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POSITIVE SOLUTIONS OF BOUNDARY-VALUE PROBLEMS FOR 2M-ORDER DIFFERENTIAL EQUATIONS

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Abstract. This article concerns the existence of positive solutions to the differential equation

$$(-1)^m x^{(2m)}(t) = f(t, x(t), x'(t), \dots, x^{(m)}(t)), \quad 0 < t < \pi,$$

subject to boundary condition

$$x^{(2i)}(0) = x^{(2i)}(\pi) = 0,$$

or to the boundary condition

$$x^{(2i)}(0) = x^{(2i+1)}(\pi) = 0,$$

for $i = 0, 1, \dots, m-1$. Sufficient conditions for the existence of at least one positive solution of each boundary-value problem are established. Motivated by references [7, 17, 21], the emphasis in this paper is that f depends on all higher-order derivatives.

1. Introduction

The study of the existence of positive solutions of boundary-value problems for second-order and higher-order ordinary differential equations has gained prominence recently and is a rapidly growing field. This happens because of the applications of this problem, especially fourth-order differential equations; see for example the articles [5, 7, 9, 12, 13, 16, 17, 19, 20, 21] and the monographs [1, 2, 3].

For the second-order case, the existence of positive solutions of boundary-value problems for nonlinear differential equations has been studied by many authors. The differential equation

$$x''(t) + f(t, x(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

subjected to different boundary conditions has received much attention. Specially in seeking conditions on the nonlinearity f for which there are at least one, at least two, or at least three positive solutions. See for example [4, 8, 10, 11, 24].

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However, there are not many publication about the existence of positive solutions of the differential equation

$$x''(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1, \tag{1.2}$$

under various boundary conditions. This because the presence of x' in the nonlinearity f causes considerable difficulties [6, 17, 18, 22].

Recently, Chyan and Henderson [7] studied the $2m$-order differential equation

$$x^{(2m)}(t) = f(t, x(t), x''(t), \dots, x^{2(m-1)}(t)), \quad 0 < t < 1, \tag{1.3}$$

with either the Lidstone boundary condition

$$x^{(2i)}(0) = x^{(2i)}(1) = 0 \quad \text{for } i = 0, 1, \dots, m-1, \tag{1.4}$$

or with the focal boundary condition

$$x^{(2i+1)}(0) = x^{(2i)}(1) = 0 \quad \text{for } i = 0, 1, \dots, m-1. \tag{1.5}$$

They proved the existence of at least one positive solution when f is either super-linear or f is sub-linear.

Similar problems were also investigated by Palamides [21] using an analysis of the corresponding field on the face-plane and the Sperner's Lemma. The method there is different from that in [7, 17]. In the papers mentioned above, the nonlinearity f depends on $x, x'', \dots, x^{(2(m-1))}$.

In this paper, we consider the $2m$-order differential equation

$$(-1)^m x^{(2m)}(t) = f(t, x(t), x'(t), \dots, x^{(m)}(t)), \quad 0 < t < \pi, \tag{1.6}$$

with either the Lidstone boundary conditions

$$x^{(2i)}(0) = x^{(2i)}(\pi) = 0 \quad \text{for } i = 0, 1, \dots, m-1, \tag{1.7}$$

or the focal boundary conditions

$$x^{(2i)}(0) = x^{(2i+1)}(\pi) = 0 \quad \text{for } i = 0, 1, \dots, m-1. \tag{1.8}$$

We assume $f : [0, \pi] \times I_0 \times I_1 \times \dots \times I_m \to [0, +\infty)$ is continuous, where $I_0 = [0, +\infty)$, $I_1 = R$, $I_2 = (-\infty, 0]$, $\dots$ for BVP (1.6)–(1.7), and $I_0 = I_1 = [0, +\infty)$, $I_2 = I_3 = (-\infty, 0]$, $\dots\dots$ for BVP (1.6) and (1.8). It is easy to check that if $x(t)$ is a positive solution of BVP (1.6)–(1.7), then

$$(-1)^m x^{(2m)}(t) \geq 0, \quad (-1)^{m-1} x^{2(m-1)}(t) \geq 0, \quad \dots \quad x(t) \geq 0$$

for $t \in [0, \pi]$ and

$$(-1)^m x^{(2m)}(t) \geq 0, \quad (-1)^m x^{(2m-1)}(t) \leq 0, \quad \dots \quad x'(t) \geq 0, \quad x(t) \geq 0$$

for $t \in [0, \pi]$ if $x(t)$ is a positive solution of BVP (1.6) and (1.8).

