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REMARKS ON LEAST ENERGY SOLUTIONS FOR QUASILINEAR ELLIPTIC PROBLEMS IN $\mathbb{R}^N$

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Abstract. In this work we establish some properties of the solutions to the quasilinear second-order problem

$$-\Delta_p w = g(w) \quad \text{in } \mathbb{R}^N$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and $1 < p \leq N$. We study a mountain pass characterization of least energy solutions of this problem. Without assuming the monotonicity of the function $t^{1-p}g(t)$, we show that the Mountain-Pass value gives the least energy level. We also prove the exponential decay of the derivatives of the solutions.

1. Introduction

In this paper, we consider the quasilinear elliptic problem

$$-\Delta_p w = g(w) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and $1 < p \leq N$. Using variational methods, more precisely by a constrained minimization argument, we show the existence of ground states solutions (or least energy solutions) for the problem (1.1) in both cases, $1 < p < N$ and $p = N$. As it is well known, in the case $1 < p < N$ the nonlinearities are required to have polynomial growth at infinity in order to define the associated functionals in Sobolev spaces. Coming to the case $p = N$, much faster growth is allowed for the nonlinearity and the Trudinger-Moser inequality in $p = N$ replaces the Sobolev imbedding theorem used for $1 < p < N$.

In our study, we prove also that the Mountain-Pass value gives the least energy level and we obtain the exponential decay of the derivatives of the solutions of problem (1.1).

The knowledge of ground states plays a role in several applications in elliptic problems. For example in the study of various types of spike solutions, ground state serves as scaled limit profile of the solution near the spike [13].

There has been recently a good amount of work on this class of problem (1.1) in the semilinear case which corresponds to the case $p = 2$, see for example [2, 3, 11]. In these papers was investigated the existence of ground state solutions using the minimization argument. The characterization of the least energy level was

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investigated by Ding Ni [9] and Rabinowitz [16], under the monotonicity condition of the function $g(t)/t$. Recently, Jeanjean and Tanaka [11] have obtained this kind of result without this monotonicity assumption.

The study of the exponential decay of the solutions, in the semilinear case, was considered by Strauss [17], Berestycki and Lions [2], among others. Gongbao and Shusen [12] have showed the exponential decay for weak solution of a class of p-Laplacian equations. Under severe restrictions about the structure of the operator and the nature of the solutions, some exponential decay results have been obtained recently by Rabier and Stuart [15]. However, on these works the decay of derivatives for the degenerate case was not shown. In the present paper we prove the exponential decay of first derivatives for all radial solution of problem (1.1) by using an appropriated test function.

The operator $-\Delta_p$ with $p \neq 2$ arises from a variety of physical phenomena. It is used in non-Newtonian fluids, in some reaction-diffusion problems, as well as in flow through porous media. It also appears in nonlinear elasticity, glaciology and petroleum extraction [1].

Several papers have appeared recently about the p-Laplacian problems involving unbounded domains, among others Serrin-Tang [18], Serrin-Zhou [19], Do Ó [10], Hebey-Demengel [4] and Jianfu and Xi Ping [21]. We referred to their references for other related results.

For easy reference we state now the assumptions that will be assumed through this paper.

(G1) $g \in C(\mathbb{I}, \mathbb{R})$ and is odd;

(G2) when $1 < p < N$ we assume that
$$\lim_{u \to +\infty} \frac{g(u)}{u^{p^*-1}} = 0 \quad \text{where} \quad p^* = \frac{Np}{N-p};$$
when $p = N$ we assume that
$$|g(u)| \leq C[\exp(\alpha_0 |u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha_0, u)],$$
for some constants α_0, $C > 0$, where
$$S_{N-2}(\alpha_0, u) = \sum_{k=0}^{N-2} \frac{\alpha_0^k}{k!} |u|^{\frac{Nk}{N-1}};$$

(G3) when $1 < p < N$ we suppose that
$$-\infty < \liminf_{u \to 0^+} \frac{g(u)}{u^{p-1}} \leq \limsup_{u \to 0^+} \frac{g(u)}{u^{p-1}} = -\nu < 0,$$
and for $p = N$
$$\lim_{u \to 0} \frac{g(u)}{|u|^{N-1}} = -\nu < 0.$$

(G4) There exists $\zeta > 0$ such that $G(\zeta) > 0$, where $G(u) = \int_0^u g(t)dt$.

Example. Let $1 < p < N$ and consider the function
$$g(u) = \lambda |u|^{q-2} u - \mu |u|^{p-2} u,$$
where λ, μ are positive constants and $1 < p < q < p^* - 1$. It is not difficult to see that g satisfies the assumptions (G1)–(G4).

Example Assume that $p = N$ and. consider the function

$$g(u) = -\mu|u|^{N-2}u + |u|^{N-1}ue^{\beta|u|^{\frac{N}{N-1}}},$$

where $\beta > 0$ and $\mu > 0$. We can see that g satisfies the assumptions (G1)–(G4).

Notation. In this paper we make use of the following notation.

- For $1 \leq p \leq \infty$, $L^p(U)$, denotes Lebesgue spaces with the norm $\|u\|_{L^p(U)}$
- $W^{1,p}(\mathbb{R}^N)$ denote Sobolev spaces with the norm $\|u\|_{W^{1,p}(\mathbb{R}^N)}$
- $W^{1,p}_r(\mathbb{R}^N)$ denotes the subspace of $W^{1,p}(\mathbb{R}^N)$ formed by the radial functions
- $C^{k,\alpha}(U)$, with k a non-negative integer and $0 \leq \alpha < 1$, denotes Hölder spaces
- C, C_0, C_1, C_2, ... denote (possibly different) positive constants
- $|A|$ denotes Lebesgue measure of the set $A \subset \mathbb{R}^N$
- ω_{N-1} is the $(N-1)$-dimensional measure of the $N-1$ unit sphere in $\mathbb{R}^N$.

Variational Formulation. We begin by recalling the following Trundiger-Moser type inequality which is crucial for our variational argument. the Trudinger-Moser inequality for $p = N$ replaces the Sobolev imbedding theorem used for $1 < p < N$.

