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LINEAR DELAY DIFFERENTIAL EQUATION WITH A POSITIVE AND A NEGATIVE TERM

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Abstract. We study a linear delay differential equation with a single positive and a single negative term. We find a necessary condition for the oscillation of all solutions. We also find sufficient conditions for oscillation, which improve the known conditions.

1. Introduction

Ladas and co-workers [3, 6, 8] studied the delay differential equation

$$\dot{u}(t) + pu(t-\tau) - qu(t-\sigma) = 0, \tag{1.1}$$

where $p, q, \tau, \sigma \in R^+$. They obtained the following set of sufficient conditions for the oscillation of all solutions to this equation:

$$\begin{gathered} p > q\,, \\ \tau \geq \sigma\,, \\ q(\tau-\sigma) \leq 1\,, \\ (p-q)\tau > \frac{1}{\mathrm{e}}[1-q(\tau-\sigma)]\,. \end{gathered} \tag{1.2}$$

Chuanxi and Ladas [4] generalized the above conditions to a non-autonomous equation in which p and q are replaced by continuous functions $P(t)$ and $Q(t)$. The delay equation with an arbitrary number of positive and negative terms has been studied, for the autonomous case, by Agwo [2] and Ahmad and Alherbi [1]. Recently Elabbasy et al. [5] have applied a technique in Li [9] to generalize conditions (1.2) for a non-autonomous equation having arbitrary number of positive and negative terms.

A common feature among results pertaining to non-autonomous generalizations of (1.1) is that, in the limit of constant coefficients, they reduce to a *stronger* version of conditions (1.2) in which the fourth condition is replaced by $(p-q)\tau > 1$. Although this set of conditions underwent a gradual improvement in [3, 6, 8], after 1991 no attempt has been successful in improving these conditions (as far as the author is aware).

The first and second conditions of (1.2) are necessary for oscillation [6, p41]. However it is obvious that the third condition is unnecessarily restrictive since, no

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matter how large the coefficient of the negative term and the difference between the delays might be, all solutions of the equation must be oscillatory provided the coefficient of the positive term is *sufficiently large*. Ideally the oscillation conditions should involve what Hunt and Yorke [7] have termed as the *torque* associated with the delay differential equation. For (1.1) this parameter is $p\tau - q\sigma$. In this paper we shall seek sufficient conditions which depend either on the difference or the ratio of $p\tau$ and $q\sigma$. We also show that the condition

$$p\tau - q\sigma > \frac{1}{e}$$

is necessary for the oscillation of all solutions of (1.1). We expect that this work will stimulate further research in the quest of *necessary and sufficient* conditions for oscillation.

2. Basic results and preliminary lemmas

The following properties of the exponential function are easily established. Let $x \geq 0$. Then

$$e^{ax} \geq ae^x + 1 - a, \quad \text{if } a \geq 1, \tag{2.1}$$

$$e^{ax} \leq ae^x + 1 - a, \quad \text{if } a < 1, \tag{2.2}$$

$$e^{ax} \geq bx + \frac{b}{a}[1 - ln\,\frac{b}{a}] \quad \text{if } a > 0 \text{ and } b > 0. \tag{2.3}$$

If $q = 0$ or $\tau = \sigma$, (1.1) reduces to a delay equation with a single positive term. If $\sigma = 0$, then a necessary and sufficient condition for oscillation is [6, p40]

$$p\tau e^{-q\tau} > \frac{1}{e}.$$

Let $0 < q < p$ and $0 < \sigma < \tau$. For $x \geq 0$, define

$$\begin{aligned} g(x) &= qe^{-\sigma x} - pe^{-\tau x}, \\ x_1 &= \frac{1}{\tau - \sigma}\ln(\frac{p}{q}), \\ x_2 &= \frac{1}{\tau - \sigma}\ln(\frac{\tau p}{\sigma q}), \\ k &= (1 - \frac{\sigma}{\tau})q(\frac{p\tau}{q\sigma})^{\frac{-\sigma}{\tau - \sigma}}. \end{aligned} \tag{2.4}$$

A simple calculation shows that g is increasing and concave on the interval $[0, x_2]$. It vanishes at x_1 and attains its maximum value, k, at x_2.

The characteristic equation corresponding to (1.1) is

$$x + pe^{-\tau x} - qe^{-\sigma x} = 0, \tag{2.5}$$

or $x = g(x)$.

It is well-known that a delay differential equation possesses a non-oscillatory solution if and only if its characteristic equation has a real root. Since $p > q$, it follows that $x = 0$ does not satisfy (2.5). We shall investigate the existence of positive or negative roots in the following lemmas.