The emphasis of this paper is that f depends on each of the m higher-order derivatives; i.e., f depends on $x, x', \dots, x^{(m)}$. To obtain the main results, we need the following notation and an abstract existence theorem, whose proof can be found in the text books [14, 23].

Definition: Let X be a real Banach space. A non-empty closed convex set $P \subset X$ is called a cone of X if it satisfies the following conditions:

(i) $x \in P$ and $\lambda \geq 0$ implies $\lambda x \in P$.

(ii) $x \in P$ and $-x \in P$ implies $x = 0$.

Every cone $P \subset X$ induces an ordering in X, which is given by $x \leq y$ if and only if $y - x \in P$.

Let X and Y be Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero, $P : X \to X$ and $Q : Y \to Y$ be projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \operatorname{Ker} Q = \operatorname{Im} L, \quad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $\operatorname{dom} L \cap \overline{\Omega} \neq \Phi$, the map $N : X \to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. Now, we present the fixed point theorem.

Theorem 1.1 ([14, 23])**.** *Let X and Y be Banach spaces, $K_1 \subset X$ and $K \subset Y$ be cones in X and Y, respectively, and the operators L and N be defined above such that $NX \subset K$, $L^{-1}(K) \subset K_1$ and $\operatorname{Ker} L = \{0\}$. Let Ω_1 and Ω_2 be open bounded subsets in X such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $N : \overline{\Omega_2} \to Y$ is L-compact on Ω_2 and there is $h \in L^{-1}(K)$ with $h \neq 0$ such that*

- (i) *$Lx \neq \lambda Nx$ for $\lambda \in (0, 1)$ and $x \in \operatorname{dom} L \cap \partial\Omega_1 \cap K_1$; $Lx - Nx \neq \lambda Lh$ for $\lambda > 0$ and $x \in \operatorname{dom} L \cap \partial\Omega_2 \cap K_1$, or*
- (ii) *$Lx - Nx \neq \lambda Lh$ for $\lambda > 0$ and $x \in \operatorname{dom} L \cap \partial\Omega_1 \cap K_1$; $Lx \neq \lambda Nx$ for $\lambda \in (0, 1)$ and $x \in \operatorname{dom} L \cap \partial\Omega_2 \cap K_1$,*

then $Lx = Nx$ has at least one solution $x \in \operatorname{dom} L \cap \big(\overline{\Omega_2}/\Omega_1\big) \cap K_1$.

2. Positive solutions of boundary-value problems

In this section, we present the main results and then give some examples to illustrate the main results.

Theorem 2.1. *Suppose*

- (A) *The following inequality holds uniformly in t:*

$$\limsup_{\sum_{i=0}^m |x_i| \to +\infty} \frac{f(t, x_0, x_1, \dots, x_m)}{\sum_{i=0}^m |x_i|} < \frac{1}{m+1}\,.$$

- (B) *The following inequality holds uniformly in t:*

$$\liminf_{\sum_{i=0}^m |x_i| \to 0} \frac{f(t, x_0, x_1, \dots, x_m)}{\sum_{i=0}^m |x_i|} > 1\,.$$

Then BVP (1.6)–(1.7) *has at least one positive solution.*

Proof. Let $X = C^m[0, \pi]$ and $Y = C^0[0, \pi]$ be endowed with the norms $\|x\| = \max\{\|x_\infty, \|x'\|_\infty, \dots, \|x^{(m)}\|_\infty\}$ and $\|x\|_\infty = \max_{t\in[0,\pi]} |x(t)|$, respectively. For $x \in Y$, denote

$$\|x\|_1 = \int_0^\pi |x(t)|dt, \qquad \|x\|_2 = \Big(\int_0^\pi |x(t)|^2 dt\Big)^{1/2}.$$

Define

$$\operatorname{dom} L = \{x \in C^{2m}[0, \pi] : x^{(2i)}(0) = x^{(2i)}(\pi) = 0,\ i = 0, 1, \dots, m - 1\}.$$

Define the linear operator $L : \operatorname{dom} L \cap X \to Y$ and the nonlinear operator $N : X \to Y$ by

$$Lx(t) = (-1)^m x^{(2m)}(t) \quad \text{for } x \in \operatorname{dom} L \cap X\,,$$
$$Nx(t) = f(t, x(t), x'(t), \dots, x^{(m)}(t)) \quad \text{for } x \in X\,.$$