Lemma 1.1. *If $N \geq 2$, $\alpha > 0$ and $u \in W^{1,N}(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \Big[\exp\big(\alpha|u|^{\frac{N}{N-1}}\big) - S_{N-2}\big(\alpha, u\big)\Big]\,dx < \infty. \tag{1.2}$$

Moreover, if $\|\nabla u\|^N_{L^N(\mathbb{R}^N)} \leq 1$, $\|u\|_{L^N(\mathbb{R}^N)} \leq M < \infty$, and $\alpha < \alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, then there exists a constant C, which depends only on N, M and α, such that

$$\int_{\mathbb{R}^N} \big[\exp\big(\alpha|u|^{\frac{N}{N-1}}\big) - S_{N-2}(\alpha, u)\big]\,dx \leq C(N, M, \alpha). \tag{1.3}$$

The proof of this lemma can be found in [10, Lemma 1].

Lemma 1.2. *Suppose that g satisfies (G1)–(G3). Then the associated energy functional of problem* (1.1)*, $I : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$, given by*

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\mathbb{R}^N} G(u)\,dx$$

is well defined and of class C^1 with

$$I'(u)v = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v\,dx - \int_{\mathbb{R}^N} g(u)v\,dx, \quad \forall v \in W^{1,p}(\mathbb{R}^N).$$

Consequently, critical points of the functional I are precisely the weak solutions of problem (1.1).

Proof. **Case:** $1 < p < N$**.** As a consequence of assumptions (G1)–(G3), with the aid of the Holder and Sobolev inequalities, we see that I and $I'(u)$ are well defined on $W^{1,p}(\mathbb{R}^N)$.

Case: $p = N$**.** From (G2) it follows that

$$|G(u)| \leq C\big[\exp\big(\alpha_1|u|^{\frac{N}{N-1}}\big) - S_{N-2}(\alpha_1, u)\big], \tag{1.4}$$

for some constants α_1, $C > 0$. Thus, by Lemma 1.1, we have $G(u) \in L^1(\mathbb{R}^N)$ for all $u \in W^{1,N}(\mathbb{R}^N)$.

Furthermore, using standard arguments [2, 8] as well as the fact that for any given strong convergent sequence (u_n) in $W^{1,N}(\mathbb{R}^N)$ there is a subsequence (u_{n_k})

and there exists $h \in W^{1,N}(\mathbb{R}^N)$ such that $|u_{n_k}(x)| \leq h(x)$ almost everywhere in $\mathbb{R}^N$, we see that I is a C^1 functional on $W^{1,N}(\mathbb{R}^N)$. $\square$

Remark 1.3. Recall that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$, and w is a solution of (1.1) with $w \in L^\infty_{loc}(\mathbb{R}^N), |\nabla w| \in L^p(\mathbb{R}^N)$ and $G(w) \in L^1(\mathbb{R}^N)$. Then w satisfies the Pohozaev-Pucci-Serrin identity [14],

$$(N-p)\int_{\mathbb{R}^N} |\nabla w|^p \, dx = Np\int_{\mathbb{R}^N} G(w)\, dx. \tag{1.5}$$

Let

$$m := \inf\big\{I(u) : u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\} \text{ and } u \text{ is a solution of (1.1)}\big\}. \tag{1.6}$$

By a *least energy solution (or ground state)* of (1.1) we mean a minimizer of m. Therefore, if w is a minimizer of (1.6) and $\bar{w}$ is any solution of (1.1) then $I(w) \leq I(\bar{w})$.

In the case $1 < p < N$, we consider the constrained minimization problem

$$M := \inf\Big\{\int_{\mathbb{R}^N} |\nabla u|^p \, dx : u \in W^{1,p}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} G(u)\, dx = 1\Big\}, \tag{1.7}$$

introduced by Coleman-Glazer and Martin [5].

Next, we establish the existence of a *least energy solution* for (1.1).

Theorem 1.4. *Let $1 < p < N$. Under the hypotheses* (G1)–(G4)*, the minimization problem (1.7) has a solution $u \in W^{1,p}(\mathbb{R}^N)$ which is positive.*

The proof of this theorem follows the same pattern as the proof of Theorem 2 in Berestycki an Lions [2].

Remark 1.5. Let u be given by Theorem 1.4. By Lagrange Multipliers Theorem there exists a multiplier μ such that (in the weak sense)

$$-\Delta_p u = \mu g(u) \quad \text{in } \mathbb{R}^N.$$

Then after some appropriated scaling $w(x) = u(\mu^{1/(1-p)}x)$ is a weak solution of (1.1).

In the case $p = N$, we consider the minimization problem

$$N := \inf\Big\{\int_{\mathbb{R}^N} |\nabla u|^p \, dx : u \in W^{1,p}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} G(u)\, dx = 0\Big\}, \tag{1.8}$$

which is motivated by the fact that if $p = N$, from the Pohozaev-Pucci-Serrin identity,

$$\int_{\mathbb{R}^N} G(u)\, dx = 0.$$

Now we state a result about the existence of *least energy solution* for (1.1). Its The proof follows the same method as in Theorem 1 by Berestycki-Gallouet-Kavian [3].

Theorem 1.6. *Let $p = N$. Under the hypotheses* (G1)–(G4) *the minimization problem* (1.8) *has a solution $u \in W^{1,N}(\mathbb{R}^N)$ which is positive.*

In Section 2, we show that under the assumptions (G1)–(G3), the functional I has the *Mountain Pass Geometry* (see Lemma 2.1 below). In particular, we can conclude that the set

$$\Gamma = \big\{\gamma \in C([0,1], W^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\big\},$$

is not empty and the *mountain pass level*

$$c := \inf_{g\in\Gamma}\max_{0\le t\le 1} J(\gamma(t)), \tag{1.9}$$

is positive.

Remark 1.7. Under the hypotheses that the function

$$s \mapsto g(s)/s \tag{1.10}$$

is increasing for $s > 0$, Ding and Ni [9] obtained the characterization

$$c = m = \inf_{v\in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} I(tv). \tag{1.11}$$

Without the monotonicity assumption (1.10), we prove that the level of the mountain pass is a critical value and the corresponding critical points are least energy solutions.

Theorem 1.8. *Let $1 < p \le N$ and assume that* (G1)–(G3). *Then $c = m$. Furthermore, for each least energy solution w of* (1.1), *there exists a path $\gamma \in \Gamma$ such that $w \in \gamma([0,1])$ and*

$$\max_{t\in[0,1]} I(\gamma(t)) = I(w).$$

It has been established in [6, 19] that for $1 < p < 2$, positive solutions of problems like (1.1) are radially symmetric around some point. In the next result, we obtain the exponential decay of positive radial solutions of (1.1) and their derivatives.

