_1}} \sqrt{\dot{x_1}^2(t) + \dot{x_2}^2(t)}dt \leq \frac{3\sqrt{E(0)}}{\sqrt{2}}t_{p_1}, \tag{2.30}$$

which in turn yields:

$$\frac{K}{3} \leq t_{p_1}. \tag{2.31}$$

Analogously, it follows from (2.14) and (2.29) that

$$\frac{\sqrt{E(t_{p_1})}}{\sqrt{2}}t_{p_1} \leq |q_0p_1| = \int_0^{t_{p_1}} \sqrt{\dot{x_1}^2(t) + \dot{x_2}^2(t)}dt \leq \frac{3\sqrt{E(0)}}{\sqrt{2}\sin(\theta_0)}, \tag{2.32}$$

which implies

$$t_{p_1} \leq \frac{3\sqrt{E(0)}}{\sin(\theta_0)\sqrt{E(t_{p_1})}}. \tag{2.33}$$

Combining (2.31) and (2.33) yields

$$\frac{K}{3} \leq t_{p_1} \leq \frac{3\sqrt{E(0)}}{\sin(\theta_0)\sqrt{E(t_{p_1})}}. \tag{2.34}$$

Since $\dot{x_2}(t)$ is strictly positive in sector I, we have $x_2(t_{p_1}) \geq x_2(0)$. Hence,

$$E(t_{p_1}) = \frac{x_1^2(t_{p_1}) + x_2^2(t_{p_1})}{2} \geq \frac{x_1^2(t_{p_1})}{2} \geq \frac{x_2^2(0)}{2} = E(0)\sin^2(\theta_0)$$

(see Figure 1), which implies that

$$\sqrt{\frac{E(0)}{E(t_{p_1})}} \leq \frac{1}{\sin(\theta_0)}. \tag{2.35}$$

Applying inequality (2.35) to (2.34) yields:

$$\frac{K}{3} \leq t_{p_1} \leq \frac{3}{\sin^2(\theta_0)}. \tag{2.36}$$

Step 5: Estimate for the time taken by the trajectory of (1.2)–(1.4) **to pass sectors** I **and** I'**.** In this step we shall derive an estimate for t_{q_1}, i.e., for the time required by the trajectory of (1.2)–(1.4), initiated at q_0, to reach point q_1 (see Figure 1). Split the trajectory in sector I' into two arcs p_1r_1 and r_1q_1 (see Figure 1). Denote the time required by the trajectory of (1.2)–(1.4) to connect the points p_1 and r_1 by $t_{p_1r_1}$ and the time required to connect the points r_1 and q_1 by $t_{r_1q_1}$.

Step 5.1: Estimate for time $t_{p_1r_1}$**.** The arc p_1r_1 can be described geometrically as a graph of function $x_2 = x_2(x_1)$, $x_1 \in [0, \sqrt{2E(t_{p_1})}\sin(\theta_0)]$. Then, applying the inequality (2.22) (see Remarks 2.1 and 2.4) to (2.37) in sector I', we obtain

$$|p_1r_1| = \int_0^{\sqrt{2E(t_{p_1})}\sin(\theta_0)} \sqrt{1+(\frac{dx_2}{dx_1})^2}dx_1 \le \frac{3\sqrt{2E(t_{p_1})}\sin(\theta_0)}{2\sin(\theta_0)} \le \frac{3\sqrt{E(t_{p_1})}}{\sqrt{2}}. \tag{2.37}$$

Since $\dot{E}(t) < 0$ for any time $t \in [t_{p_1}, t_{p_1} + t_{p_1r_1}]$, we have from (2.10),

$$\sqrt{\dot{x_1}^2(t)+\dot{x_2}^2(t)} \ge \sqrt{\frac{x_1^2(t)+x_2^2(t)}{4}} = \frac{\sqrt{E(t)}}{\sqrt{2}} \ge \frac{\sqrt{E(t_{p_1}+t_{p_1r_1})}}{\sqrt{2}}. \tag{2.38}$$