Then the differential equation (1.6) can be written as $Lx = Nx$. It is easy to see that $KerL = \{0\}$ and $ImL = Y$. Define the projectors $P : X \to X$ by $Px(t) = 0$ for all $t \in [0, \pi]$ and $Q : Y \to Y$ by $Qy(t) = 0$ for all $t \in [0, \pi]$, respectively. So L is a Fredholm operator of index zero, and $L^{-1} : Y \to X \cap \operatorname{dom} L$ can be written by

$$L^{-1}y(t) = \int_0^\pi G_m(s,t)y(s)ds,$$

where

$$G_0(s,t) = \begin{cases} \frac{s(\pi-t)}{\pi}, & 0 \le s \le t \le \pi \\ \frac{t(\pi-s)}{\pi}, & 0 \le t \le s \le \pi, \end{cases}$$

$$G_k(s,t) = \int_0^\pi G_0(s,u)G_{k-1}(u,t)du \quad \text{for } k = 1, \dots, m\,.$$

It is easy to check that L^{-1} is completely continuous, together with that $N : X \to Y$ is continuous and bounded, it follows that N is L-compact. We divide the proof into two steps.

Step 1. Prove the first part of (ii) in Theorem 1.1. By (B), there is $r > 0$ such that if $\sum_{i=0}^m |x_i| \le r$, then

$$f(t, x_0, x_1, \dots, x_m) > \sum_{i=0}^m |x_i| \ge x_0\,.$$

Choose

$$\Omega_1 = \{x \in X : \|x\| \le r/(m+1)\},$$

$$K_1 = \{x \in \operatorname{dom} L \cap X : x(t) \ge 0 \text{ and } (-1)^m x^{(2m)}(t) \ge 0 \text{ for } t \in [0,\pi]\},$$

$$K = \{x \in Y : x(t) \ge 0 \text{ for } t \in [0,\pi]\}.$$

Then $\operatorname{Ker} L = \{0\}$, $NX \subset K$, $L^{-1}(K) \subset K_1$ and $K_1 \subset X$ and $K \subset Y$ are cones.

If $x \in \operatorname{dom} L \cap \partial\Omega_1 \cap K_1$, then $\|x\| \le r/(m+1)$, so

$$\sum_{i=0}^m |x^{(i)}(t)| \le \sum_{i=0}^m \|x^{(i)}\|_\infty \le (m+1)\|x\| \le r\,.$$

It follows that

$$f(t, x(t), x'(t), \dots, x^{(m)}(t)) \ge x(t) \; for \; t \in [0,\pi]. \tag{2.1}$$

Thus

$$\sin t f(t, x(t), x'(t), \dots, x^{(m)}(t)) \ge x(t)\sin t \quad \text{for } t \in [0,\pi].$$

Integrating the above inequality from 0 to π,, we obtain

$$\begin{aligned} \int_0^\pi \sin t f(t, x(t), x'(t), \dots, x^{(m)}(t))dt &\ge \int_0^\pi \sin t x(t)dt \\ &= -\cos t x(t)\,|_0^\pi + \int_0^\pi x'(t)\cos t dt \\ &= \sin t x'(t)\,|_0^\pi - \int_0^\pi \sin t x''(t)dt \\ &= \dots \\ &= \int_0^\pi \sin t(-1)^m x^{(2m)}(t)dt. \end{aligned}$$

i.e.,

$$\int_0^\pi \sin t Nx(t)dt \ge \int_0^\pi \sin t Lx(t)dt. \tag{2.2}$$

On the other hand, let $h(t)$ be the unique solution of the following problem (it is easy to know, from [7], that it has unique solution)

$$\begin{gathered}(-1)^m x^{(2m)}(t) = 1, \quad 0 < t < \pi,\\ x^{(2i)}(0) = x^{(2i)}(\pi) = 0 \quad i = 0, 1, \dots, m-1.\end{gathered}$$

Then $h \in \operatorname{dom} L$ and $Lh(t) = 1$. We will prove that

$$Lx - Nx \ne \lambda Lh$$

for $\lambda > 0$ and $x \in \operatorname{dom} L \cap \partial\Omega_1 \cap K_1$. In fact, if there is $\lambda_1 > 0$ and $x_1 \in \operatorname{dom} L \cap \partial\Omega_1 \cap K_1$ such that

$$Lx_1 - Nx_1 = \lambda_1 Lh,$$

then

$$\begin{aligned}\int_0^\pi \sin t Lx_1(t)dt &= \int_0^\pi \sin t Nx_1(t)dt + \lambda_1 \int_0^\pi \sin t dt\\ &> \int_0^\pi \sin t Nx_1(t)dt,\end{aligned}$$

which contradicts (2.2). So the first part of (ii) in Theorem 1.1 is satisfied.
Step 2. Prove the second part of (ii) in Theorem 1.1. Choose $1/(m+1) > \epsilon > 0$ and $M > 0$ such that

