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GLOBAL ATTRACTORS FOR A CLASS OF DEGENERATE DIFFUSION EQUATIONS

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Abstract. In this paper we give two existence results for a class of degenerate diffusion equations with p-Laplacian. One is on a unique global strong solution, and the other is on a global attractor. It is also shown that the global attractor coincides with the unstable set of the set of all stationary solutions. As a by-product, an a-priori estimate for solutions of the corresponding elliptic equations is obtained.

1. Introduction and Results

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain of class C^2 with boundary $\partial\Omega$. We consider the degenerate diffusion equation

$$\begin{gathered} u_t = \lambda\Delta_p u + f(u), \quad (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty), \end{gathered} \tag{1.1}$$

with the initial condition $u(x,0) = u_0(x)$ in Ω, where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 2$. We assume that $f \in C^1(\mathbb{R})$, $f(0) = 0$, and one of the following two conditions is satisfied:

(F1) $p > N$ and $\limsup_{|s|\to+\infty} \frac{f'(s)}{(p-1)|s|^{p-2}} < \lambda_1\lambda$;

(F2) $p \leq N$ and $\sup_{s\in\mathbb{R}} f'(s) < +\infty$,

where λ_1 is the first eigenvalue of $-\Delta_p$ with the Dirichlet boundary condition and is characterized by $\lambda_1 = \inf\{\|\nabla u\|_p^p/\|u\|_p^p; u \in W_0^{1,p}(\Omega)\setminus\{0\}\} \in (0,+\infty)$. For example of f satisfying the above conditions, we can give $f(s) = s$; $f(s) = |s|^{q-2}s(1-|s|^r)$ with $q \geq 2$ and $r > 0$; $f(s) = |s|^{q-2}s$ with $q \in (2,p)$ and $p > N$; and $f(s) = |s|^{p-2}s$ with $p > N$ when $\lambda > 1/\lambda_1$. It is important that we assume only one sided boundedness on f'.

For semilinear parabolic equations, i.e. $p = 2$, there are many studies on the existence of global attractors and on the existence of solutions; see for example Temam [14]. A fundamental result in this field appeared on the paper [8] by Marion. He assumes that f has a polynomial growth nonlinearity and becomes negative for sufficiently large u, and that f' is bounded from above. Under these conditions, he showed that a global attractor of (1.1) in $L^2(\Omega)$ exists and is bounded in $L^\infty(\Omega)$.

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In fact, the boundedness can be proved even in $D(\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$; see [14]. Moreover, it was also proved that the fractal and Hausdorff dimensions of the global attractor are both finite, which roughly means that solutions of (1.1) eventually behave with a finite number of "degrees of freedom" as $t \to +\infty$. The analysis for the dimensions relies on the method of linearization, which is very operative tool to investigate the time-local behavior of solutions.

This article concerns the degenerate case; i.e., $p > 2$. We start with the existence of solutions for (1.1). The Galerkin method is well-known for constructing (weak) solutions of partial differential equations (see e.g., Tsutsumi [15]). However, the method of monotone operators gives a more straightforward proof of the existence of (strong) solutions than the Galerkin method, when available. Indeed, Ôtani [11] extended the abstract theory of monotone operators of Brézis [1] to nonlinear evolution equations with a difference term of subdifferentials, and then succeeded in obtaining better properties of solutions of $u_t = \Delta_p u + |u|^{q-2}u$ than those had been known in [15] by the Galerkin method. He also proved in [12] that the solution converges to the set of all stationary solutions (c.f., Theorem 1.3 below). For our first result, we use the method in [11] to establish the existence of unique global strong solutions of (1.1) and give regularity properties. The definition of (global) strong solutions is given in Definition 2.1, below.

Theorem 1.1. *Let $N \geq 1$, $p > 2$ and $f \in C^1(\mathbb{R})$ with $f(0) = 0$. Assume that either (F1) or (F2) is satisfied. Let $u_0 \in L^2(\Omega)$. Then for any $T > 0$ there exists a unique strong solution $u \in C([0,T]; L^2(\Omega))$ of (1.1) in $[0,T]$ with $u(0) = u_0$, and u can be extended to a global strong solution which is denoted again by u. Moreover, u satisfies*

$$u \in C^{0,1}_{\rm loc}((0,+\infty); L^2(\Omega)) \cap C^{0,\frac{1}{p}}_{\rm loc}((0,+\infty); W_0^{1,p}(\Omega)), \tag{1.2}$$

$$u \in C^{\alpha}(\overline{\Omega} \times [\delta, T]) \quad \text{for all } \alpha \in (0,1), \tag{1.3}$$

$$\nabla u \in C^{\alpha}(\overline{\Omega} \times [\delta, T]) \quad \text{for some } \alpha \in (0,1), \tag{1.4}$$

$$u_t \in L^2(\delta, +\infty; L^2(\Omega)),\ u \in L^{\infty}(\delta, +\infty; W_0^{1,p}(\Omega)), \tag{1.5}$$

$$t^{1-\frac{1}{\sigma}} u_t,\ t^{1-\frac{1}{\sigma}} \Delta_p u \in L^{\sigma}(0,T; L^2(\Omega)) \quad \text{for all } \sigma \in [2,+\infty], \tag{1.6}$$

where δ and T $(0 < \delta < T < +\infty)$ are arbitrary.

Remark 1.2. Under the assumption (F1), the uniqueness and local existence of strong solutions with $u(0) = u_0 \in W_0^{1,p}(\Omega)$ follow from Ishii [7, Theorem 3.3], since f is locally Lipschitz in $L^2(\Omega)$ with the domain $W_0^{1,p}(\Omega)$. However, it seems that his proof can not be applied in case of (F2). We give a unified proof for "weak" reactions (F1) and (F2), and obtain some regularity properties for initial data in $L^2(\Omega)$.