Hence, in view of (2.37),

$$\frac{\sqrt{E(t_{p_1}+t_{p_1r_1})}}{\sqrt{2}}t_{p_1r_1} \le |p_1r_1| = \int_{t_{p_1}}^{t_{p_1}+t_{p_1r_1}} \sqrt{\dot{x_1}^2(t)+\dot{x_2}^2(t)}dt \le 3\frac{\sqrt{E(t_{p_1})}}{\sqrt{2}}, \tag{2.39}$$

which yields

$$t_{p_1r_1} \le \frac{3\sqrt{E(t_{p_1})}}{\sqrt{E(t_{p_1}+t_{p_1r_1})}}. \tag{2.40}$$

Since $\dot{x_2}(t)$ is positive, we have $x_2(t_{p_1}+t_{p_1r_1}) \ge x_2(t_{p_1})$. Hence,

$$E(t_{p_1}+t_{p_1r_1}) = \frac{x_2^2(t_{p_1}+t_{p_1r_1})}{2} \ge \frac{x_2^2(t_{p_1})}{2} = E(t_{p_1})\cos^2(\theta_0),$$

which implies

$$\sqrt{\frac{E(t_{p_1})}{E(t_{p_1}+t_{p_1r_1})}} \le \frac{1}{\cos(\theta_0)}. \tag{2.41}$$

Finally, combining (2.40) and (2.41), we obtain the following estimate for $t_{p_1r_1}$,

$$t_{p_1r_1} \le \frac{3}{\cos(\theta_0)}. \tag{2.42}$$

Step 5.2: Estimate for the time $t_{r_1q_1}$**.** Once again, the arc r_1q_1 can be described geometrically as the graph of a function $x_2 = x_2(x_1)$, with x_1 in the interval $[-\sqrt{2E(t_{p_1}+t_{p_1r_1}+t_{r_1q_1})}\sin(\theta_0), 0]$. Then

$$|r_1q_1| = \int_{-\sqrt{2E(t_{p_1}+t_{p_1r_1}+t_{r_1q_1})}\sin(\theta_0)}^{0} \sqrt{1+(\frac{dx_2}{dx_1})^2}dx_1. \tag{2.43}$$

Note that $x_1(t)$, $\dot{x_1}(t)$, and $\dot{x_2}(t)$ are strictly negative on the interval $t \in [t_{p_1} + t_{p_1r_1}, t_{p_1} + t_{p_1r_1} + t_{r_1q_1}]$, where $x_2(t)$ is positive. Since it follows from (1.2)–(1.4) and (2.12) (see Remark 2.3) that

$$-\dot{x_2} \le -\frac{3x_1}{2} \quad \text{and} \quad \dot{x_1} = -x_2,$$

we have

$$0 \le \frac{\dot{x_2}}{\dot{x_1}} \le \frac{3x_1(t)}{-2x_2(t)}, \quad t \in [t_{p_1} + t_{p_1 r_1}, t_{p_1} + t_{p_1 r_1} + t_{r_1 q_1}]. \tag{2.44}$$

Therefore,

$$1+(\frac{dx_2}{dx_1})^2 \le \frac{9(x_1^2(t)+x_2^2(t))}{4x_2^2(t)} \le \frac{9E(t)}{2x_2^2(t)}, \quad t \in [t_{p_1}+t_{p_1 r_1}, t_{p_1}+t_{p_1 r_1}+t_{r_1 q_1}]. \tag{2.45}$$

Using (2.45) and the fact that $E(t)$ is a decreasing function of t, we obtain

$$\sqrt{1+(\frac{dx_2}{dx_1})^2} \le \frac{3\sqrt{E(t_{p_1}+t_{p_1 r_1})}}{\sqrt{2}x_2(t)}. \tag{2.46}$$