$$f(t, x_0, x_1, x_2, \dots, x_m) \le \big(\frac{1}{m+1} - \epsilon\big) \sum_{i=0}^m |x_i| + M \tag{2.3}$$

for all $t \in [0, \pi]$ and $x_i \in I_i$ for $i = 0, \dots, m$. In fact, from (A), there is $H > 0$ such that

$$f(t, x_0, x_1, x_2, \dots, x_m) \le \big(\frac{1}{m+1} - \epsilon\big) \sum_{i=0}^m |x_i|$$

for $t \in [0, \pi]$ and $\sum_{i=0}^m |x_i| \ge H$, where $x_i \in I_i$ for $i = 0, 1, \dots, m$. Let

$$M = \max_{t\in[0,\pi], \sum_{i=0}^m |x_i| \le H} f(t, x_0, x_1, \dots, x_m),$$

then we have (2.3). So for $x \in \operatorname{dom} L \cap K_1$, we have

$$f(t, x(t), x'(t), \dots, x^{(m)}(t)) \le \big(\frac{1}{m+1} - \epsilon\big)\Big(\sum_{i=1}^m |x_i| + x(t)\Big) + M.$$

In order to get Ω_2, we now prove that the set

$$S = \{x \in \operatorname{dom} L \cap K_1,\ Lx = \lambda Nx,\ 0 < \lambda < 1\}$$

is bounded. In fact, if S is unbounded, then there is $\lambda \in (0, 1)$, and $x \in S$ such that x satisfies

$$(-1)^m x^{(2m)}(t) = \lambda f(t, x(t), x'(t), \dots, x^{(m)}(t)),\ t \in [0, \pi]. \tag{2.4}$$

Thus

$$\begin{aligned}(-1)^m x^{(2m)}(t)x(t) &= \lambda x(t) f(t, x(t), x'(t), \dots, x^{(m)}(t)) \\ &\leq \lambda\Big(\frac{1}{m+1} - \epsilon\Big)\Big(x^2(t) + \sum_{i=1}^m x(t)|x^{(i)}(t)|\Big) + x(t)M.\end{aligned}$$

Integrating above inequality from 0 to π, we get

$$\begin{aligned}&\int_0^\pi (-1)^m x(t)x^{(2m)}(t)dt \\ &\leq \lambda\Big(\frac{1}{m+1} - \epsilon\Big)\int_0^\pi \Big(x^2(t) + \sum_{i=1}^m x(t)|x^{(i)}(t)|\Big)dt + M\int_0^\pi x(t)dt.\end{aligned}$$

Since

$$\begin{aligned}(-1)^m \int_0^\pi x(t)x^{(2m)}(t)dt &= (-1)^m \int_0^\pi x(t)dx^{(2m-1)}(t) \\ &= (-1)^m x(t)x^{(2m-1)}\Big|_0^\pi + (-1)^{m-1}\int_0^\pi x^{(2m-1)}(t)x'(t)dt \\ &= (-1)^{m-1}\int_0^\pi x'(t)dx^{(2m-2)}(t) \\ &= \dots \\ &= \int_0^\pi \Big(x^{(m)}(t)\Big)^2 dt,\end{aligned}$$

we obtain

$$\begin{aligned}\|x^{(m)}\|_2^2 &\leq \lambda\big(\frac{1}{m+1} - \epsilon\big)\Big[\int_0^\pi x^2(t)dt + \sum_{i=1}^m \int_0^\pi x(t)|x^{(i)}(t)|dt\Big] + M\int_0^\pi x(t)dt \\ &\leq \lambda\big(\frac{1}{m+1} - \epsilon\big)\Big(\|x\|_2^2 + \sum_{i=1}^m \|x\|_2\|x^{(i)}\|_2\Big) + \pi M\|x\|_\infty.\end{aligned}$$

Since $x(t) \sim \sum_{n=1}^\infty a_n \sin nt$, where a_n is the Fourier coefficient of x and

$$x'(t) \sim \sum_{n=1}^\infty n a_n \cos nt,$$

by Parseval equality, $\|x\|_2 \leq \|x'\|_2$. Similarly, we have

$$\|x\|_2 \leq \|x'\|_2 \leq \dots \leq \|x^{(m)}\|_2.$$

Again,

$$\begin{aligned}|x(t)| = |x(t) - x(0)| &= \Big|\int_0^t x'(s)ds\Big| \\ &\leq \int_0^t |x'(s)|ds \leq \int_0^\pi |x'(s)|ds \\ &\leq \Big(\int_0^\pi |x'(t)|^2 dt \int_0^\pi dt\Big)^{1/2} = \pi^{1/2}\|x'\|_2.\end{aligned}$$