Due to Theorem 1.1, (1.1) produces a nonlinear semigroup on $L^2(\Omega)$ and it is significant to investigate the asymptotic behavior of solutions, which induces us to study a global attractor. Global attractors for degenerate diffusion equations with a Lipschitz perturbation have been discussed in [14, Section III.5]. A few years ago, Carvalho, Cholewa and Dlotko [3] proved the existence results of solutions and a global attractor for abstract evolution equations with a maximal monotone operator and a globally Lipschitz perturbation, which involve [14]. Our following theorem extends their results to non globally Lipschitz perturbation, and furthermore we give regularity results of the global attractor and its characterization by the set of all stationary solutions for (1.1) (though it is assumed that the diffusion term is represented by a subdifferential of functions).

Theorem 1.3. *Suppose the same conditions as in Theorem 1.1. Then there exists a connected global attractor $\mathcal{A}_\lambda$ in $L^2(\Omega)$ of (1.1). $\mathcal{A}_\lambda$ and $\{\Delta_p\phi; \phi \in \mathcal{A}_\lambda\}$ are bounded in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and in $L^2(\Omega)$, respectively. Moreover, $\mathcal{A}_\lambda = \mathcal{M}_+(\mathcal{E}_\lambda)$, where $\mathcal{E}_\lambda$ consists of all solutions $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of*

$$\lambda\Delta_p\phi + f(\phi) = 0, \quad x \in \Omega, \tag{1.7}$$

and $\mathcal{M}_+(\mathcal{E}_\lambda)$ is the unstable set of $\mathcal{E}_\lambda$. (For the definition of unstable sets, see Definition 2.3)

Theorem 1.3 assures that, for instance, the equation $u_t = \lambda\Delta_p u + u$ has a global attractor for all $\lambda > 0$. However, the equation $u_t = \lambda\Delta u + u$ no longer has a global attractor for $\lambda \le 1/\lambda_1$. This is the simplest and most remarkable distinction on asymptotic behavior of solutions between two cases $p > 2$ and $p = 2$ (see also [3]).

A self-evident fact that $\mathcal{A}_\lambda$ contains $\mathcal{E}_\lambda$ prompts us to observe as follows. It is known that $\mathcal{E}_\lambda$ is generally contained in $C^{1,\alpha}$ and not to C^2 even if Ω and f are in C^∞. Indeed, we can explicitly compose such solutions (see Takeuchi and Yamada [13, Remark 3.2]). Therefore we can not expect that $\mathcal{A}_\lambda$ is included in C^2, though Ω and f are both very smooth. Next, if $\mathcal{E}_\lambda$ is discrete, fortunately, then the global attractor can be exactly represented by the union of the unstable sets associated to the functions in $\mathcal{E}_\lambda$, i.e., $\mathcal{A}_\lambda = \bigcup_{\phi\in\mathcal{E}_\lambda} \mathcal{M}_+(\phi)$ (c.f., Temam [14, Theorem VII.4.1]). However, since $\mathcal{E}_\lambda$ often includes some continua (c.f., [13, Theorems 3.1–3.3]), we have no conclusion about it from the abstract theory for dynamical systems.

In addition, concerning the p-Laplacian, we note that there is no guarantee for the validity of linearization. This seems to be the reason why equations with the p-Laplacian are not extensively treated in terms of dynamical systems.

Remark 1.4. Dung [6] has obtained the ultimately uniform boundedness of solutions and gradients for degenerate parabolic systems including (1.1) with bounded initial data, and shown the existence of a global attractor in the space of bounded continuous functions only under the Neumann boundary conditions. Note that we are not subject to the boundedness for initial data.

Remark 1.5. It is possible to relax the assumptions on f if one pays no attention to the uniqueness of solutions. Even in this case, we may be able to show only the existence of global attractors, which is especially defined for multivalued semiflow (see Valero [16]).

As a by-product of Theorem 1.3, an a-priori (uniform) estimate for solutions of the elliptic equation (1.7) is immediately deduced.

Corollary 1.6. *Suppose the same conditions as in Theorem 1.1. Then there exists a positive constant M_λ such that $\|\phi\|_{C^{1,\alpha}(\overline{\Omega})} \le M_\lambda$ for all $\phi \in \mathcal{E}_\lambda$.*

The contents of this paper are as follows. Section 2 is devoted to the preliminaries in which we define strong solutions and global attractors, and give some lemmas. We will prove Theorems 1.1 and 1.3 in Sections 3 and 4, respectively.

2. Preliminaries

In this section we give some definitions and elementary lemmas. Throughout this paper, $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$, $1 \le p \le \infty$, are the usual Lebesgue and Sobolev spaces with norms $\|\cdot\|_p$ and $\|\nabla\cdot\|_p$, respectively. The scalar product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. $C^\alpha(\overline{\Omega}\times[\delta,T])$, $0<\alpha<1$, is the Hölder space with norm

$$[u]_{\alpha,\overline{\Omega}\times[\delta,T]} = \sup_{(x,t)\in\overline{\Omega}\times[\delta,T]} |u(x,t)| + \sup_{(x,t),(y,\tau)\in\overline{\Omega}\times[\delta,T]} \frac{|u(x,t)-u(y,\tau)|}{|x-y|^\alpha + |t-\tau|^{\alpha/p}}.$$

Also, $C^{1,\alpha}(\overline{\Omega})$, $0<\alpha<1$, is the usual Hölder space.

Definition 2.1. A function $u \in C([0,T];L^2(\Omega))$ is called a *strong solution of* (1.1) *in* $[0,T]$ *with* $u(0)=u_0$ if u is locally absolutely continuous on $(0,T)$, $u(t) \in W_0^{1,p}(\Omega)$, $\Delta_p u(t) \in L^2(\Omega)$, $f(u(t)) \in L^2(\Omega)$ for a.a. $t \in (0,T)$ and u satisfies

$$\begin{gathered} u_t = \lambda\Delta_p u + f(u) \quad \text{a.e. in } \Omega\times(0,T), \\ u(0) = u_0 \quad \text{a.e. in } \Omega. \end{gathered}$$

Moreover, we say that a function $u \in C([0,+\infty);L^2(\Omega))$ is a *global strong solution of* (1.1) if u is a strong solution of (1.1) in $[0,T]$ with $u(0)=u_0$ for every $T>0$.