Since $\dot{x_2}(t)$ is strictly negative, for t in $[t_{p_1} + t_{p_1 r_1}, t_{p_1} + t_{p_1 r_1} + t_{r_1 q_1}]$,

$$\sqrt{2E(t_{p_1}+t_{p_1 r_1}+t_{r_1 q_1})}\cos(\theta_0) \le x_2(t) \le \sqrt{2E(t_{p_1}+t_{p_1 r_1})}. \tag{2.47}$$

Applying (2.47) to (2.46) gives

$$\sqrt{1+(\frac{dx_2}{dx_1})^2} \le \frac{3\sqrt{E(t_{p_1}+t_{p_1 r_1})}}{\sqrt{2}\sqrt{2E(t_{p_1}+t_{p_1 r_1}+t_{r_1 q_1})}\cos(\theta_0)}, \tag{2.48}$$

for $t \in [t_{p_1} + t_{p_1 r_1}, t_{p_1} + t_{p_1 r_1} + t_{r_1 q_1}]$. Thus applying inequality (2.48) to (2.43), we obtain an upper estimate for the length of the arc $r_1 q_1$:

$$\begin{aligned} |r_1 q_1| &= \int_{-\sqrt{2E(t_{p_1}+t_{p_1 r_1}+t_{r_1 q_1})}\sin(\theta_0)}^{0} \sqrt{1+(\frac{dx_2}{dx_1})^2}dx_1 \\ &\le \frac{3\sqrt{2E(t_{p_1}+t_{p_1 r_1})}\tan(\theta_0)}{\sqrt{2}}. \end{aligned} \tag{2.49}$$

Since $\dot{E}(t) < 0$ for any $t \in [t_{p_1} + t_{p_1 r_1}, t_{p_1} + t_{p_1 r_1} + t_{r_1 q_1}]$, making use of (2.10) (see Remark 2.3), we derive that

$$\sqrt{\dot{x_1}^2(t)+\dot{x_2}^2(t)} \ge \sqrt{\frac{x_1^2(t)+x_2^2(t)}{4}} \ge \frac{\sqrt{E(t_{p_1}+t_{p_1 r_1}+t_{r_1 q_1})}}{\sqrt{2}}. \tag{2.50}$$

Using inequalities (2.49) and (2.50), we obtain

$$\begin{aligned} \frac{\sqrt{E(t_{p_1}+t_{p_1 r_1}+t_{r_1 q_1})}}{\sqrt{2}} t_{r_1 q_1} \le |r_1 q_1| &= \int_{t_{p_1}+t_{p_1 r_1}}^{t_{p_1}+t_{p_1 r_1}+t_{r_1 q_1}} \sqrt{\dot{x_1}^2(t)+\dot{x_2}^2(t)}dt \\ &\le \frac{3\sqrt{E(t_{p_1}+t_{p_1 r_1})}\tan(\theta_0)}{\sqrt{2}}. \end{aligned} \tag{2.51}$$

which, in turn, provides

$$t_{r_1 q_1} \le \frac{3\tan(\theta_0)\sqrt{E(t_{p_1}+t_{p_1 r_1})}}{\sqrt{E(t_{p_1}+t_{p_1 r_1}+t_{r_1 q_1})}}. \tag{2.52}$$

Once again, integrating (2.4) over $(t_{p_1} + t_{p_1 r_1}, t_{p_1} + t_{p_1 r_1} + t_{r_1 q_1})$ yields (compare with (2.6))

$$E(t_{p_1}+t_{p_1 r_1}) = E(t_{p_1}+t_{p_1 r_1}+t_{r_1 q_1}) + \int_{t_{p_1}+t_{p_1 r_1}}^{t_{p_1}+t_{p_1 r_1}+t_{r_1 q_1}} x_1^2 x_2^2 dt. \tag{2.53}$$