Theorem 1.9. *Problem* (1.1) *has a positive radial solution $w \in C^{1,\alpha}(\mathbb{R}^N) \cap W_r^{1,p}(\mathbb{R}^N)$ such that*

(i) *There exists $r_o > 0$ such that $w'(r) \le 0$ for $r \ge r_o$ and $w \in C^2(r_o,\infty)$*
(ii) *w and its first derivatives decay exponentially, i.e., there exist $C > 0$, $\delta > 0$ such that*
$$|D^\alpha w(x)| \le Ce^{-\delta|x|}, \quad \text{if } |\alpha| \le 1 \tag{1.12}$$
(iii) *Moreover, w is a solution with minimal energy, i.e., $0 < I(w) \le I(v)$ for any positive solution v of* (1.1).

In the classical case, when $p = 2$, Problem (1.1) reduces to

$$-\Delta u = g(u) \quad \text{in} \quad \mathbb{R}^N$$

which has been treated by several authors [2, 3, 5, 17]. Our result can be considered as an extension of the classical case.

2. Characterization of Mountain Pass Level

The main goal of this section is to present the proof of Theorem 1.8. For this end we use arguments similar in spirit to those addressed in [11]. We divide the prove in two steps.

First, we prove the *Mountain Pass Geometry* for the energy functional I. More precisely, we have the following lemma.

Lemma 2.1 (Geometrical Mountain-Pass structure)**.** *The functional I satisfies the following three conditions:*

(i) $I(0) = 0$.
(ii) *There exist ρ, $\alpha > 0$, such that $I(u) \ge \alpha$ if $\|u\|_{W^{1,p}(\mathbb{R}^N)} = \rho$.*

(iii) *There is* $u_o \in W^{1,p}(\mathbb{R}^N)$ *such that* $\|u_0\|_{W^{1,p}(\mathbb{R}^N)} > \rho$ *and* $I(u_0) < 0$.

Proof. Statement (i) is trivial. To show **(ii)**, we consider two cases:
Case : $1 < p < N$**.** By our assumptions (G1)–(G3), for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$g(s) \leq (\epsilon - \nu)s^{p-1} + C_\epsilon s^{p^*-1}, \quad \text{for } s \geq 0. \tag{2.1}$$

Since g is an odd function, we have

$$G(s) \leq \frac{1}{p}(\epsilon - \nu)|s|^p + C'_\epsilon |s|^{p^*}, \quad \text{for all } s \in \mathbb{R}.$$

In view of embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ we have

$$\begin{aligned} I(u) &\geq \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \frac{\nu - \epsilon}{p}\int_{\mathbb{R}^N} |u|^p\,dx - \frac{1}{p^*}C'_\epsilon \int_{\mathbb{R}^N} |u|^{p^*}\,dx \\ &\geq \frac{1}{p}\min\{1, \nu - \epsilon\}\|u\|^p_{W^{1,p}(\mathbb{R}^N)} - \frac{1}{p^*}C'_\epsilon \|u\|^{p^*}_{L^{p^*}(\mathbb{R}^N)} \\ &\geq \frac{1}{p}\min\{1, \nu - \epsilon\}\|u\|^p_{W^{1,p}(\mathbb{R}^N)} - C''_\epsilon \|u\|^{p^*}_{W^{1,p}(\mathbb{R}^N)}, \end{aligned}$$

for all $u \in W^{1,p}(\mathbb{R}^N)$. This implies (ii).
Case: $p = N$**.** From (G3), given $\epsilon > 0$ there is $\delta > 0$ such that

$$G(u) \leq \frac{\epsilon - \nu}{N}|u|^N, \quad \text{if } |u| \leq \delta.$$

On the other hand, for $q > N$, by (G2), there is a constant $C = C(q, \delta)$ such that

$$G(u) \leq C|u|^q\big[\exp(\beta|u|^{\frac{N}{N-1}}) - S_{N-2}(\beta, u)\big], \quad \text{if } |u| \geq \delta.$$

These two estimates yield

$$G(u) \leq \frac{\epsilon - \nu}{N}|u|^N + C|u|^q\big[\exp\big(\beta|u|^{\frac{N}{N-1}}\big) - S_{N-2}(\beta, u)\big].$$

In what follows we make use of the inequality (to be proved later)

$$\int_{\mathbb{R}^N} |u|^q\big[\exp(\beta|u|^{\frac{N}{N-1}}) - S_{N-2}(\beta, u)\big]\,dx \leq C(\beta, N)\|u\|^q_{W^{1,N}(\mathbb{R}^N)}, \tag{2.2}$$

provided that $\|u\|_{W^{1,N}(\mathbb{R}^N)} \leq M$, where M is sufficiently small. Under this assumption, we have

$$\begin{aligned} I(u) &\geq \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N\,dx - \frac{(\epsilon - \nu}{N})\|u\|^N_{L^N(\mathbb{R}^N)} - C\|u\|^q_{W^{1,N}(\mathbb{R}^N)} \\ &\geq C_1\|u\|^N_{W^{1,N}(\mathbb{R}^N)} - C\|u\|^q_{W^{1,N}(\mathbb{R}^N)}. \end{aligned}$$

Thus, since $\varepsilon > 0$ and $q > N$, we may choose α, $\rho > 0$ such that $I(u) \geq \alpha$ if $\|u\|_{W^{1,N}(\mathbb{R}^N)} = \rho$. Hence (ii) holds.