Then we obtain $\|x\|_\infty \leq \pi^{1/2}\|x'\|_2$. Thus

$$\|x^{(m)}\|_2^2 \leq \lambda\big(\frac{1}{m+1} - \epsilon\big)(m+1)\|x^{(m)}\|_2^2 + M\pi^{3/2}\|x^{(m)}\|_2.$$

Hence

$$\|x^{(m)}\|_2 \le \frac{M\pi^{3/2}}{\epsilon(m+1)} =: c_1.$$

Thus, we obtain

$$\|x\|_\infty \le \pi^{1/2}\|x^{(m)}\|_2 \le \frac{M\pi^2}{\epsilon(m+1)} =: c_2\,,$$

$$\|x^{(i)}\|_2 \le \|x^{(m)}\|_2 \le \frac{M\pi^{3/2}}{\epsilon(m+1)} =: c_1 \quad \text{for } i = 0, 1, \dots, m.$$

Similarly, we have

$$\|x^{(i)}\|_\infty \le \pi^{1/2}\|x^{(i+1)}\|_2 \le \frac{M\pi^2}{\epsilon(m+1)} = c_2 \quad \text{for } i = 1, \dots, m-1.$$

¿From (2.3),

$$\begin{aligned}
|x^{(2m)}(t)| &\le \big(\frac{1}{m+1} - \epsilon\big)\Big(x(t) + \sum_{i=1}^{m}|x^{(i)}(t)|\Big) + M\\
&\le \big(\frac{1}{m+1} - \epsilon\big)\Big(\|x\|_\infty + \frac{m}{2} + \frac{1}{2}\sum_{i=1}^{m}|x^{(i)}(t)|^2\Big) + M\\
&\le \big(\frac{1}{m+1} - \epsilon\big)\Big(c_2 + \frac{m}{2} + \frac{1}{2}\sum_{i=1}^{m}|x^{(i)}(t)|^2\Big) + M.
\end{aligned}$$

Integrating above inequality from 0 to π, we get

$$\|x^{(2m)}\|_1 \le \pi\big(\frac{1}{m+1} - \epsilon\big)\big(c_2 + \frac{m}{2}\big) + \frac{1}{2}\big(\frac{1}{m+1} - \epsilon\big)c_1^2 + M\pi =: c_3.$$

Since $x^{(2m-2)}(0) = x^{(2m-2)}(\pi) = 0$, there is $\xi \in [0,\pi]$ such that $x^{(2m-1)}(\xi) = 0$, thus

$$|x^{(2m-1)}(t)| \le \|x^{(2m)}\|_1.$$

So $\|x^{(2m-1)}\|_\infty \le c_3$. Similarly, one gets

$$\|x^{(2i-1)}\|_\infty \le c_3, \quad i = 1, \dots, m-1.$$

This implies $\|x\| \le \max\{c_3, c_2, c_1\} + 1$ for all $x \in S$.

Choose $R > \max\{\max\{c_1, c_2, c_3\} + 1, r/(2m+1)\}$. Let

$$\Omega_2 = \{x \in X : \|x\| < R\}.$$

Then $S \subset \Omega_2$. So $Lx \ne \lambda Nx$ for $\lambda \in (0,1)$ and $x \in \operatorname{dom} L \cap \partial\Omega_2 \cap K_1$. Thus by Theorem 1.1, $Lx = Nx$ has at least one solution $x \in \operatorname{dom} L \cap \big(\overline{\Omega_2}/\Omega_1\big) \cap K_1$. x is a solution of BVP (1.6)–(1.7).

Next, we prove that $x(t) > 0$ for $t \in [0,\pi]$. Since $(-1)^m x^{(2m)}(t) \ge 0$ for all $t \in [0,\pi]$, together with the boundary value conditions (1.7), we get $x(t) \ge 0$ and $x''(t) \le 0$ for all $t \in [0,\pi]$. If there is $t_0 \in (0,\pi)$ such that $x(t_0) = 0$, then the concavity of $x(t)$ implies

$$\begin{aligned} 0 = x(t_0) &= x\Big(\frac{\pi - t_0}{\pi - t} t + \frac{t_0 - t}{\pi - t}\pi\Big) \\ &\geq \frac{\pi - t_0}{\pi - t} x(t) + \frac{t_0 - t}{\pi - t} x(\pi) \\ &= \frac{\pi - t_0}{\pi - t} x(t). \end{aligned}$$

