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ASYMPTOTIC BEHAVIOR FOR A HEAT CONDUCTION PROBLEM WITH PERFECT-CONTACT BOUNDARY CONDITION

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Abstract. In this paper we consider the heat conduction problem for a slab represented by the interval $[0,1]$. The initial temperature is a positive constant, the flux at the left end is also a positive constant, and at the right end there is a perfect contact condition: $u_x(1,t)+\gamma u_t(1,t)=0$. We analyze the asymptotic behavior of these problems as γ approaches infinity, and present some numerical calculations.

1. Introduction and preliminaries

When two bodies A and B are in perfect thermal contact at a boundary S, the boundary conditions are

$$K_1\frac{\partial V}{\partial \eta}=K_2\frac{\partial v}{\partial \eta},\quad \text{on } S,$$
$$V=v,\quad \text{on } S;$$

where V, v are the temperatures of A and B, K_1 and K_2 are their thermal conductivities, and η is the outer unit normal. Assuming that $K_1 \gg K_2$, we can consider that $V=V(t)=v|_S$. Then by means of an energy balance, we get a boundary condition for v:

$$K_2\int\int_S \frac{\partial v}{\partial \eta}dS+Mc'\frac{\partial v}{\partial t}=0, \tag{1.1}$$

where M and c' denote the mass and the specific heat of the body A. This kind of boundary condition has been widely investigated; see for example [3, 4, 5].

We consider a one-dimensional slab $[0,\ell]$ with its face $y=\ell$ in perfect thermal contact with mass M_f per unit area of a well-stirred fluid (or a perfect conductor) of specific heat c_f. In this case the condition (1.1) is given by

$$kv_y(\ell,\tau)+M_f c_f v_\tau(\ell,\tau)=0. \tag{1.2}$$

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This condition appears in several interesting applications such as heat condensers [3]. We consider the heat conduction problem

$$\begin{gathered} kv_{yy} = \rho c v_\tau, \quad \text{in } [0,\ell]\times(0,\mathcal{T}], \\ v(y,0) = V_0 > 0, \quad 0 \le y \le \ell, \\ kv_y(0,\tau) = q_0 > 0, \quad 0 < \tau \le \mathcal{T}, \\ kv_y(\ell,\tau) + M_f c_f v_\tau(\ell,\tau) = 0, \quad 0 < \tau \le \mathcal{T}, \end{gathered} \tag{1.3}$$

where k is the thermal conductivity, ρ the density, q_0 the heat flux, and c the specific heat of the material. All of these constants are positive. With the change of variables

$$x = \frac{y}{\ell}, \quad t = \frac{k\tau}{\rho c \ell^2}, \quad v(y,\tau) = cu(x,t).$$

problem (1.3) is transformed into the problem

$$\begin{gathered} u_{xx} = u_t, \quad \text{in } [0,1]\times(0,T], \\ u(x,0) = M > 0, \quad 0 \le x \le 1, \\ u_x(0,t) = q > 0, \quad 0 < t \le T, \\ \gamma u_t(1,t) + u_x(1,t) = 0, \quad 0 < t \le T, \end{gathered} \tag{1.4}$$

where

$$M = cV_0, \quad q = \frac{c\ell q_0}{k}, \quad \gamma = \frac{M_f c_f}{\rho c}, \quad T = \frac{k\mathcal{T}}{\rho c \ell^2}\,.$$

Roughly speaking, we expect that as $\gamma \to +\infty$ the solution to (1.4) converge to the solution of the problem

$$\begin{gathered} u_{xx} = u_t, \quad \text{in } [0,1]\times(0,T], \\ u(x,0) = M > 0, \quad 0 \le x \le 1, \\ u_x(0,t) = q > 0, \quad 0 < t \le T, \\ u_t(1,t) = 0. \quad 0 < t \le T. \end{gathered} \tag{1.5}$$

In the case of models of heat conduction in material media it is natural to attempt to determine the temporary range of validity (i.e. the solution remains positive). Here an important limitation of this range is imposed by the change of phase phenomena. An extensive bibliography on phase-change problem can be found in [6].

In [7] the authors studied this problem with temperature and convective boundary conditions at $x = 1$. They obtained an explicit expression for the approximation of the time of phase change t_{ch} for the problem (1.5), namely

$$t_{ch} = \Big(\frac{\sqrt{\pi}M}{2q}\Big)^2. \tag{1.6}$$

Here we prove that the solution of the problem (1.4) converges to the solution of the problem (1.5) using (1.6). This relation was obtained in [8] using Laplace transforms.