Definition 2.2. Let $\{S(t)\}_{t\ge0}$ be a semigroup on $L^2(\Omega)$. An *attractor* for the semigroup $\{S(t)\}_{t\ge0}$ is a set $\mathcal{A}\subset L^2(\Omega)$ satisfying the following two properties:

1. $\mathcal{A}$ is an invariant set under $\{S(t)\}_{t\ge0}$, i.e., $S(t)\mathcal{A}=\mathcal{A}$ for all $t\ge0$, and
2. $\mathcal{A}$ possesses an open neighborhood $\mathcal{U}$ such that for every $u_0\in\mathcal{U}$, $S(t)u_0$ converges to $\mathcal{A}$ as $t\to+\infty$:

$$\inf_{y\in\mathcal{A}} \|S(t)u_0 - y\|_2 \to 0 \quad \text{as } t\to+\infty.$$

We say that $\mathcal{A}\subset L^2(\Omega)$ is a *global attractor* for the semigroup $\{S(t)\}_{t\ge0}$ if $\mathcal{A}$ is a compact attractor that attracts any bounded sets of $L^2(\Omega)$:

$$\sup_{x\in S(t)B}\inf_{y\in\mathcal{A}} \|x-y\|_2 \to 0 \quad \text{as } t\to+\infty$$

for any bounded set $B\subset L^2(\Omega)$.

Definition 2.3. The *unstable set* $\mathcal{M}_+(X)$ of $X\subset L^2(\Omega)$ is the (possibly empty) set of points u_* which belong to a complete orbit $\{u(t);t\in\mathbb{R}\}$ such that

$$\inf_{y\in X} \|u(t)-y\|_2 \to 0 \quad \text{as } t\to-\infty.$$

Lemma 2.4 (Ghidaglia's inequality)**.** *Let $y(\cdot)$ be a positive absolutely continuous function on $(0,+\infty)$ which satisfies*

$$y' + \gamma y^{\frac{p}{2}} \le \delta$$

with $p>2$, $\gamma>0$ and $\delta\ge0$. Then for $t>0$

$$y(t) \le \Big(\frac{\delta}{\gamma}\Big)^{\frac{2}{p}} + \Big(\frac{\gamma(p-2)}{2}t\Big)^{-\frac{2}{p-2}}.$$

For the proof of this lemma, see Temam [14, Lemma III.5.1].

Define a function $(u-M)_+ := \max\{u-M,0\}$ for a function u and a constant M, and $\chi[u>\alpha]$ denotes the characteristic function of the set $\{x\in\Omega;u(x)>\alpha\}$.

Lemma 2.5. *Let $\{k_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of nonnegative numbers. Then for any $u \in L^2(\Omega)$*

$$\Big(\int_\Omega u(u-k_{n+1})_+ dx\Big)^{1/2} \le \frac{\|(u-k_n)_+\|_2}{1-\frac{k_n}{k_{n+1}}}. \tag{2.1}$$

Proof. Easily we obtain estimates that yield

$$\begin{aligned}
\big(1-\frac{k_n}{k_{n+1}}\big)^2 \int_\Omega u(u-k_{n+1})_+ dx &\le \int_\Omega \big(u-k_n\frac{u}{k_{n+1}}\big)^2 \cdot \chi[u>k_{n+1}]dx \\
&\le \int_\Omega (u-k_n)^2 \cdot \chi[u>k_{n+1}]dx \\
&\le \|(u-k_n)_+\|_2^2,
\end{aligned}$$

which implies (2.1). □

Lemma 2.6. *Let $\{t_n\}_{n=0}^{\infty}$ and $\{k_n\}_{n=0}^{\infty}$ be strictly increasing sequences of nonnegative numbers. Then for any $u \in L^\infty_{\rm loc}(0,+\infty;L^2(\Omega)) \cap L^p_{\rm loc}(0,+\infty;W_0^{1,p}(\Omega))$, the function*

$$Y_n(t) = \int_{t_n}^t \|(u-k_n)_+(s)\|_2^2 ds, \quad t>t_n, \tag{2.2}$$

satisfies

$$Y_{n+1}^{q/2} \le \frac{C_0\big(\operatorname{ess\,sup}_{t_{n+1}<s<t}\|(u-k_{n+1})_+(s)\|_2^2\big)^{p/N}\int_{t_{n+1}}^t \|\nabla((u-k_{n+1})_+)(s)\|_p^p ds}{(k_{n+1}-k_n)^{q-2}} Y_n^{\frac{q-2}{2}} \tag{2.3}$$

for all $t>t_{n+1}$ and some constant $C_0>0$, where $q=(N+2)p/N$.

Proof. By the Hölder and the Gagliardo-Nirenberg inequality

$$\begin{aligned}
Y_{n+1}^{q/2} &= \Big(\int_{t_{n+1}}^t \int_\Omega (u-k_{n+1})_+^2 \cdot \chi[u>k_{n+1}]dxds\Big)^{q/2} \\
&\le \int_{t_{n+1}}^t \|(u-k_{n+1})_+\|_q^q ds \cdot |A_{n+1}|^{\frac{q-2}{2}} \\
&\le C_0\Big(\operatorname*{ess\,sup}_{(t_{n+1},t)}\|(u-k_{n+1})_+\|_2^2\Big)^{p/N}\int_{t_{n+1}}^t \|\nabla((u-k_{n+1})_+)\|_p^p ds \cdot |A_{n+1}|^{\frac{q-2}{2}},
\end{aligned}$$

where $|A_{n+1}|$ denotes the Lebesgue measure of $\{(x,s)\in\Omega\times[t_n,t]; u(x,s)>k_{n+1}\}$. Combining this with

$$Y_n \ge \int_{t_n}^t \int_\Omega (u-k_n)_+^2 \cdot \chi[u>k_{n+1}]\,dx\,ds \ge (k_{n+1}-k_n)^2|A_{n+1}|,$$

we obtain (2.3). □

Finally we provide a simple, but nice bright lemma.