This implies that $x(t) = 0$ for all $t \in [0, \pi]$, which contradicts $x \in \overline{\Omega_2}/\Omega_1$. The proof is complete. □

For our convenience, we introduce the following notation:

$$\Delta_1 = \max_{t\in[0,\pi]} \int_0^\pi G_m(s,t)ds,$$

$$\Delta_2 = \max\Big\{\Delta_1,\ \max_{t\in[0,\pi]} \Big(\int_0^t \frac{s}{\pi} G_{m-1}(s,t)ds + \int_t^\pi (1 - \frac{s}{\pi}) G_{m-1}(s,t)ds\Big)\Big\},$$

$$\Delta_3 = \max\Big\{\Delta_2,\ \max_{t\in[0,\pi]} \int_0^\pi G_{m-1}(s,t)ds\Big\},$$

$$\dots$$

$$\Delta_m = \max\Big\{\Delta_{m-1},\ \max_{t\in[0,\pi]} \Big(\int_0^t \frac{s}{\pi} G_{m/2}(s,t)ds + \int_t^\pi (1 - \frac{s}{\pi}) G_{m/2}(s,t)ds\Big)\Big\}$$

if m is an even integer,

$$\Delta_m = \max\Big\{\Delta_m,\ \max_{t\in[0,\pi]} \int_0^\pi G_{(m-1)/2}(s,t)ds\Big\}, \quad \text{if } m \text{ is an odd integer.}$$

Clearly, we have $\Delta_m \geq \Delta_i$ for $i = 1, 2, \dots, m$.

Theorem 2.2. *Assume the following two conditions are satisfied:*

- (C) *The inequality $f(t, x_0, x_1, \dots, x_m) \geq x_0$ holds for all $(x_0, x_1, \dots, x_m)$ in R^{m+1} and all t in $[0, \pi]$.*
- (D) *The following inequality holds uniformly for $t \in [0, \pi]$:*

$$\limsup_{\sum_{i=0}^m |x_i| \to 0} \frac{f(t, x_0, x_1, \dots, x_m)}{\sum_{i=0}^m |x_i|} < \frac{1}{(m+1)\Delta_m}.$$

Then BVP (1.6)–(1.7) *has at least one positive solution.*

Proof. We divide the proof of the theorem into two steps.

Step 1. To prove the first part of (i), choose $r > 0$ and $\delta \in (0, 1/[(m+1)\Delta_m]$ such that

$$f(t, x_0, x_1, \dots, x_m) \leq \delta \sum_{i=0}^m |x_i| \tag{2.5}$$

for $t \in [0, \pi]$ and $(x_0, x_1, \dots, x_m) \in R^{m+1}$ with $\sum_{i=0}^m |x_i| \leq r$. Let

$$\Omega_1 = \{\, x \in \operatorname{dom} L \cap K_1,\ \|x\| < \frac{r}{m+1}\}.$$

For $x \in \partial\Omega_1$, we have $\|x\| = \frac{r}{m+1}$, then

$$\sum_{i=0}^m |x^{(i)}(t)| \leq \sum_{i=0}^m \|x^{(i)}\|_\infty \leq (m+1)\|x\| = r.$$

So, we get

$$f(t,x(t),x'(t),\dots,x^{(m)}(t))\le \delta\sum_{i=1}^{m}|x^{(i)}(t)|,\quad \text{for } t\in[0,\pi].$$

If $Lx=\lambda Nx$ with $\lambda\in(0,1)$ has a solution $x\in \operatorname{dom}L\cap K_1\cap\partial\Omega_1$, then

$$x(t)=\lambda L^{-1}Nx(t)=\lambda\int_0^{\pi}G_m(t,s)f(s,x(s),x'(s),\dots,x^{(m)}(s))ds.$$

Hence, we get

$$\begin{aligned}\|x\|_\infty&=\lambda\max_{t\in[0,\pi]}\int_0^{\pi}G_m(t,s)f(s,x(s),x'(s),\dots,x^{(m)}(s))ds\\&\le\delta\max_{t\in[0,\pi]}\int_0^{\pi}G_m(t,s)\sum_{i=0}^{m}|x^{(i)}(s)|ds\\&\le\delta\Delta_1(m+1)\|x\|.\end{aligned}$$