Now, we prove inequality (2.2). We may assume $u \geq 0$, since we can replace u by $|u|$ without causing any increase in the integral of the gradient. Here, we make use of Schwarz symmetrization method. We begin by recalling some basic properties: let $1 \leq p \leq \infty$ and $u \in L^p(\mathbb{R}^N)$ such that $u \geq 0$. Thus, there is a unique nonnegative function $u^* \in L^p(\mathbb{R}^N)$, called the Schwarz symmetrization of u, such that it depends only on $|x|$, u^* is a decreasing function of $|x|$; for all $\lambda > 0$

$$|\,\{x : u^*(x) \geq \lambda\}\,| = |\,\{x : u(x) \geq \lambda\}\,|$$

and there exists $R_\lambda > 0$ such that $\{x : u^* \geq \lambda\}$ is the ball $B[0, R_\lambda]$ of radius R_λ centered at origin. Moreover, if $G : [0, +\infty) \to [0, +\infty)$ is a continuous and increasing function such that $G(0) = 0$. Then, we have

$$\int_{\mathbb{R}^N} G(u^*(x))dx = \int_{\mathbb{R}^N} G(u(x))\,dx.$$

Moreover, if $u \in W^{1,p}(\mathbb{R}^N)$ then $u^* \in W^{1,p}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |\nabla u^*|^p(x)\,dx \leq \int_{\mathbb{R}^N} |\nabla u|^p(x)\,dx.$$

Thus, we can write

$$\int_{\mathbb{R}^N} \big[\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u)\big]\,dx = \int_{\mathbb{R}^N} \big[\exp(\alpha|u^*|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u^*)\big]\,dx,$$

Letting $R(\beta, u) = \exp(\beta|u|^{\frac{N}{N-1}}) - S_{N-2}(\beta, u)$, we have

$$\int_{\mathbb{R}^N} R(\beta, u)|u|^q\,dx = \int_{\mathbb{R}^N} R(\beta, u^*)|u^*|^q\,dx$$

and

$$\int_{\mathbb{R}^N} R(\beta, u^*)|u^*|^q\,dx = \int_{|x|\leq\sigma} R(\beta, u^*)|u^*|^q\,dx + \int_{|x|\geq\sigma} R(\beta, u^*)|u^*|^q\,dx, \tag{2.3}$$

where σ is a number to be determined later.

Let us recall two elementary inequalities. Using the fact that the function $h : (0, +\infty) \to \mathbb{R}$ given by $h(t) = [(t + 1)^{\frac{N}{N-1}} - t^{\frac{N}{N-1}} - 1]/t^{\frac{1}{N-1}}$ is bounded, we have a positive constant $A = A(N)$ such that

$$(u + v)^{\frac{N}{N-1}} \leq u^{\frac{N}{N-1}} + Au^{\frac{1}{N-1}}v + v^{\frac{N}{N-1}}, \quad \forall u, v \geq 0. \tag{2.4}$$

If γ and γ' are positive real numbers such that $\gamma + \gamma' = 1$, then for all $\varepsilon > 0$, we have

$$u^\gamma v^{\gamma'} \leq \varepsilon u + \varepsilon^{-\frac{\gamma}{\gamma'}}v, \quad \forall u, v \geq 0, \tag{2.5}$$

because $g : [0, +\infty) \to \mathbb{R}$, given by $g(t) = t^\gamma - \varepsilon t$, is bounded.

Let $v(x) = u^*(x) - u^*(rx_0)$ where x_0 is some fixed unit vector in $\mathbb{R}^N$. Notice that $v \in W_0^{1,N}(B(0, r))$. Here, $B(0, r)$ denotes the ball of radius r centered at the origin of $\mathbb{R}^N$. Now, from (2.4) and (2.5), we have, respectively,

$$|u^*|^{\frac{N}{N-1}} = |v + u^*(rx_0)|^{\frac{N}{N-1}} \leq v^{\frac{N}{N-1}} + Av^{\frac{1}{N-1}}u^*(rx_0) + u^*(rx_0)^{\frac{N}{N-1}},$$

$$v^{\frac{1}{N-1}}u^*(rx_0) = (v^{\frac{N}{N-1}})^{1/N}(u^*(rx_0)^{\frac{N}{N-1}})^{\frac{N-1}{N}} \leq \frac{\varepsilon}{A}v^{\frac{N}{N-1}} + (\frac{\varepsilon}{A})^{\frac{1}{1-N}}u^*(rx_0)^{\frac{N}{N-1}},$$

and hence,

$$|u^*|^{\frac{N}{N-1}} \leq (1 + \varepsilon)v^{\frac{N}{N-1}} + K(\varepsilon, N)u^*(rx_0)^{\frac{N}{N-1}},$$

where $K(\varepsilon, N) = A^{\frac{N}{N-1}}\varepsilon^{\frac{1}{1-N}} + 1$. Therefore,

$$\int_{|x|\leq r} \exp(\alpha|u^*|^{\frac{N}{N-1}}) \leq \exp\big(K(\varepsilon, N)u^*(rx_0)^{\frac{N}{N-1}}\big) \int_{|x|\leq r} \exp\big(\alpha|(1 + \varepsilon)v|^{\frac{N}{N-1}}\big),$$

which, in view of Trudinger-Moser inequality, implies,

$$\int_{|x|\leq r} \exp\big(\alpha|u^*|^{\frac{N}{N-1}}\big) < \infty, \quad \forall u \in W^{1,N}(\mathbb{R}^N), \quad \forall\alpha > 0. \tag{2.6}$$

Furthermore, taking $\epsilon > 0$ such that $(1+\varepsilon)\alpha < \alpha_N$, we obtain

$$\begin{aligned}\int_{|x|\le r} \exp(\alpha|u^*|^{\frac{N}{N-1}}) &\le C(N)\frac{\omega_{N-1}}{N} r^N \exp(K(\epsilon,N) u^*(rx_0)^{\frac{N}{N-1}}) \\ &\le C(N)\frac{\omega_{N-1}}{N} r^N \exp\big(\big(\frac{NM^N}{\omega_{N-1}}\big)^{\frac{1}{N-1}} \frac{K(\epsilon,N)}{r^{\frac{N}{N-1}}}\big),\end{aligned} \tag{2.7}$$

for all $u \in W^{1,N}(\mathbb{R}^N)$ such that $\|\nabla u\|^N_{L^N(\mathbb{R}^N)} \le 1$ and $\|u\|_{L^N(\mathbb{R}^N)} \le M$, where in the last inequality we have used Radial Lemma A.IV in [2]:

$$|u^*(x)| \le |x|^{-1}\big(\frac{N}{\omega_{N-1}}\big)^{1/N} \|u^*\|_{L^N(\mathbb{R}^N)}, \quad \forall x \ne 0.$$