Observe that

$$x_2^2(t) \leq \frac{1}{2} \quad \text{and} \quad 0 \leq x_1^2(t) \leq 2E(t_{p_1} + t_{p_1 r_1} + t_{r_1 q_1}) \sin^2(\theta_0)$$

for any $t \in [t_{p_1} + t_{p_1 r_1}, t_{p_1} + t_{p_1 r_1} + t_{r_1 q_1}]$. Hence, it follows from (2.53) that

$$E(t_{p_1} + t_{p_1 r_1}) \leq E(t_{p_1} + t_{p_1 r_1} + t_{r_1 q_1})(1 + t_{r_1 q_1} \sin^2(\theta_0)). \tag{2.54}$$

Hence,

$$\sqrt{\frac{E(t_{p_1} + t_{p_1 r_1})}{E(t_{p_1} + t_{p_1 r_1} + t_{r_1 q_1})}} \leq \sqrt{1 + \sin^2(\theta_0) t_{r_1 q_1}}. \tag{2.55}$$

Applying inequality (2.55) to (2.52) yields:

$$0 < t_{r_1 q_1} \leq 3 \tan(\theta_0) \sqrt{1 + \sin^2(\theta_0) t_{r_1 q_1}}, \tag{2.56}$$

which gives the following estimate for $t_{r_1 q_1}$:

$$0 \leq t_{r_1 q_1} \leq C = \frac{9 \tan^2(\theta_0) \sin^2(\theta_0) + 3 \tan(\theta_0) \sqrt{9 \tan^2(\theta_0) \sin^4(\theta_0) + 4}}{2}. \tag{2.57}$$

Step 5.3: Estimate for time t_{q_1}. Combining (2.36), (2.42) and (2.57) yields

$$\frac{K}{3} \leq t_{q_1} = t_{p_1} + t_{p_1 r_1} + t_{r_1 q_1} \leq \frac{3}{\sin^2(\theta_0)} + \frac{3}{\cos(\theta_0)} + C. \tag{2.58}$$

Step 6: Estimates for $E(t_{p_1})$ and $E(t_{q_1})$. Substituting the lower bound for t_{p_1} from (2.36) into (2.7) and recalling that since $E(t)$ is decreasing in time, we obtain from (2.7) that

$$E(t_{q_1}) \leq E(t_{p_1}) \leq \frac{E(0)}{1 + 4E(0)\delta^2 \frac{K}{3}}. \tag{2.59}$$

Step 7: Estimate for $E(t)$ for any time $t > 0$. It is not hard to see that in the next two sectors II and II' one can apply the same strategy as described in the above steps 1-6, namely, leading to estimate (2.59). Denote the time required by the trajectory of (1.2)–(1.4) to reach points p_2 and q_2 from q_0 respectively by t_{p_2} and t_{q_2}. Then, similarly as in (2.59) and (2.36), (2.58) (see Remark 2.3), we obtain:

$$E(t_{q_2}) \leq E(t_{p_2}) \leq \frac{E(t_{q_1})}{1 + 4E(t_{q_1})\delta^2 (t_{p_2} - t_{q_1})} \leq \frac{E(t_{q_1})}{1 + 4E(t_{q_1})\delta^2 \frac{K}{3}}, \tag{2.60}$$

where, analogously to (2.58), we have:

$$\frac{K}{3} \leq t_{p_2} - t_{q_1} \leq t_{q_2} - t_{q_1} \leq \frac{3}{\sin^2(\theta_0)} + \frac{3}{\cos(\theta_0)} + C.$$

Remark 2.5. Consider the function $f(x) = \frac{x}{1+\kappa x}$, with κ a positive constant which is associated with the right hand sides in (2.59), (2.60), considered as functions of energy. Since $f'(x) > 0$ for all $x > 0$, $f(x)$ is a monotonically increasing function.