Next we prove that the solution to the problem (1.4) converges to the solution of the problem (1.5) in the L^∞ norm. In fact, by using a numerical scheme we visualize this convergence. The formulation and the results of the numerical scheme are provided.

2. Asymptotic behavior of problem (1.4)

Lemma 2.1. *The solution to* (1.4) *satisfies*

1. *If $u_x(x,t) \geq 0$ then $u(0,t) \leq u(x,t)$, for $0 \leq x \leq 1$, $0 < t \leq T$.*
2. *For all $(x,t) \in [0,1] \times (0,T]$, $u_t(x,t) \leq 0$.*

Proof. Let $v = u_x$. Then v satisfies

$$\begin{gathered} v_{xx} = v_t, \quad \text{in } [0,1] \times (0,T], \\ v(x,0) = 0, \quad 0 \leq x \leq 1, \\ v(0,t) = q, \quad 0 < t \leq T, \\ v(1,t) + \gamma v_x(1,t) = 0, \quad 0 < t \leq T. \end{gathered} \tag{2.1}$$

By the maximum principle [10], $\min v(x,t) = \min\{q, 0, v(1,t)\}$ for $0 \leq x \leq 1$ and $t > 0$. Assuming that $v(1,t) < 0$ (we remark that $q > 0$) it follows that $\min v(x,t) = v(1,t)$. By Hopf's lemma (see [10]), $v_x(1,t) < 0$, which contradicts the last equation in (2.1) ($\gamma > 0$). Therefore, $u_x(x,t) \geq 0$. This proves part 1.

Let $v^\varepsilon(x,t) = u(x,t+\varepsilon) - u(x,t)$. Hence

$$\begin{gathered} v^\varepsilon_{xx} = v^\varepsilon_t, \quad \text{in } D = [0,1] \times (0,T], \\ v^\varepsilon(x,0) = u(x,\varepsilon) - M, \quad 0 \leq x \leq 1, \\ v^\varepsilon_x(0,t) = 0, \quad ,0 < t \leq T \\ v^\varepsilon_x(1,t) + \gamma v^\varepsilon_t(1,t) = 0, \quad t > 0. \end{gathered} \tag{2.2}$$

Let us show that $u(x,\varepsilon) - M \leq 0$ for all $\varepsilon \geq 0$. By the maximum principle and Hopf's lemma we have

$$\max v^\varepsilon(x,t) = \max\{u(x,\varepsilon) - M, v^\varepsilon(1,t)\},$$

for $0 \leq x \leq 1$ and $0 < t \leq T$. Assuming that $\max v^\varepsilon(x,t) = v^\varepsilon(1,t_0) > 0$, then it follows that $v^\varepsilon_x(1,t_0) > 0$. From (2.2) it follows that $v^\varepsilon_t(1,t_0) < 0$, which implies that $v^\varepsilon(1,t)$ decreases in $(t_0 - \varepsilon, t_0)$. This contradiction proves that that $v^\varepsilon(x,t) \leq 0$. Hence

$$\lim_{\varepsilon \to 0} \frac{v^\varepsilon(x,t)}{\varepsilon} = u_t(x,t) \leq 0,$$

which completes the proof. □

Since $u(x,t) - M$ satisfies (1.4) with zero initial condition, by the maximum principle and Hopf's lemma, it follows that $u(x,\varepsilon) - M \leq 0$ for all $\varepsilon \geq 0$.

Lemma 2.2.
1. *Let u_{γ_i} be solutions to Problem* (1.4) *with γ_1 and γ_2 respectively. If $\gamma_1 \leq \gamma_2$ then $u_{\gamma_1} \leq u_{\gamma_2}$.*
2. *Let u_∞ be the solution to Problem* (1.5). *Then $u_\gamma \leq u_\infty$ for all $\gamma > 0$.*

Proof. Let $z = u_{\gamma_2} - u_{\gamma_1}$. Then for $x = 1$ the function z satisfies

$$\begin{aligned} z_x(1,t) &= u_{\gamma_{2x}}(1,t) - u_{\gamma_{1x}}(1,t) \\ &= -\gamma_2 u_{\gamma_{2t}}(1,t) + \gamma_1 u_{\gamma_{1t}}(1,t) \\ &= -\gamma_2 (u_{\gamma_{2t}}(1,t) - u_{\gamma_{1t}}(1,t)) + (\gamma_1 - \gamma_2) u_{\gamma_{1t}}(1,t) \\ &= -\gamma_2 z_t(1,t) + (\gamma_1 - \gamma_2) u_{\gamma_{1t}}(1,t). \end{aligned}$$