Now, we estimate (2.3). Using the Hölder inequality we obtain

$$\begin{aligned}\int_{|x|\le\sigma} R(\beta,u^*)|u^*|^q\,dx &\le \int_{|x|\le\sigma} [\exp(\beta|u^*|^{\frac{N}{N-1}})]|u^*|^q\,dx \\ &\le \Big(\int_{|x|\le\sigma} \exp(\beta r|u^*|^{\frac{N}{N-1}})\,dx\Big)^{1/r}\Big(\int_{|x|\le\sigma}|u^*|^{qs}\,dx\Big)^{1/s},\end{aligned}$$

where $1/r + 1/s = 1$. In view, of (2.7) we get

$$\int_{|x|\le\sigma} \exp(\beta r|u^*|^{\frac{N}{N-1}})\,dx \le C(\beta,N)$$

if $\|u\|_{W^{1,N}(\mathbb{R}^N)} \le M$, where M is such that $\beta r M^{\frac{N}{N-1}} < \alpha_N$. Thus, using the continuous imbedding $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^{qs}(\mathbb{R}^N)$, we have

$$\int_{|x|\le\sigma} R(\beta,u^*)|u^*|^q\,dx \le C(\beta,N)\|u\|^q_{W^{1,N}(\mathbb{R}^N)}. \tag{2.8}$$

On the other hand, the Radial Lemma leads to

$$\begin{aligned}&\int_{|x|\ge\sigma} |u^*|^{\frac{N}{N-1}k}|u^*|^q\,dx \\ &\le \Big(\big(\frac{N}{\omega_{N-1}}\big)^{1/N}\|u^*\|_{L^N(\mathbb{R}^N)}\Big)^{\frac{N}{N-1}k} \int_{|x|\ge\sigma} \frac{|u^*|^q}{|x|^{\frac{N}{N-1}k}}\,dx \\ &\le \Big(\big(\frac{N}{\omega_{N-1}}\big)^{1/N}\|u^*\|_{L^N(\mathbb{R}^N)}\Big)^{\frac{N}{N-1}k} \Big(\int_{|x|\ge\sigma} \frac{dx}{|x|^{\frac{N}{N-1}kr}}\Big)^{\frac{1}{r}}\Big(\int_{|x|\ge\sigma}|u^*|^{qs}\,dx\Big)^{1/s} \\ &\le \omega_{N-1}\sigma^N\Big(\frac{(\frac{N}{w_{N-1}})^{1/N}\|u^*\|_{L^N(\mathbb{R}^N)}}{\sigma^r}\Big)^{\frac{N}{N-1}k}\|u\|^q_{L^{sq}(\mathbb{R}^N)} \\ &\le C(N,M)\|u\|^q_{W^{1,N}(\mathbb{R}^N)},\end{aligned}$$

for all $k \ge N$, where $\sigma^r = M_0(\frac{N}{\omega_{N-1}})^{1/N}$ and $\|u\|_{L^N(\mathbb{R}^N)} \le M_0 = \lambda_1(N)^{1/N}M$. We also have that if $\|u^*\|^q_{W^{1,N}(\mathbb{R}^N)} \le M$,

$$\begin{aligned}\int_{|x|\ge\sigma} |u^*|^N|u^*|^q\,dx &\le \Big(\int_{|x|\ge\sigma}|u^*|^{Nr}\,dx\Big)^{1/r}\Big(\int_{|x|\ge\sigma}|u^*|^{qs}\,dx\Big)^{1/s} \\ &\le \|u^*\|_{L^{Nr}(\mathbb{R}^N)\|u^*\|^q_{L^{qs}(\mathbb{R}^N)}} \\ &\le C(N,M)\|u^*\|^q_{W^{1,N}(\mathbb{R}^N)},\end{aligned}$$

which is shown via the continuous imbedding $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^{Nr}(\mathbb{R}^N)$. Therefore,

$$\int_{|x|\geq\sigma} R_N(\beta, u^*)|u^*|^q \, dx \leq C(N, M) \exp(\beta)\|u\|^q_{W^{1,N}(\mathbb{R}^N)}. \tag{2.9}$$

Finally, the combination of estimates (2.8)-(2.9) and (2.3) implies that (2.2) is holds.

Now we prove (iii). Since $I(0) = 0$, by (ii) we have $I(u) > 0$ for all $0 < \|u\|_{W^{1,p}(\mathbb{R}^N)} \leq \rho_0$. Thus, ir suffices to show that $\Gamma \neq \emptyset$. This will be done in the next Lemma. □

Lemma 2.2. *There exists* γ *in the set*

$$\Gamma = \{\gamma \in C([0, 1], W^{1,p}(\mathbb{R}^N) : \gamma(0) = 0 \textit{ and } I(\gamma(1)) < 0\},$$

such that

$$w \in \gamma([0, 1]) \quad \textit{and} \quad \max_{t\in[0,1]} I(\gamma(t)) = m, \tag{2.10}$$

where w *is a given least energy.*

Proof. Let w be a given *least energy solution* of (1.1. In the case $1 < p < N$, we consider the curve $\gamma : [0, \infty) \to W^{1,p}(\mathbb{R}^N)$ defined by

$$\gamma(t)(x) = \begin{cases} w(x/t) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

It is not difficult to see that

(i) $\|\gamma(t)\|^p_{W^{1,p}(\mathbb{R}^N)} = t^{N-p}\|\nabla w\|^p_{L^p(\mathbb{R}^N)} + t^N\|w\|^p_{L^p(\mathbb{R}^N)}$

(ii) $I(\gamma(t)) = \frac{t^{N-p}}{p}\|\nabla w\|^p_{L^p(\mathbb{R}^N)} - t^N \int_{\mathbb{R}^N} G(w)dx = \frac{t^N}{p}\|\nabla w\|^p_{L^p(\mathbb{R}^N)}\Big(\frac{1}{t^p} - \frac{N-p}{N}\Big)$, where in the above term we have used the Pohozaev-Pucci-Serrin identity.

Using (i), we have

$$\lim_{t\to 0} \|\gamma(t)\|_{W^{1,p}(\mathbb{R}^N)} = 0,$$

which implies that γ is continuous. From (ii) and $1 < p < N$, we obtain a value $L > 0$ such that $I(\gamma(L)) < 0$. These facts together with a suitable scale change in t, imply that there exists the desired path $\gamma \in \Gamma$.