Lemma 2.7. *Let $\{Y_n\}_{n=0}^{\infty}$ be a sequence of positive numbers, satisfying that there exist $a>0$, $b>1$ and $\theta>0$ such that*

$$Y_{n+1} \le ab^n Y_n^{1+\theta}, \ n=0,1,2,\dots. \tag{2.4}$$

Then $Y_0 \le a^{-1/\theta}b^{-1/\theta^2}$ implies that $Y_n \to 0$ as $n\to+\infty$.

Proof. The lemma is introduced in the book of DiBenedetto [5, Lemma I.4.1] without its proof. Though it is proved easily, we show it here for confirmation. Using the recursive inequality (2.4) repeatedly, we have

$$Y_n \le a^{\frac{(1+\theta)^n-1}{\theta}} b^{\frac{(1+\theta)^n-1-\theta n}{\theta^2}} Y_0^{(1+\theta)^n} \le a^{-\frac{1}{\theta}} b^{-\frac{1+\theta n}{\theta^2}} \to 0$$

as $n \to +\infty$. □

3. Proof of Theorem 1.1

Let κ be a positive constant such that

$$\kappa < 1 - \max\Big\{ \limsup_{|s|\to+\infty} \frac{f'(s)}{\lambda_1\lambda(p-1)|s|^{p-2}},\ 0\Big\} \quad \text{when (F1) is satisfied;}$$

$$\kappa = 1 \quad \text{when (F2) is satisfied.}$$

Then there exists a constant $C_1 > 0$ such that $f'(s) \le (1-\kappa)\lambda_1\lambda(p-1)|s|^{p-2} + C_1$ for all $s \in \mathbb{R}$. Putting

$$g(s) := (1-\kappa)\lambda_1\lambda|s|^{p-2}s + C_1 s - f(s),$$

we can see that $g \in C^1(\mathbb{R})$, $g(0) = 0$, g is nondecreasing on $\mathbb{R}$, and equation (1.1) can be represented by

$$u_t - \lambda\Delta_p u + g(u) = (1-\kappa)\lambda_1\lambda|u|^{p-2}u + C_1 u. \tag{3.1}$$

Defining the following proper lower semi-continuous convex functions on $L^2(\Omega)$:

$$\varphi_1(u) := \begin{cases} \frac{\lambda}{p}\|\nabla u\|_p^p, & u \in W_0^{1,p}(\Omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

$$\varphi_2(u) := \begin{cases} \int_\Omega \int_0^u g(s)dsdx, & u \in L^2(\Omega) \quad \text{with } \int_0^u g(s)ds \in L^1(\Omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\psi(u) := \begin{cases} \frac{(1-\kappa)\lambda_1\lambda}{p}\|u\|_p^p + \frac{C_1}{2}\|u\|_2^2, & u \in L^p(\Omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

we rewrite (3.1) as

$$u_t + \partial\varphi_1(u) + \partial\varphi_2(u) = \partial\psi(u) \quad \text{in } (0,+\infty), \tag{3.2}$$

where $\partial\varphi(u)$ denotes the subdifferential of φ at u. Since $\partial\varphi_1 + \partial\varphi_2$ is m-accretive (maximal monotone) in $L^2(\Omega)$ (see Brézis, Crandall and Pazy [2, Theorem 3.1] and Okazawa [9, Theorem 1]), it follows that $\partial\varphi_1 + \partial\varphi_2 = \partial\varphi$, where $\varphi = \varphi_1 + \varphi_2$. Hence (3.2) is rewritten as

$$u_t + \partial\varphi(u) = \partial\psi(u) \quad \text{in } (0,+\infty). \tag{3.3}$$

The next lemma holds the key to establishing the existence of global strong solutions of (3.3).

Lemma 3.1. *Let κ, C_1, φ, φ_1 and ψ be as above. Then*

$$\|\partial\psi(u)\|_2 \le C_1\|u\|_2 + C_2(\varphi_1(u))^{1-\frac{1}{p}} \quad \textit{for all } u \in W_0^{1,p}(\Omega), \tag{3.4}$$

$$\psi(u) \le (1-\kappa)\varphi_1(u) + \frac{C_1}{2}\|u\|_2^2 \quad \textit{for all } u \in W_0^{1,p}(\Omega), \tag{3.5}$$

$$(\partial\psi(u), u) \le (1-\kappa)(\partial\varphi_1(u), u) + C_1\|u\|_2^2 \quad \textit{for all } u \in D(\partial\varphi_1) \tag{3.6}$$

for some constant $C_2 > 0$.

Proof of Lemma 3.1. It is clear that (3.4) and (3.5) follow from Sobolev's embedding theorem and Poincaré's inequality, respectively. Also, we can obtain by (3.5)

$$(\partial\psi(u), u) = p\Big(\psi(u) - (\frac{1}{2} - \frac{1}{p})C_1\|u\|_2^2\Big) \le (1-\kappa)p\varphi_1(u) + C_1\|u\|_2^2,$$

which proves (3.6). □

The set $\{u \in L^2(\Omega);\ \varphi(u) + \|u\|_2 \le L\}$ is compact in $L^2(\Omega)$ for every $L < +\infty$ by Rellich's theorem. Therefore, by the same argument as in the proof of Ôtani [11, Theorem 5.3] (use (3.5) and (3.6) instead of (5.11) in [11]), we see that for any $u_0 \in L^2(\Omega)$ and for any $T > 0$ there exists a strong solution $u \in C([0,T]; L^2(\Omega))$ of (1.1) in $[0,T]$ with $u(0) = u_0$ such that

$$\partial\varphi(u),\ \partial\psi(u) \in L^2(\delta, T; L^2(\Omega)) \quad \text{for all } \delta \in (0,T).$$

We need the following lemmas to prove (1.2)–(1.6) and the uniqueness.