By Lemma 2.1, $z(x,t)$ satisfies:

$$\begin{gathered}
z_{xx} = z_t, \quad \text{in } [0,1]\times(0,\leq T], \\
z(x,0) = 0, \quad 0 \leq x \leq 1, \\
z_x(0,t) = 0, \quad 0 < t \leq T, \\
z_x(1,t) + \gamma_2 z_t(1,t) \geq 0, \quad 0 < t \leq T.
\end{gathered} \tag{2.3}$$

By the maximum principle and Hopf's lemma, for $0 \leq x \leq 1$ and $0 < t \leq T$ we have

$$\min z(x,t) = \min\{0, z(1,t)\}.$$

Assume that $\min z(x,t) = z(1,t_0) < 0$, then using Hopf's lemma,

$$z_x(1,t_0) < 0.$$

From (2.3) ($\gamma_2 > 0$) it follows that $z_t(1,t_0) > 0$. This implies that $z(1,t)$ increases in $(t_0-\varepsilon, t_0)$, which contradicts that $z(1,t_0)$ is a minimum. Therefore we obtain that $z(x,t) \geq 0$.

Setting $z = u_\infty - u_\gamma$, we obtain

$$\begin{aligned}
z_x(1,t) + \gamma z_t(1,t) &= u_{\infty_x}(1,t) - u_{\gamma_x}(1,t) + \gamma(u_{\infty_t}(1,t) - u_{\gamma_t}(1,t)), \\
&= -(u_{\gamma_x}(1,t) + \gamma u_{\gamma_t}(1,t)) + u_{\infty_x}(1,t) + \gamma u_{\infty_t}(1,t) \\
&= u_{\infty_x}(1,t).
\end{aligned}$$

The function $\theta(x,t) = u_{\infty_x}(x,t)$ satisfies the heat conduction problem:

$$\begin{gathered}
\theta_{xx} = \theta_t, \quad D = \{(x,t) : 0 \leq x \leq 1, 0 < t \leq T\}, \\
\theta(x,0) = 0, \quad 0 \leq x \leq 1, \\
\theta(0,t) = q, \quad 0 < t \leq T, \\
\theta_x(1,t) = 0, \quad 0 < t \leq T,
\end{gathered}$$

Using the maximum principle and Hopf's lemma we obtain

$$0 \leq \theta(x,t) = u_{\infty_x}(x,t) \leq q.$$

From the above inequality, we conclude that

$$0 \leq z_x(1,t) + \gamma z_t(1,t) \leq q.$$

Using maximum principle and Hopf's lemma, we deduce that $z(x,t) \geq 0$. $\square$

Remark From lemma 2.2, we can assure the existence of a function $u^*(x,t)$ such that $\lim_{\gamma\to\infty} u_\gamma(x,t) = u^*(x,t)$ a.e. $x \in [0,1]$, and $u^*(x,t) \leq u_\infty(x,t)$.

Let $\|\cdot\|_\infty$ be the $L^\infty([0,1]\times[0,T])$ norm:

$$\|u(x,t)\|_\infty = \sup_{[0,1]\times[0,T]} |u(x,t)|.$$

In the next lemma, we prove that the convergence is uniform and that $u^*(x,t) = u_\infty(x,t)$ a.e. $x \in [0,1]$. For this proof we use Laplace Transforms [1].

Lemma 2.3. *With the above notation,*

$$\|u_\infty(x,t) - u_\gamma(x,t)\|_\infty \leq \frac{qT}{\gamma}. \tag{2.4}$$

Proof. Taking $w = u_\infty - u_\gamma$, we have

$$w_{xx} = w_t, \quad [0,1]\times(0,T], \tag{2.5}$$
$$w(x,0) = 0, \quad 0 \le x \le 1, \tag{2.6}$$
$$w_x(0,t) = 0, \quad 0 < t \le T, \tag{2.7}$$
$$0 \le w_x(1,t) + \gamma w_t(1,t) \le q, \quad 0 < t \le T. \tag{2.8}$$

Note that since u_∞ and u_γ are increasing functions and $u_\gamma \le u_\infty$, we have

$$\|u_\infty(x,t) - u_\gamma(x,t)\|_\infty = u_\infty(1,t) - u_\gamma(1,t), \tag{2.9}$$

Applying the Laplace Transform,

$$W_{xx}(s,x) - sW(s,x) = 0, \tag{2.10}$$
$$W_x(s,0) = 0, \tag{2.11}$$
$$0 < W_x(s,1) + s\gamma W(1,t) < \frac{q}{s}, \tag{2.12}$$

where s is a positive parameter. The general solution to this problem is

$$W(s,x) = C(s,q,\gamma)\cosh(\sqrt{s}x). \tag{2.13}$$

Replacing (2.13) in (2.12),

$$C(s,q,\gamma) \le \frac{q}{s(\sqrt{s}\sinh(\sqrt{s}) + \gamma s\cosh(\sqrt{s}))} \le \frac{q}{\gamma s^2\cosh(\sqrt{s})}.$$