It is easy to check that

$$\begin{aligned}\|x'\|_\infty&=\lambda\max_{t\in[0,\pi]}\Big[-\int_0^{t}\frac{s}{\pi}G_{m-1}(t,s)f(s,x(s),x'(s),\dots,x^{(m)}(s))ds\\&\quad+\int_t^{\pi}\big(1-\frac{s}{\pi}\big)G_{m-1}(t,s)f(s,x(s),x'(s),\dots,x^{(m)}(s))ds\Big]\\&\le\max_{t\in[0,\pi]}\Big(\int_0^{t}\frac{s}{\pi}G_{m-1}(t,s)f(s,x(s),x'(s),\dots,x^{(m)}(s))ds\\&\quad+\int_t^{\pi}\big(1-\frac{s}{\pi}\big)G_{m-1}(t,s)f(s,x(s),x'(s),\dots,x^{(m)}(s))ds\Big)\\&\le\max_{t\in[0,\pi]}\Big(\int_0^{t}\frac{s}{\pi}G_{m-1}(t,s)ds+\int_t^{\pi}\big(1-\frac{s}{\pi}\big)G_{m-1}(t,s)ds\Big)\delta(m+1)\|x\|\\&\le\Delta_2\delta(m+1)\|x\|.\end{aligned}$$

Finally, we can get $\|x^{(m)}\|_\infty\le\delta\Delta_m(m+1)\|x\|$. Hence, we have

$$\|x\|\le\delta\Delta_m(m+1)\|x\|.$$

Thus $(m+1)\delta\Delta_m\ge1$, which contradicts $\delta\in(0,1/[\Delta_m(m+1)])$. The first step is complete.

Step 2. Choose Ω_2 sufficiently large such that $\Omega_1\subset\overline{\Omega_1}\subset\Omega_2$, by condition (C), we have that

$$f(t,x_0,x_1,\dots,x_m)\ge x_0$$

holds for all $t\in[0,\pi]$ all $(x_0,x_1,\dots,x_m)\in R^{m+1}$. Hence,

$$f(t,x(t),x'(t),\dots,x^{(m)}(t))\ge x(t)$$

holds for all $t\in[0,\pi]$, i.e. (2.1) holds. Similar to Step 1 in Theorem 1.1, we can get a contradiction, hence the second part of (i) in Theorem 1.1 is satisfied. It follows from (i) of Theorem 1.1 that BVP (1.6) and (1.8) has at least one positive solution $x(t)$. The proof is complete. □

Remark. Consider the boundary-value problem

$$\begin{gathered}(-1)^m x^{(2m)}(t) = f(t, x(t), x'(t), \dots, x^{(m)}(t)), \quad 0 < t < T,\\ x^{(2i)}(0) = x^{(2i)}(T) = 0 \quad \text{for } i = 0, 1, \dots, m-1,\end{gathered} \tag{2.6}$$

where $T > 0$ is a constant, f and m are defined in (1.6)–(1.7). Let $s = \pi t/T$, we transform BVP (2.6) into a BVP similar to BVP (1.6)–(1.7). Then a similar existence result can be obtained.

Theorem 2.3. *Suppose* (*A*) *and* (*B*) *of Theorem 2.1 hold. Then BVP* (1.6) *and* (1.8) *has at least one positive solution.*

Proof. Consider the boundary-value problem

$$\begin{gathered}(-1)^m x^{(2m)}(t) = \begin{cases} f(t, x(t), x'(t), \dots, x^{(m)}(t)), & \text{for } 0 \le t \le \pi,\\ f(2\pi - t, x(2\pi - t), -x'(2\pi - t), \dots, & \\ (-1)^m x^{(m)}(2\pi - t)) & \text{for } \pi \le t \le 2\pi,\end{cases}\\ x^{(2i)}(0) = x^{(2i)}(2\pi) = 0 \quad \text{for } i = 0, 1, \dots, m-1\,.\end{gathered}$$

This problem is exactly similar to that of Theorem 2.1, we can obtain at least one positive solution $x(t)$, which is defined on $[0, 2\pi]$, of above BVP and so $x(t)(t \in [0, \pi])$ is a positive solution of BVP (1.6) and (1.8). The proof completed. □

Theorem 2.4. *Suppose Conditions* (*C*) *and* (*D*) *of Theorem 2.2 hold. Then BVP* (1.6) *and* (1.8) *has at least one positive solution.*

The proof is similar to that of Theorem 2.3 and is omitted. Next, we present two examples to illustrate the main results.

Example 2.5. Consider the boundary-value problem

$$\begin{gathered}x^{(4)}(t) = f(t, x(t), x'(t), x''(t)), \quad 0 < t < \pi,\\ x(0) = x''(0) = x(\pi) = x''(\pi) = 0,\end{gathered} \tag{2.7}$$

where f is a nonnegative continuous function. From Theorem 2.1, if

$$\limsup_{|x|+|y|+|z| \to \infty} \frac{f(t, x, y, z)}{|x| + |y| + |z|} < \frac{1}{3},$$

and

$$\liminf_{|x|+|y|+|z| \to 0} \frac{f(t, x, y, z)}{|x| + |y| + |z|} > 1$$

hold uniformly, then (2.7) has at least one positive solution.