Lemma 3.2. *Take* $T > 0$ *and* $\delta \in (0,T]$. *Let* u *be a strong solution of* (1.1) *in* $[0,T]$ *obtained as above. Then there exist positive constants* C_3 *and* C_4 *independent of* T *such that*

$$\frac{1}{2}\|u(T)\|_2^2 + \kappa\int_0^T \varphi(u(t))dt \le C_3T + \frac{1}{2}\|u_0\|_2^2, \tag{3.7}$$

$$\int_\delta^T \|u_t(t)\|_2^2dt + \frac{\kappa}{2}\varphi(u(T)) \le \varphi(u(\delta)) + C_4, \tag{3.8}$$

$$\int_0^T t\|u_t(t)\|_2^2dt + \kappa T\varphi_1(u(T)) \le c(T, \|u_0\|_2), \tag{3.9}$$

$$t^{1-\frac{1}{\sigma}}(\varphi_1(u))^{1-\frac{1}{p}} \in L^\sigma(0,T), \tag{3.10}$$

where $c(T, \|u_0\|_2) = (C_1T + 1/\kappa)(C_3T + \|u_0\|_2^2/2)$ *and* $\sigma \in [1, +\infty]$ *is arbitrary.*

Proof of Lemma 3.2. Taking the scalar product of (3.3) in $L^2(\Omega)$ with u and using (3.6) and some inequalities with $p > 2$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + (\partial\varphi(u), u) &= (\partial\psi(u), u)\\ &\le (1-\kappa)(\partial\varphi_1(u), u) + C_3 + \frac{\kappa(p-1)\lambda}{p}\|\nabla u\|_p^p\\ &= \big(1 - \frac{\kappa}{p}\big)(\partial\varphi_1(u), u) + C_3;\end{aligned}$$

so that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \frac{\kappa}{p}(\partial\varphi_1(u), u) + (\partial\varphi_2(u), u) \le C_3.$$

Since $(\partial\varphi_1(u), u) = p\varphi_1(u)$ and $(\partial\varphi_2(u), u) \ge \varphi_2(u)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \kappa\varphi(u(t)) \le C_3 \quad \text{for a.a. } t > 0. \tag{3.11}$$

Integrating this inequality gives (3.7).

Next, setting $J(u(t)) := \varphi(u(t)) - \psi(u(t))$, we see from (3.3) that

$$\frac{d}{dt}J(u(t)) = (\partial\varphi(u(t)) - \partial\psi(u(t)), u_t(t)) = -\|u_t(t)\|_2^2. \tag{3.12}$$

Integrating it over $[\delta, T]$ $(0 < \delta \le T)$ and using (3.5), we have

$$\begin{aligned}\int_\delta^T \|u_t(t)\|_2^2 dt + \varphi(u(T)) - \varphi(u(\delta)) &= \psi(u(T)) - \psi(u(\delta))\\ &\le (1-\kappa)\varphi(u(T)) + \frac{C_1}{2}\|u(T)\|_2^2\\ &\le (1-\kappa)\varphi(u(T)) + \frac{\kappa}{2}\varphi(u(T)) + C_4,\end{aligned}$$

which implies (3.8). Here we used Sobolev's embedding theorem and Young's inequality in the last inequality.

Multiplying (3.12) by $t \ge 0$ and integrating it over $[0, \tau]$, we have

$$\int_0^\tau t\|u_t(t)\|_2^2 dt + \tau J(u(\tau)) = \int_0^\tau J(u(t))dt.$$

Since $\kappa\varphi_1(u) - C_1\|u\|_2^2/2 \le J(u)$ $(\le \varphi(u))$ by (3.5), it follows that

$$\int_0^\tau t\|u_t(t)\|_2^2 dt + \kappa\tau\varphi_1(u(\tau)) \le \frac{C_1}{2}\tau\|u(\tau)\|_2^2 + \int_0^\tau \varphi(u(t))dt.$$

Setting $\tau = T$ and applying (3.7) to the right-hand side, we obtain (3.9), and

$$\tau^{1-\frac{1}{\sigma}}(\varphi_1(u(\tau)))^{1-\frac{1}{p}} \le \left(\frac{c(T, \|u_0\|_2)}{\kappa}\right)^{1-\frac{1}{p}} \frac{1}{\tau^{\frac{1}{\sigma}-\frac{1}{p}}} \in L^\sigma(0, T).$$

This is nothing but (3.10). □

Lemma 3.3. *Let u and $c(\cdot,\cdot)$ be as in Lemma 3.2. Then for any $t \in (0, T]$ there exists a constant $L(t) > 0$ such that*

$$\|u_t(t)\|_2 \le e^{L(t)}\|u_t(\delta)\|_2 \quad \text{for a.a. } t \ge \delta, \tag{3.13}$$

$$t^2\|u_t(t)\|_2^2 \le 2c(t, \|u_0\|_2)e^{2L(t)}. \tag{3.14}$$

Proof of Lemma 3.3. Let $0 < h < 1$. Then the monotonicity of $\partial\varphi$ implies that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u(t+h) - u(t)\|_2^2\\ &\le (\partial\psi(u(t+h)) - \partial\psi(u(t)), u(t+h) - u(t))\\ &\le (1-\kappa)\lambda_1\lambda(p-1)\int_\Omega \max\{|u(t+h)|^{p-2}, |u(t)|^{p-2}\}|u(t+h) - u(t)|^2\,dx\\ &\quad + C_1\|u(t+h) - u(t)\|_2^2\\ &\le K(\|\nabla u(t+h)\|_p^{p-2}, \|\nabla u(t)\|_p^{p-2})\|u(t+h) - u(t)\|_2^2,\end{aligned}$$

where $K(a, b) := (1-\kappa)\lambda_1\lambda(p-1)C_5 \max\{a, b\} + C_1$. Note that C_5 is given by the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ only if $p > N$, otherwise $\kappa = 1$. Applying Gronwall's inequality to the preceding estimate yields that for all $\delta > 0$

$$\|u(t+h) - u(t)\|_2 \le e^{\int_\delta^t K(\|\nabla u(s+h)\|_p^{p-2}, \|\nabla u(s)\|_p^{p-2})ds}\|u(\delta+h) - u(\delta)\|_2, \tag{3.15}$$

where $\int_\delta^t K ds$ is bounded with respect to h by (3.7) in Lemma 3.2. Dividing (3.15) by h and letting $h \to +0$, we obtain (3.13).