Since $E(t_{q_1}) \leq E(t_{p_1})$ and (2.59) holds, Remark 2.5 implies

$$E(t_{q_2}) \leq \frac{\frac{E(0)}{1+4E(0)\delta^2 \frac{K}{3}}}{1 + 4\frac{E(0)}{1+4E(0)\delta^2 \frac{K}{3}} \delta^2 \frac{K}{3}} = \frac{E(0)}{1 + \frac{2K}{3} 4E(0)\delta^2}. \tag{2.61}$$

Denote the following subsequent intersecting points of the trajectory at hand with the lines $OA_6, OA_7, \dots$ by $p_3, q_3, \dots, p_n, q_n, \dots$. Analogously, we have

$$E(t_{q_n}) \leq E(t_{p_n}) \leq \frac{E(0)}{1 + \frac{nK}{3} 4E(0)\delta^2}, \quad \forall n = 1, \dots. \tag{2.62}$$

Consider any $t > 0$. Then there exists a positive integer n such that

$$t_{q_{n-1}} \leq t \leq t_{q_n} \leq \frac{3n}{\sin^2(\theta_0)} + \frac{3n}{\cos(\theta_0)} + nC, \tag{2.63}$$

where, thus,

$$\frac{t}{\frac{3}{\sin^2(\theta_0)} + \frac{3}{\cos(\theta_0)} + C} \leq n. \tag{2.64}$$

Note that, since $t > t_{q_{n-1}}$ and $E(t)$ decreases in time, (2.62) also implies that

$$E(t) \leq E(t_{q_{n-1}}) \leq \frac{E(0)}{1 + \frac{(n-1)K}{3} 4E(0)\delta^2}. \tag{2.65}$$

Combined with (2.63), this gives the following energy decay estimate for any $t > 0$ in the case when the initial state (x_{10}, x_{20}) lies in B on the line OA_1 as shown on Figure 1:

$$E(t) \leq \frac{E(0)}{1 - \frac{4KE(0)\delta^2}{3} + \frac{4KE(0)\delta^2}{3}(\frac{3}{\sin^2(\theta_0)} + \frac{3}{\cos(\theta_0)} + C)^{-1} t}. \tag{2.66}$$

Step 8: The general case. Consider any initial point (x_{10}, x_{20}). Since (1.2)–(1.4) has an isolated equilibrium point $(0, 0)$, by Poincare-Bendixson theorem (see [7]), the trajectory to (1.2)–(1.4) starting from (x_{10}, x_{20}) after some finite time will enter the ball B and intersect the line OA_1. Denote the time of this intersection by t_0. Then, by the scheme discussed in steps 1-7, we have

$$E(t) \leq \frac{E(t_0)}{1 - \frac{4KE(t_0)\delta^2}{3} + \alpha E(t_0)(t - t_0)}, \quad t \geq t_0, \tag{2.67}$$

where

$$\alpha = \frac{4K \sin^2(\theta_0) \cos(\theta_0) \sin^4(\theta_0)}{3C \sin^2(\theta_0) \cos(\theta_0) + 9 \sin^2(\theta_0) + 9 \cos(\theta_0)}.$$

Using (2.1) (i.e., $E(t_0) \leq \frac{1}{4}$) and the argument of Remark 2.5, we derive from (2.67) that

$$\begin{aligned} E(t) &\leq \frac{E(t_0)}{1 - \frac{4E(t_0)K\delta^2}{3} + \alpha E(t_0)(t - t_0)} \\ &\leq \frac{E(t_0)}{1 - \frac{K\delta^2}{3} + \alpha E(t_0)(t - t_0)} \\ &\leq \frac{E(0)}{1 - \frac{K\delta^2}{3} + \alpha E(0)(1 - \frac{t_0}{t})t} \quad t > t_0. \end{aligned} \tag{2.68}$$

Choose $\bar{t} = 2t_0$ and denote

$$\beta = 1 - \frac{K \sin^4(\theta_0)}{3}, \quad \gamma = \frac{\alpha}{2}. \tag{2.69}$$

Recall that $\delta = \sin^2(\theta_0)$ and θ_0 is any number in $(0, \frac{\pi}{4})$.

Then (2.68) implies (1.2). This concludes the proof of Theorem 1.1.

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