In the case $p = N$, we choose real numbers $0 < t_0 < 1 < t_1 < \theta_1$ so that a curve γ, constituted of three pieces defined below, gives a desired path:

$$\gamma(\theta) = \begin{cases} \theta\omega_{t_0} & \text{if } \theta \in [0, t_0], \\ \theta\omega_\theta & \text{if } \theta \in [t_0, t_1], \\ \theta\omega_{t_1} & \text{if } \theta \in [t_1, \theta_1], \end{cases}$$

where $w_t(x) = w(x/t)$. Since w is a weak solution we have

$$\int_{\mathbb{R}^N} g(w)w \, dx = \|\nabla w\|^N_{L^N(\mathbb{R}^N)} > 0.$$

Thus we can find $\theta_1 > 1$ such that

$$\int_{\mathbb{R}^N} g(\theta w)w \, dx > 0 \quad \text{for all } \theta \in [1, \theta_1].$$

Next we set $\varphi(s) = g(s)/s^{N-1}$. By assumption (G3) we have $\varphi \in C(\mathbb{R}, \mathbb{R})$. Therefore,

$$\int_{\mathbb{R}^N} \varphi(\theta w)w^N \, dx > 0 \quad \text{for all } \theta \in [1, \theta_1]. \tag{2.11}$$

Now note that

$$\begin{aligned}\frac{d}{d\theta}I(\theta w_t) &= I'(\theta w_t)w_t\\ &= \theta^{N-1}\Big(\|\nabla w_t\|^N_{L^N(\mathbb{R}^N)} - \int_{\mathbb{R}^N}\varphi(\theta w_t)w_t^N\,dx\Big)\\ &= \theta^{N-1}\Big(\|\nabla w\|^N_{L^N(\mathbb{R}^N)} - t^N\int_{\mathbb{R}^N}\varphi(\theta w)w^N\,dx\Big).\end{aligned}$$

Choosing $t_0 \in (0,1)$ sufficiently small, we have

$$\|\nabla w_t\|^N_{L^N(\mathbb{R}^N)} - t_0^N\int_{\mathbb{R}^N}\varphi(\theta w)w^N\,dx > 0 \quad \text{for all } \theta \in [1,\theta_1]. \tag{2.12}$$

By (2.11), we can also choose $t_1 > 1$ such that for all $\theta \in [1,\theta_1]$,

$$\|\nabla w\|^N_{L^N(\mathbb{R}^N)} - t_1^N\int_{\mathbb{R}^N}\varphi(\theta w)w^2\,dx \le -\frac{1}{\theta_1 - 1}\|\nabla w\|^N_{L^N(\mathbb{R}^N)}\,. \tag{2.13}$$

Thus we can see by (2.12) that the function $I(\gamma(\theta))$ is increasing on the interval $[0,t_0]$ and takes its maximal at $\theta = 1$. By Pohozaev-Pucci-Serrin identity we have $\int_{\mathbb{R}^N} G(w) = 0$. Consequently

$$I(w_{t_1}) = I(w) = \frac{1}{N}\|\nabla w\|^N_{L^N(\mathbb{R}^N)}.$$

Now note that

$$\begin{aligned}I(\theta_1 w_{t_1}) &= I(w_{t_1}) + \int_1^{\theta_1}\frac{d}{dt}I(\theta w_{t_1})d\theta\\ &\le \frac{1}{N}\|\nabla w\|^N_{L^N(\mathbb{R}^N)} - \frac{1}{\theta_1 - 1}\int_1^{\theta_1}\|\nabla w\|^N_{L^N(\mathbb{R}^N)}d\theta\\ &< (\frac{1}{N} - 1)\|\nabla w\|^N_{L^N(\mathbb{R}^N)} < 0.\end{aligned}$$

Thus, we have obtained the desired curve. □

As consequence of Lemma 2.2 we have the following important step of the proof of Theorem 1.8.

Corollary 2.3. *With c and m as defined in* (1.11) *and* (1.6)*, we have $c \le m$.*

In view of the Pohozaev-Pucci-Serrin identity we have

Lemma 2.4. *For $1 < p \le N$, we obtain*

$$m = \inf_{u\in\mathcal{P}} I(u), \tag{2.14}$$

where

$$\mathcal{P} = \Big\{u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\} : (N-p)\int_{\mathbb{R}^N}|\nabla u|^p\,dx = Np\int_{\mathbb{R}^N}G(u)\,dx\Big\}.$$

Proof. For the case $1 < p < N$, we introduce the set

$$\mathcal{S} = \Big\{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N}G(u)\,dx = 1\Big\},$$

which is in one-to-one correspondence with the set $\mathcal{P}$ via the map $\Phi : \mathcal{S} \to \mathcal{P}$: $\Phi(u)(x) = u(x/t_u)$ with $t_u = \big(\frac{N-p}{Np}\big)^{1/p}\|\nabla u\|_{L^p(\mathbb{R}^N)}$. Thus,

$$\inf_{u\in\mathcal{P}} I(u) = \inf_{u\in\mathcal{S}} I(\Phi(u)).$$

Next we prove that $\inf_{u\in\mathcal{S}} I(\Phi(u)) = m$. From Theorem 1.4, there exists $u_0 \in W^{1,p}(\mathbb{R}^N)$ such that

$$M = \inf_{u\in\mathcal{S}} \int_{\mathbb{R}^N} |\nabla u|^p \, dx = \int_{\mathbb{R}^N} |\nabla u_0|^p \, dx.$$

After a suitable scale change, $\Phi(u_0)$ becomes a least energy solution; that is, $I(\Phi(u_0)) = m$. By the Pohozaev-Pucci-Serrin identity,

$$\begin{aligned} I(\Phi(u_0)) &= \frac{1}{p} t_{u_0}^{N-p} \|\nabla u_0\|_{L^p(\mathbb{R}^N)}^p - t_{u_0}^N \int_{\mathbb{R}^N} G(u_0)\, dx \\ &= \frac{1}{N}\Big(\frac{N-p}{Np}\Big)^{(N-p)/p} \|\nabla u_0\|_{L^p(\mathbb{R}^N)}^N \\ &= \inf_{u\in S} \frac{1}{N}\Big(\frac{N-p}{Np}\Big)^{\frac{N-p}{p}} \|\nabla u\|_{L^p(\mathbb{R}^N)}^N. \end{aligned}$$

Thus we have (2.14) in the case $1 < p < N$.