Applying (3.13): $\|u_t(T)\|_2 \le e^{L(T)}\|u_t(t)\|_2$ $(0 < t \le T)$ to the integrand of the first term on the left-hand side of (3.9), we obtain

$$e^{-2L(T)}\|u_t(T)\|_2^2 \int_0^T t dt \le c(T, \|u_0\|_2),$$

and hence (3.14) follows. □

Lemma 3.4. *Let u be as in Lemma 3.2. Then for any $T > 0$ there exists a constant $k_T > 0$ such that $u(t) \in L^\infty(\Omega)$ and*

$$\|u(t)\|_\infty \le k_T \quad \text{for all } t \in [\tfrac{T}{2}, T]. \tag{3.16}$$

Proof of Lemma 3.4. In case of (F1), the assertion is trivial by (3.8) with Sobolev's embedding theorem. We consider the case (F2). However, we note that the following proof does not need the condition $p \le N$ in (F2).

The key to the proof of (3.16) is to deduce a global iterative inequality (c.f., DiBenedetto [5, Chapter V]). Take any $T > 0$ and $k > 0$. Define sequences $\{t_n\}_{n=0}^\infty$, $\{k_n\}_{n=0}^\infty$ of nonnegative numbers and a sequence of functions $\{\zeta_n\}_{n=0}^\infty$ as follows:

$$t_n = \frac{T}{2}\big(1 - \frac{1}{2^n}\big), \quad k_n = k\big(1 - \frac{1}{2^n}\big) \quad \text{and}$$

$$\zeta_n(t) = \begin{cases} 0, & 0 \le t \le t_n, \\ \frac{t-t_n}{t_{n+1}-t_n}, & t_n < t < t_{n+1}, \\ 1, & t_{n+1} \le t \le T. \end{cases}$$

Differentiating $\|(u-k_{n+1})_+(s)\|_2^2\zeta_n(s)$ with respect to s and using (3.1) with $\kappa = 1$ and $(u - k_{n+1})_+g(u) \ge 0$, we obtain

$$\begin{aligned}
&\frac{d}{ds}(\|(u - k_{n+1})_+\|_2^2\zeta_n) + 2\lambda\|\nabla((u - k_{n+1})_+)\|_p^p\zeta_n \\
&\le \|(u - k_{n+1})_+\|_2^2\zeta_n' + 2C_1 \int_\Omega u(u - k_{n+1})_+\zeta_n dx.
\end{aligned}$$

Integrating this over $[t_n, t]$ with $t_{n+1} \le t \le T$ and noting the properties of ζ_n, we have

$$\begin{aligned}
&\|(u - k_{n+1})_+(t)\|_2^2 + 2\lambda \int_{t_{n+1}}^t \|\nabla((u - k_{n+1})_+)\|_p^p ds \\
&\le \frac{2^{n+2}}{T} \int_{t_n}^t \|(u - k_{n+1})_+\|_2^2 ds + 2C_1 \int_{t_n}^t \int_\Omega u(u - k_{n+1})_+ dx ds \\
&\le \frac{2^{n+2}}{T} \int_{t_n}^t \|(u - k_n)_+\|_2^2 ds + 2C_1(2^{n+1} - 1)^2 \int_{t_n}^t \|(u - k_n)_+\|_2^2 ds,
\end{aligned}$$

where we used an obvious inequality and Lemma 2.5 in the second inequality. Thus

$$\begin{aligned}
&\sup_{[t_{n+1},T]} \|(u - k_{n+1})_+\|_2^2 + 2\lambda \int_{t_{n+1}}^T \|\nabla((u - k_{n+1})_+)\|_p^p ds \\
&\le C_6 \left(1 + \frac{1}{T}\right) 4^n \int_{t_n}^T \|(u - k_n)_+\|_2^2 ds
\end{aligned} \tag{3.17}$$

for some constant $C_6 > 0$. Now we put Y_n as (2.2) with $t = T$ and it follows from (3.17) and (2.3) in Lemma 2.6 that (2.4) in Lemma 2.7 is satisfied with

$$a = a_k = \frac{C_7}{k^{\frac{2}{q}(q-2)}}\big(1+\frac{1}{T}\big)^{\frac{2}{q}(1+\frac{p}{N})},$$
$$b = 4^{1+\frac{2p}{Nq}} \ (>1), \quad \theta = \frac{2p}{Nq}, \quad q = \frac{(N+2)p}{N},$$

where C_7 is a positive constant. Since it is possible to take $k = k_T$ sufficiently large as

$$Y_0 \le \int_0^T \|u(s)\|_2^2 ds \le a_k^{-\frac{1}{\theta}} b^{-\frac{1}{\theta^2}}, \tag{3.18}$$

Lemma 2.7 gives $Y_n \to 0$ as $n \to +\infty$. Hence

$$\int_{\frac{T}{2}}^T \|(u-k_T)_+\|_2^2 ds = 0,$$

which implies that $u \le k_T$ a.e. in $\Omega \times [T/2, T]$. The same argument holds true for $-u$ so that Lemma 3.4 is established. □

Now we are in a position to complete the proof of Theorem 1.1. By Lemma 3.4 we see that $f(u) \in L^\infty(\delta, T; L^\infty(\Omega))$, and hence DiBenedetto [5, Theorems X.1.1 and X.1.2] (see also Chen and DiBenedetto [4, Theorems 1 and 2]) yields (1.3), (1.4) and

$$\|u(t)\|_{C^{1,\alpha}(\overline{\Omega})} \le \gamma(p, N, \delta, T) \quad \text{for all } t \in [\delta, T], \tag{3.19}$$

where $\gamma(p, N, \delta, T)$ depends also on $\int_\delta^T \|\nabla u(t)\|_p^p dt$.