Therefore,

$$W(s,x) \le \frac{q}{\gamma s^2}. \tag{2.14}$$

To obtain a bound for $w(1,t)$, we apply the inverse Laplace Transform at $x = 1$ to (2.14):

$$w(1,t) \le \frac{qt}{\gamma}. \tag{2.15}$$

From (2.9) and (2.15) we obtain

$$\|u_\infty(x,t) - u_\gamma(x,t)\|_\infty \le \frac{qT}{\gamma}, \tag{2.16}$$

which proves the proof. □

3. Numerical scheme and results

We preset a short description of our numerical scheme, for problem (1.4), and refer the reader to [2] for more details. First, we consider the weak formulation for the problem (1.4):

$$\begin{aligned}\int_0^1 u_t(x,t)\phi(x)dx + \int_0^1 u_x(x,t)\phi_x(x)dx &= u_x(1,t)\phi(1) - u_x(0,t)\phi(0)\\ &= -\gamma u_t(1,t)\phi(1) - q\phi(0),\end{aligned}$$

where $\phi(x)$ belongs to $H^1(0,1)$. We will consider a finite element method for the discretization of the space variable.

Let $x_i = i/N$ for $0 \le i \le N$ be a partition of the interval $[0,1]$ into subintervals $I_i = [x_i, x_{i+1}]$, of length $h = 1/N$. Let V_h the set of continuous functions which are linear on each I_i. We consider the basis functions of V_h taking as usual ϕ_i, with $\phi_i(x_j) = \delta_{ij}$. We define a partition $\{0 = t_0 < t_1 < \dots < t_M = T\}$ of the interval $[0,T]$, with equal subintervals $\Delta t = t_k - t_{k-1}$ and $k = 1,\dots,M$.

We consider the following approximations for $u(x,t_k)$ and $u_t(x,t_k)$:
$$u(x,t_k) \approx \sum_{i=0}^{N} U_i^k \phi_i(x)$$
$$u_t(x,t_k) \approx \frac{1}{\Delta t}\sum_{i=0}^{N}(U_i^k - U_i^{k-1})\phi_i(x).$$
By using these approximations in the weak formulation, we obtain the following linear system for $U^k = (U_0^k, \dots, U_N^k)$:
$$AU^k = BU^{k-1} + C \quad \text{for} \quad k = 1, 2\dots$$
$$U_i^0 = (x_i, 0).$$
where A and B are symmetric tridiagonal matrices and $C = (-q\Delta t, 0, \dots, 0)^T$. The coefficients of these matrices are:
$$A_{ij} = \begin{cases} \frac{h}{3} + \frac{\Delta t}{h} & \text{if } j = i = 1 \\ \frac{2h}{3} + \frac{2\Delta t}{h} & \text{if } j = i \text{ for } i = 2, \dots, N \\ \frac{h}{3} + \frac{\Delta t}{h} - \gamma & \text{if } j = i = N+1 \\ \frac{h}{6} - \frac{\Delta t}{h} & \text{if } j = i+1 \text{ for } \quad i = 1, \dots, N. \end{cases}$$
and
$$B_{ij} = \begin{cases} \frac{h}{3} & \text{if } j = i = 1 \\ \frac{2h}{3} & \text{if } j = i \text{ for } i = 2, \dots, N \\ \frac{h}{3} - \gamma & \text{if } j = i = N+1 \\ \frac{h}{6} & \text{if } j = i+1 \text{ for } i = 1, \dots, N. \end{cases}$$
In the case of problem (1.5) we obtain the similar linear system for the discrete scheme where we replace the last file in the matrices A and B for $(0, \dots, 0, 1)$. Now, we show examples that verify the theoretical results obtained above. For all examples we set $h = \Delta t = 10^{-3}$.

Example 3.1. In this first example, we show that the solution to problem (1.4) satisfies the hypotheses of Lemma 2.1 (i.e. $u_t \le 0$, $u_x \ge 0$). We set $q = 10$, $M = 100$ and $\gamma = 25$ in the problem (1.4). In figure 1 we show the solution for different times t_j for $j = 1, 2, 3, 4$. We plot the temperature $u(x,t)$ with respect to x for different t.

Example 3.2. We take the following values for the data in the problem (1.4): $q = 10$, $M = 100$, $\gamma_1 = 1$, $\gamma_2 = 25$ and $\gamma_3 = 50$. Figures 2 and 3 show the convergence when $\gamma \to \infty$ at $x = 0$ and $x = 1$.