Example 2.6. Consider the boundary-value problem

$$\begin{gathered}x^{(6)}(t) = -\frac{2}{1 + |x(t)| + |x'(t)| + x''(t)| + |x'''(t)|}, \quad 0 < t < \pi,\\ x(0) = x''(0) = x''''(0) = x(\pi) = x''(\pi) = x''''(\pi) = 0.\end{gathered} \tag{2.8}$$

It is easy to check that all conditions of Theorem 2.1 are satisfied. So (2.8) has at least one positive solution.

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References

[1] R. P. Agarwal, *Boundary Value Problems for Higher Order Differential Equations*, World Scientific, Singapore, 1986.

[2] R. P. Agarwal, *Focal Boundary Value Problems for Differential and Difference Equations*, Kluwer, Dordrecht, 1998.

[3] R. P. Agarwal, D. O'Regan, P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[4] R. I. Avery, C. J. Chyan, J. Henderson, *Twin positive solutions of boundary value problems for ordinary differential and finite difference equations*, Comput. Math. Appl., 42(2001), 695-704.

[5] C. F. Beards, *Vibration Analysis with Applications to Control Systems*, Edward Arnold, London, (1995).

[6] D. Cao, R. Ma, *Positive solutions to a second-order multi-point boundary value problem*, Electronic J. Diff. Equs., 2000(2000), 65:1-8.

[7] C. J. Chyan, J. Henderson, *Positive solutions of $2m^{th}$-order boundary-value problems*, Appl. math. Letters, 15(2002), 767-774.

[8] J. M. Davis, P. W. Eloe, J. Henderson, *Triple positive solutions and dependence on higher-order derivatives*, J. Math. Anal. Appl., 237(1999), 710-720.

[9] E. Dulacska, *Soil settlement effects on buildings, In developments in Geotechnical Engineering*, Volume 69, Elsevier, Amsterdam, (1992).

[10] L. Erbe, H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc., 120(1994), 743-748.

[11] L. Erbe, M. Tang, *Existence and multiplicity of positive solutions to a nonliear boundary value problem*, Diff. Equns. Dynam. Syst., 4(1996), 313-320.

[12] J. R. Graef, B. Yang, *Existence and non-existence of positive solutions of fourth-order nonlinear boundary value problems*, Appl. Anal., 74(2000)201-204.

[13] J. R. Graef, B. Yang, *On a nonlinear boundary value problem for fourth-order differential equations*, Appl. Anal., 72(1999)439-448.

[14] D. Guo, J. Sun, Zh. Liu, *Functional Methods for Nonlinear Ordinary Differential Equations(in Chinese)*, Science and Technical Press of Shandong, Jinan, (1995), 45-108.

[15] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, (1988).

[16] J. Henderson, H. B. Thomson, *Multiple symmetric positive solutions for a second-order boundary value prolem*, Proc. Amer. Math. Soc., 128(2000), 2373-2379.

[17] J. Henderson, *Existence of multiple solutions for second order boundary-value problems*, J. Diff. Equns., 166(2000), 443-454.

[18] G. L. Karakostas, P. Ch. Tsamatos, *Sufficient conditions for the existence of nonnegative solutions of a nonlocal boundary value problem*, Appl. Math. Letters, 2002, 15:401-407.

[19] R. Ma, H. Wang, *On the existence of positive solutions of fourth-order differential equations*, Appl. Anal., 59(1995)225-231.

[20] E. H. Mansfield, *The bending and stretching of plates, In international series of monographs on Aeronautics and Astronautics*, Volume 6, Pergmon, New York, (1964).

[21] P. K. Palamides, *Positive solutions for higher-order Lidstone boundary value problems: A new apploach Via Sperner's Lemma*, Comput. Math. Appl., 42(2001), 75-89.

[22] P. K. Palamides, *Positive and monotone solutions of an m-point boundary value problem*, Electronic J. Diff. Equs., 2002(2002), 1-16.

[23] N. Wang, *Toplogical Degree and its Applications(in Chinese)*, Chinese Math. Annals, 8A(1987), 311-318.

[24] P. J. Y. Wong, *Triple positive solutions of conjugate boundary value problems*, Comput. Math. Applic., 36(1998), 19-35.

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