The first claim of (1.6) is proved by (3.9) in Lemma 3.2 and (3.14) in Lemma 3.3 because $\|t^{1-1/\sigma} u_t\|_2^\sigma = \|\sqrt{t} u_t\|_2^2 \|t u_t\|_2^{\sigma-2}$. Since $(\partial\varphi_1(u), \partial\varphi_2(u)) \ge 0$ (see [2, p.138, l.6] and [9, (5)]), we have

$$\begin{aligned} \|\partial\varphi_1(u)\|_2 &\le \|\partial\varphi(u)\|_2 \\ &\le \|u_t\|_2 + \|\partial\psi(u)\|_2 \quad \text{by (3.3)} \\ &\le \|u_t\|_2 + C_1\|u\|_2 + C_2(\varphi_1(u))^{1-\frac{1}{p}} \quad \text{by (3.4)}. \end{aligned} \tag{3.20}$$

Multiplying (3.20) by $t^{1-1/\sigma}$ and integrating it over $[0, T]$, we obtain the second claim of (1.6) by virtue of the first one and (3.10).

In view of (3.13) in Lemma 3.3 we have the first claim of (1.2). It follows from Tartar's inequality that

$$\begin{aligned} &\|\nabla u(t) - \nabla u(s)\|_p^p \\ &\le 2^{p-2} \int_\Omega (|\nabla u(t)|^{p-2}\nabla u(t) - |\nabla u(s)|^{p-2}\nabla u(s)) \cdot (\nabla u(t) - \nabla u(s))\, dx \\ &\le 2^{p-2}(\|\Delta_p u(t)\|_2 + \|\Delta_p u(s)\|_2)\|u(t) - u(s)\|_2. \end{aligned}$$

Noting that $\Delta_p u \in L^\infty(\delta, T; L^2(\Omega))$ (see (1.6) with $\sigma = +\infty$) and applying the first claim to the right-hand side, we obtain (1.2) (c.f., Okazawa and Yokota [10] for (1.2) in case (F2) is satisfied).

The uniqueness of solutions of (1.1) in $[0, T]$ is proved as follows. Let u and v be strong solutions of (1.1) in $[0, T]$ with $u(0) = u_0 \in L^2(\Omega)$ and $v(0) = v_0 \in L^2(\Omega)$, respectively. As in the proof of (3.15), we have

$$\|u(t) - v(t)\|_2 \le e^{\int_0^t K(\|\nabla u(s)\|_p^{p-2}, \|\nabla v(s)\|_p^{p-2}) ds} \|u_0 - v_0\|_2.$$

This implies the uniqueness of solutions of (1.1) in $[0,T]$.

Finally, since $T>0$ is arbitrary, we see that u can be extended uniquely to a global strong solution of (1.1). Noting that C_4 in (3.8) of Lemma 3.2 is independent of T, we obtain (1.5).

Remark 3.5. To prove the first claim of (1.2), we have used (3.13). If (3.14) is employed instead of (3.13), then we see that

$$\|u(t)-u(s)\|_2 \le e^{L(T)}\sqrt{2c(T,\|u_0\|_2)}\cdot|\log t-\log s|,\quad t,\ s\in(0,T].$$

Remark 3.6. Let $f(u)$ be replaced by the spatially inhomogeneous reaction $f(x,u)$. If $f\in C^1(\overline{\Omega}\times\mathbb{R})$, $f(x,0)=0$ for every $x\in\Omega$ and either (F1) or (F2) is satisfied uniformly with respect to $x\in\Omega$, then under some condition on $\nabla_x f$ (see Okazawa [9]), we can prove the unique existence of global strong solutions of (1.1) with $f(u)$ replaced by $f(x,u)$.

4. Proof of Theorem 1.3

Thanks to Theorem 1.1, an operator $S(t):L^2(\Omega)\to L^2(\Omega)$ for each $t\ge 0$ is well defined by $S(t)u_0=u(t;u_0)$. Then it is easy to verify that the family of operators $\{S(t)\}_{t\ge0}$ enjoys the C^0-*semigroup* properties on $L^2(\Omega)$, that is, $\{S(t)\}_{t\ge0}$ is a semigroup and the mapping $(t,u_0)\mapsto S(t)u_0$ from $(0,+\infty)\times L^2(\Omega)$ into $L^2(\Omega)$ is continuous.

Proof of Theorem 1.3. Let κ, C_1 and C_3 be the same constants defined in the proof of Theorem 1.1. It is sufficient to show the existence of a compact absorbing set in $L^2(\Omega)$ for the semigroup $\{S(t)\}_{t\ge0}$ (see, e.g., Temam [14, Theorem I.1.1]).