For the case $p = N$, we have

$$\mathcal{P} = \Big\{ u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} G(u)\, dx = 0 \Big\}.$$

Thus

$$\begin{aligned} \inf_{u\in\mathcal{P}} I(u) &= \frac{1}{N} \inf \Big\{ \int_{\mathbb{R}^N} |\nabla u|^N \, dx : u \in W^{1,p}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} G(u)\, dx = 0 \Big\} \\ &= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_0|^N \, dx\,, \end{aligned}$$

where in the last equality we have used Theorem 1.6. On the other hand

$$\int_{\mathbb{R}^N} |\nabla u_0|^N \, dx = NI(u_0) = Nm.$$

Thus we have (2.14) in the case $p = N$. Therefore the proof of Lemma 2.14 is complete. □

To complete the proof of Theorem 1.8, in view of Corollary 2.3 and Lemma 2.4, it only remains to prove that $m \le c$, which is a consequence of the following result.

Lemma 2.5. *For all $\gamma \in \Gamma$, $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$.*

Proof. **Case:** $1 < p < N$**.** We consider the functional

$$P(u) = \frac{N-p}{p} \|\nabla u\|_{L^p(\mathbb{R}^N)}^p - \int_{\mathbb{R}^N} G(u)\, dx = NI(u) - \|\nabla u\|_{L^p(\mathbb{R}^N)}^p,$$

defined in $W^{1,p}(\mathbb{R}^N)$. Using (2.1), it is not difficult to see that there exists $\rho_0 > 0$ such that

$$P(u) > 0 \quad \text{for } 0 < \|u\|_{W^{1,p}(\mathbb{R}^N)} \le \rho_0.$$

For each $\gamma \in \Gamma$ we have $P(\gamma(1)) = NI(\gamma(1)) - \|\nabla\gamma(1)\|_{L^p(\mathbb{R}^N)}^p \le NI(\gamma(1)) < 0$ and $\gamma(0) = 0$. Thus there exists $t_0 \in [0,1]$ such that

$$\|\gamma(t_0)\|_{W^{1,p}(\mathbb{R}^N)} > \rho_0, \quad \text{and} \quad P(\gamma(t_0)) = 0.$$

Therefore, $\gamma(t_0) \in \gamma([0,1]) \cap \mathcal{P}$.

Case: $p = N$. We consider $\rho \in C_0^\infty(\mathbb{R}^N, [0,\infty))$ such that $\int_{\mathbb{R}^N} \rho(x)\,dx = 1$. For $\gamma \in \Gamma$ and $\epsilon > 0$, we define $\gamma_\epsilon : [0,1] \to W^{1,N}(\mathbb{R}^N)$ given by

$$\gamma_\epsilon(t)(x) = \int_{\mathbb{R}^N} \rho\big(\frac{x-y}{\epsilon}\big)\gamma(t)(y)dy.$$

It is easy to see that the function γ_ϵ satisfies the following three properties:

(i) $\gamma_\epsilon(t) \in L^\infty(\mathbb{R}^N)$, for all $t \in [0,1]$
(ii) $\gamma_\epsilon \in C([0,1], L^\infty(\mathbb{R}^N))$
(iii) $\max_{t\in[0,1]} \|\gamma_\epsilon(t) - \gamma(t)\|_{W^{1,N}(\mathbb{R}^N)} \to 0$ as $\epsilon \to 0$.

Now, using assumption (G3) there exists $\rho_0 > 0$ such that

$$P(u) > 0 \text{ if } 0 < \|u\|_\infty \le \rho_0. \tag{2.15}$$

By (iii), we have $P(\gamma(1)) \le NI(\gamma_\epsilon(1)) < 0$ and $\gamma(0) = 0$ for all $\epsilon > 0$. Thus, using (2.15) and (ii) we obtain that $P(\gamma_\epsilon(t)) > 0$ for $t > 0$ sufficiently small. Therefore, we can find $t_\epsilon \in [0,1]$ such that

$$\|\gamma_\epsilon(t_\epsilon)\|_\infty > \rho_0, \quad P(\gamma_\epsilon(t_\epsilon)) = 0.$$

That is, $\gamma_\epsilon(t_\epsilon) \in \mathcal{P}$. We extract a subsequence $\epsilon_n \to 0$ such that $t_{\epsilon_n} \to t_0$. From (ii)-(iii) it follows that

$$\|\gamma_\epsilon(t_{\epsilon_n}) - \gamma(t_0)\|_{W^{1,N}(\mathbb{R}^N)} \to 0, \quad P(\gamma(t_0)) = 0\,.$$

Now we claim that $\gamma(t_0) \neq 0$. Indeed, by Theorem 1.6,

$$\inf_{u\in\mathcal{P}} \|\nabla u\|^N_{L^N(\mathbb{R}^N)} = 2m > 0.$$

Therefore, $\|u\|_{W^{1,N}(\mathbb{R}^N)} \ge (Nm)^{1/N}$ for all $u \in \mathcal{P}$. In particular,

$$\|\gamma_\epsilon(t_{\epsilon_n})\|_{W^{1,N}(\mathbb{R}^N)} \ge (Nm)^{1/N}.$$

Consequently, $\|\gamma(t_0)\|_{W^{1,N}(\mathbb{R}^N)} \ge (Nm)^{1/N} > 0$. Thus $\gamma(t_0) \in \gamma([0,1]) \cap \mathcal{P}$ and $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$. This, show the Lemma in the case $p = N$. □

Proof of Theorem 1.8. By Corollary 2.3 we have $c \le m$. On the outer hand, Lemmas 2.4 and 2.5 imply

$$m = \inf_{u\in\mathcal{P}} I(u) \le c.$$

Thus, the proof of Theorem is complete. □

3. Asymptotic Behavior

In this section we show the decay at infinity of the weak solution and its derivatives.

Proof of Theorem 1.9. The exponential decay of w at infinity is already known [12, Theorem 2.3]. We show first that there exists $r_o > 0$ such that $w'(r) \le 0$ for $r \ge r_o$. Indeed, since w has exponential decay at infinity, it follows form (G1) that there exists $r_1 > 0$ such that

$$\int_{r_1}^\infty r^{N-1}|w'|^{p-2}w'\varphi'\,dr = \int_{r_1}^\infty r^{N-1}g(u(r))\varphi\,dr < 0 \tag{3.1}$$

for all $0 \le \varphi \in W_r^{1,p}(0,+\infty)$ with $\operatorname{supp}\varphi \subset (r_1,\infty)$. The result then follows by contradiction. Take $r_o > r_1 + 1$ and suppose that exists $r' \ge r_o$ such that,