We observe that the solution $u_\gamma \approx u_\infty$ for large values of γ.

Example 3.3. To present numerical evidence of Lemma 2.3, we take $q = 10$, $M = 100$ and $T = 10$. We consider the parameter $\gamma \to +\infty$ and we show that $\|u_\infty - u_\gamma\|_\infty$ is bounded for $f(\gamma) = \frac{qT}{\gamma}$. This example shows that the bound $f(\gamma) = \frac{qT}{\gamma}$ actually estimates $\|u_\infty(x,t) - u_\gamma(x,t)\|_\infty$.

Concluding Remarks. We have proved that the solution to the problem (1.4) converge in L^∞-norm to the solution of problem (1.5). Moreover we have illustrated this convergence and the properties of the solution of problem (1.4) using a finite element method in the space variable.

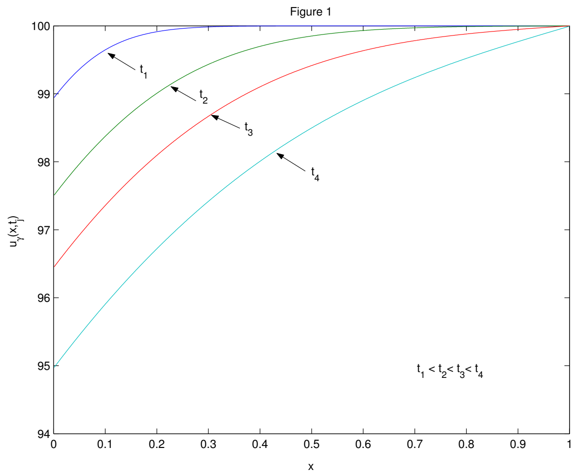

FIGURE 1. Solutions to (1.4)

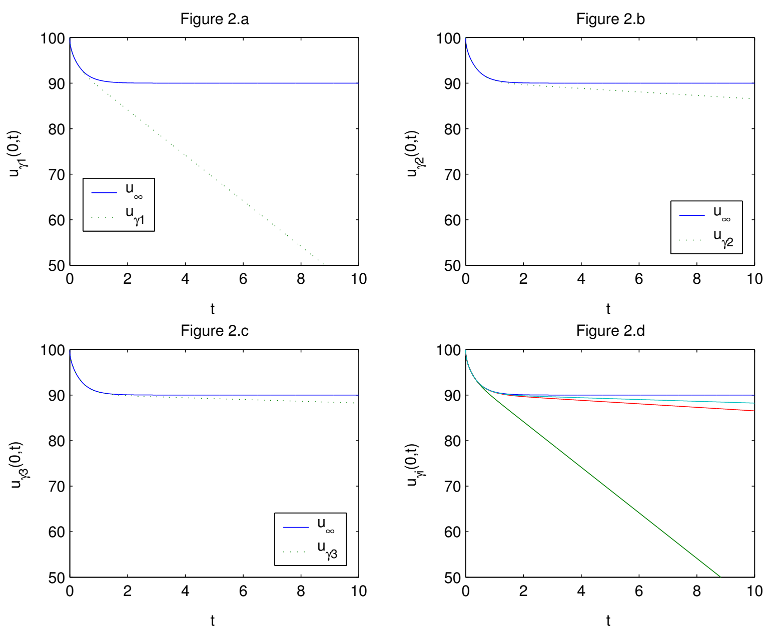

FIGURE 2. Convergence at $x = 0$

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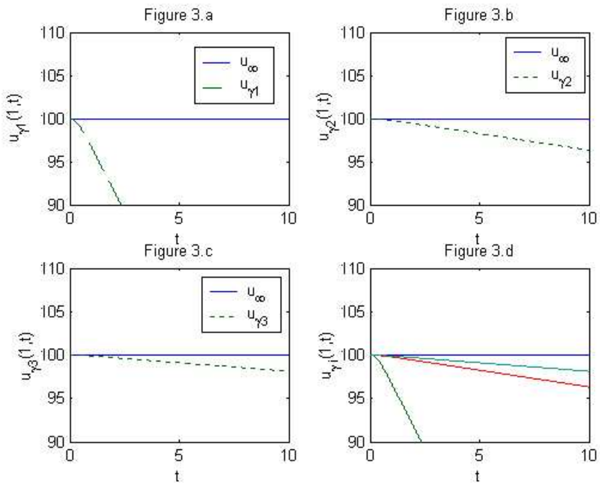

FIGURE 3. Convergence at $x = 1$

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