From (3.11) in the proof of Lemma 3.2, in particular

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2+C_8\|u(t)\|_2^p\le C_3$$

for some $C_8>0$. Hence, Ghidaglia's inequality (Lemma 2.4 with $y(t)=\|u(t)\|_2^2$) gives

$$\|u(t)\|_2^2\le\Big(\frac{C_3}{C_8}\Big)^{\frac{2}{p}}+(C_8(p-2)t)^{-\frac{2}{p-2}}\quad\text{for all } t>0. \tag{4.1}$$

Next, it follows from (3.12) that $J(u(t))$ is nonincreasing in $t>0$ and hence

$$J(u(t+1))\le\int_t^{t+1}J(u(s))ds\le\int_t^{t+1}\varphi(u(s))ds \tag{4.2}$$

when $t>0$. By (3.5) in Lemma 3.1, we obtain $J(u(t+1))\ge\kappa\varphi_1(u(t+1))-C_1\|u(t+1)\|_2^2/2$. Moreover, integrating (3.11) over $[t,t+1]$ gives $\kappa\int_t^{t+1}\varphi(u(s))ds\le C_3+\|u(t)\|_2^2/2$. Applying these two inequalities to (4.2), we have

$$2\kappa^2\varphi_1(u(t+1))\le 2C_3+\|u(t)\|_2^2+\kappa C_1\|u(t+1)\|_2^2;$$

and hence (4.1) yields that there exist positive constants C_9 and C_{10} such that

$$\varphi_1(u(t))\le C_9+C_{10}((p-2)(t-1))^{-\frac{2}{p-2}}\quad\text{for all } t>1. \tag{4.3}$$

Since C_9 and C_{10} are independent of the solution, (4.3) implies that there exists a number $t_0>1$ such that $S(t)B\subset B_{\rho_0}(0)$ for any bounded set $B\subset L^2(\Omega)$ and $t\ge t_0$, where $B_{\rho_0}(0)=\{u\in W_0^{1,p}(\Omega);\|\nabla u\|_p\le\rho_0\}$ and $\lambda\rho_0^p/p>C_7$. Therefore

$B_{\rho_0}(0)$ is a compact absorbing set in $L^2(\Omega)$ and $\mathcal{A}_\lambda = \bigcap_{t\geq 0}\overline{\bigcup_{s\geq t} S(s)B_{\rho_0}(0)}$ is a connected global attractor in $L^2(\Omega)$. In addition, for all $\phi \in \mathcal{A}_\lambda$

$$\|\phi\|_2^2 \leq \Big(\frac{C_3}{C_8}\Big)^{2/p} \quad \text{and} \quad \|\nabla\phi\|_p^p \leq \frac{pC_9}{\lambda} < \rho_0^p. \tag{4.4}$$

Indeed, by the invariance property of global attractor, for every $\phi \in \mathcal{A}_\lambda$ there exists a $u_n \in \mathcal{A}_\lambda$ such that $S(n)u_n = u(n;u_n) = \phi$. Applying (4.1) and (4.3) to $u(t) = S(t)u_n$, and setting $t = n \to +\infty$, we obtain these estimates.

We will prove the boundedness of $\mathcal{A}_\lambda$ in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. It follows from (3.19) and (4.3) that

$$\|u(t)\|_{C^{1,\alpha}(\overline{\Omega})} \leq \gamma(p,N) \quad \text{for all } t \in [1,2], \tag{4.5}$$

where α and $\gamma(p,N)$ are independent of the solution. Now take any $\phi \in \mathcal{A}_\lambda$ and u_2 be as above. Applying (4.5) to $S(t)u_2$, we see that $\|S(t)u_2\|_{C^{1,\alpha}(\overline{\Omega})} \leq \gamma(p,N)$ for all $t \in [1,2]$. Setting $t = 2$, we obtain

$$\|\phi\|_{C^{1,\alpha}(\overline{\Omega})} \leq \gamma(p,N) \quad \text{for all } \phi \in \mathcal{A}_\lambda;$$

that is, $\mathcal{A}_\lambda$ is bounded in $C^{1,\alpha}(\overline{\Omega})$.

The boundedness of $\{\Delta_p\phi; \phi \in \mathcal{A}_\lambda\}$ in $L^2(\Omega)$ is also shown in a similar way. The solution $u(t;u_2) \in \mathcal{A}_\lambda$ satisfies (3.20). Multiplying it by t yields

$$\begin{aligned}
\|t\partial\varphi_1(u)\|_2 &\leq \|tu_t\|_2 + C_1t\|u\|_2 + C_2t(\varphi_1(u))^{1-\frac{1}{p}} \\
&\leq \|tu_t\|_2 + C_1t\|u\|_2 + \frac{C_2t}{p} + \frac{C_2(p-1)}{p}t\varphi_1(u) \\
&\leq \tilde{c}(t, \|u_0\|_2),
\end{aligned}$$

where $\tilde{c}(\cdot,\cdot)$ is a continuous function and increasing with respect to the first variable, determined by (3.14), (3.7) and (3.9). Hence, $\lambda\|\Delta_p u(t;u_2)\|_2 \leq \tilde{c}(2, \|u_2\|_2)$ for all $t \in [1,2]$. Since $u_2 \in \mathcal{A}_\lambda$ and (4.4) is satisfied, there exists a constant $C_{11} > 0$ such that $\|\Delta_p u(t;u_2)\|_2 \leq C_{11}$ for all $t \in [1,2]$. Setting $t = 2$, we conclude that

$$\|\Delta_p\phi\|_2 \leq C_{11} \quad \text{for all } \phi \in \mathcal{A}_\lambda.$$

Finally we show that $\mathcal{A}_\lambda = \mathcal{M}_+(\mathcal{E}_\lambda)$. Since $\mathcal{A}_\lambda$ is relatively compact in $C^1(\overline{\Omega})$, the function $J(u) = \varphi(u) - \psi(u)$ is continuous on $\mathcal{A}_\lambda$ with respect to the $L^2(\Omega)$-topology. This fact and (3.12) mean that $J : \mathcal{A}_\lambda \to \mathbb{R}$ is a Lyapunov function of $S(\cdot)$. Therefore it follows from [14, Theorem VII.4.1] that $\mathcal{A}_\lambda$ coincides with the unstable set of $\mathcal{E}_\lambda$. $\square$

Remark 4.1. The absorbing time t_0 of the absorbing set $B_{\rho_0}(0)$ is independent of the set of initial data B. Indeed, (4.3) implies that all solutions belong to $B_{\rho_0}(0)$ uniformly with respect to the initial data when $t \geq t_0$, where

$$t_0 := 1 + \frac{1}{p-2}\Big(\frac{pC_{10}}{\lambda\rho_0^p - pC_9}\Big)^{(p-2)/2}.$$

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