$w'(r') > 0$. Since w' is continuous, there exists $\delta > 0$ such that $w'(r) > 0$ for $r \in (r'-\delta, r'+\delta)$. Choosing the test function

$$\varphi(r) = \begin{cases} 0 & \text{if } 0 \le r \le r'-\delta, \\ \frac{w(r'+\delta)}{2\delta}(r - r' + \delta) & \text{if } r'-\delta < r \le r'+\delta, \\ w(r) & \text{if } r \ge r'+\delta \end{cases}$$

in (3.1) we have

$$\int_{r'-\delta}^{r'+\delta} r^{N-1}|w'|^{p-2}w'(r)\,dr < 0.$$

This is a contradiction. Therefore, there exists $r_o > 0$ such that $w'(r) \le 0$ for $r \ge r_o$. Next, we show that w' has exponential decay. Since w is radial,

$$\int_0^\infty r^{N-1}|w'|^{p-2}w'\varphi'\,dr = \int_0^\infty h(r)\varphi\,dr \quad \forall\, \varphi \in W_r^{1,p}(\mathbb{R}^N), \tag{3.2}$$

where $h(r) = r^{N-1}g(w(r))$. If $u(r) = \int_r^\infty h(s)ds$ we have $u'(r) = -h(r)$. Consequently, if $v(r) = r^{N-1}|w'(r)|^{p-2}w' - u(r)$ we have

$$\int_0^\infty v(s)\varphi'(s)ds = 0 \qquad \forall \;\; \varphi \in W_r^{1,p}(0,\infty).$$

Therefore, by [4, Lemma VIII.1], there exists a constant C such that

$$r^{N-1}|w'|^{p-2}w' = C + u(r)\,. \tag{3.3}$$

We claim that $C = 0$. Indeed, suppose that $C \neq 0$. By the exponential decay of u and (3.3), there exists a constant $C_1 > 0$ such that for r sufficiently large

$$r^{N-1}|w'(r)|^{p-1} \ge C - ce^{-\theta r} \ge C_1/r,$$

that is,

$$|w'(r)| \ge \frac{C_1}{r^\alpha}, \tag{3.4}$$

where $\alpha = \frac{N}{p-1}$. Since $p \le N$ we have $\alpha > 1$. Integrating (3.4) from R to r and using the fact of that $w'(r) \le 0$ for $r \ge r_o$, we obtain

$$-w(r) + w(R) \ge \frac{C_1}{1-\alpha}\big(\frac{1}{r^{\alpha-1}} - \frac{1}{R^{\alpha-1}}\big). \tag{3.5}$$

Letting $r \to \infty$ in (3.5) we have

$$w(R) \ge \frac{C_1}{(\alpha-1)}\frac{1}{R^{\alpha-1}},$$

for R sufficiently large. This contradicts the exponential decay of w. Therefore,

$$r^{N-1}|w'|^{p-2}w' = u(r). \tag{3.6}$$

It follows from (3.6) that w' has exponential decay. Moreover, $w \in C^2(r_o,\infty)$. This completes the proof of Theorem 1.9. $\square$

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References

[1] D. Arcoya, J. Diaz, L. Tello , *S-Shaped bifurcation branch in a quasilinear multivalued model arising in climatology*, J. Diff. Equation **150** (1998), 215-225.

[2] H. Berestycki, P. L. Lions, *Nonlinear scalar field equations.* **I**. *Existence of ground state*, Arch Rational Mech. Anal. **82** (1983), 313–346.

[3] H. Berestycki, T. Gallouet O. Kavian, *Equations de Champs scalaries euclidiens non lineaires dans le plan*, C. R. Acad. Sci; Paris Ser. I Math. **297** (1983), 307–310.

[4] H. Brezis, *Analyse Fonctionelle Théorie et Applications*, Masson, Paris (1987).

[5] S. Coleman, V. Glazer, A. Martin, *Action minima among solution to a class of euclidean scalar field equations*, Comm. Math. Phys. **58** (1978), 211–221.

[6] L. Damascelli, F. Pacella, M. Ramaswamy, *Symmetry of ground states of p-Laplace equations via the moving plane method.* Arch. Ration. Mech. Anal. **148** (1999), no. 4, 291–308.

[7] F. Demegel, E. Hebey, *On some nonlinear equations involving the p-laplacian critical Sobolev growth*, Advance in Differential Equations **4** (1998), 533–574.

[8] D. G. de Figueiredo, *Lectures on the Ekeland variational principle with applications and detours*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 81. Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin, (1989).

[9] Y. Ding, W. Y. Ni, *On the existence of positive entire solutions of a semilinear elliptic equations*, Arch. Rat. Mech. Mech. Anal. **91** (1986), 283–308.

[10] J. M. do Ó, *N-Laplacian equations in* $\mathbb{R}^N$ *with critical growth*, Abstract and App. Analysis **2** (1997), 301–315.

[11] L. Jeanjean, K. Tanaka, *A remark on least energy solutions in* $\mathbb{R}^N$, Proc. Amer. Math. Soc. 131 (2003), 2399–2408

[12] G. Li, S. Yan, *Eigenvalue problems for quasilinear elliptic equations on* $\mathbb{R}^N$, Commun. in Partial Differential Equations **14** (1989), 1291–1314.

[13] W. M. Ni, I. Takagi, *On the Shape of Least-Energy Solutions to a Semilnear Neumann Problem*, Comm. Pure Appl. Math. **44** (1991) 819-851.

[14] P. Pucci, J. Serrin, *A general variational identity*, Indiana Univ. Math. J. **35** (1986) 681–703.

[15] P. J. Rabier, C. A. Stuart, *Exponential decay of the solutions of quasilinear second-order equations and Pohozaev identities*, J. Diff. Equations **167** (2000) 199–234.

[16] P. H. Rabinowitz, *On class of nonlinear Schrodinger equations*, Z. Angew. Math. Phys. **43** (1992) 272–291.

[17] W. A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977) 149–162.

[18] J. Serrin, M. Tang, *Uniqueness of ground states for quasilinear elliptic equations*, Indiana Univ. Math. J. **49** (2000) 897–923.

[19] J. Serrin, H. Zou, *Symmetry of ground state of quasilinear elliptic equations*, Arch. Rational Mech. Anal. **148** (1999) 265–290.

[20] J. L. Vasquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984) 191–202.

[21] J. Yang, X. Zhou, *On the existence of nontrivial solution of a quasilinear elliptic boundary value problem for unbounded domains*, Acta Math. Sci. **7** (1987) 447–459.

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