Lemma 2.1. *The characteristic equation does not have a positive root if $x_1 \geq k$ i.e.*

$$\ln(\frac{p}{q}) \geq \frac{(\tau - \sigma)^2}{\tau}q(\frac{p\tau}{q\sigma})^{\frac{-\sigma}{\tau - \sigma}}.$$

Proof. Let $f(x) = g(x) - x$. Then f is negative on $[0, x_1]$ and for $c > 0$, we have

$$f(x_1 + c) = g(x_1 + c) - x_1 - c \le k - x_1 - c < 0\,.$$

Thus f remains negative on $[0, \infty)$ implying that the characteristic equation does not have a positive root. □

Lemma 2.2. *The characteristic equation has no positive root if*

$$\ln(\frac{\mathrm{p}}{\mathrm{q}}) > \frac{(\tau-\sigma)^2}{\tau} q(\frac{\mathrm{p}}{\mathrm{q}})^{\frac{-\sigma}{\tau-\sigma}} - \frac{(\tau-\sigma)}{\tau}.$$

Proof. Since g is concave on $[x_1, x_2]$, there is at most one point, call it c, in $[x_1, x_2]$ where the tangent to the curve $y = g(x)$ is parallel to the line $y = x$. The intercept of the tangent with the $x-$axis is $c - g(c)$. If this number is positive, the line $y = x$ does not intersect the curve $y = g(x)$. This implies no positive root of the characteristic equation if

$$g(c) < c, \quad \text{or} \quad qe^{-\sigma c} - pe^{-\tau c} < c. \tag{2.6}$$

Since the slope at the point $(c, g(c))$ is unity, we have

$$-\sigma q e^{-\sigma c} + p\tau e^{-\tau c} = 1. \tag{2.7}$$

From (2.6)–(2.7), we get

$$-\tau c + q(\tau-\sigma)e^{-\sigma c} < 1\,. \tag{2.8}$$

The expression on the left side of (2.8), as a function of c, is decreasing. Hence it will hold on the entire interval $[x_1, x_2]$ provided it does so for $c = x_1$. This gives

$$\frac{-\tau}{\tau-\sigma}\ln(\frac{\mathrm{p}}{\mathrm{q}}) + q(\tau-\sigma)e^{\frac{-\sigma}{\tau-\sigma}\ln(\frac{\mathrm{p}}{\mathrm{q}})} < 1, \tag{2.9}$$

or

$$\ln(\frac{\mathrm{p}}{\mathrm{q}}) > \frac{(\tau-\sigma)^2}{\tau} q(\frac{\mathrm{p}}{\mathrm{q}})^{\frac{-\sigma}{\tau-\sigma}} - \frac{(\tau-\sigma)}{\tau}.$$

□

Lemma 2.3. *The characteristic equation does not have a negative root if* $p\tau - q\sigma > 1/b$ *or* $e^{q(\tau-\sigma)-1}$, *where* b *is the larger root of the equation*

$$x(1 - \ln x) = \frac{\mathrm{q}(\tau-\sigma)}{\mathrm{p}\tau - \mathrm{q}\sigma},$$

and $1 < b \le e$.

Proof. A negative root of the characteristic equation (2.5) is equivalent to a positive root of the equation

$$-x + pe^{\tau x} - qe^{\sigma x} = 0\,. \tag{2.10}$$

In the above equation, the change of variable $y = \tau x$ yields

$$-y + p\tau e^{y} - q\tau e^{\frac{\sigma}{\tau}y} = 0\,.$$

Suppose $y > 0$ is a root of this equation. On making use of (2.2), we get

$$0 \ge -y + p\tau e^{y} - q\tau(\frac{\sigma}{\tau}\,e^{y} + 1 - \frac{\sigma}{\tau}) = -y + (p\tau - q\sigma)e^{y} - q(\tau-\sigma). \tag{2.11}$$

Now we use (2.3) to obtain, for arbitrary $b > 0$,

$$0 \ge -y - q(\tau-\sigma) + (p\tau - q\sigma)[by + b(1 - \ln b)]. \tag{2.12}$$

Choose b such that $1 < b \le e$ and

$$b(1 - \ln b) = \frac{\mathrm{q}(\tau - \sigma)}{p\tau - q\sigma}.$$

Inequality (2.12) becomes $0 \ge y[-1 + b(p\tau - q\sigma)]$. Since $y > 0$, there will be a contradiction if $p\tau - q\sigma > 1/b$.

Now define $z = y + q(\tau - \sigma)$. Inequality (2.11) becomes

$$\begin{aligned} 0 &\ge -z + (p\tau - q\sigma)e^{-q(\tau-\sigma)}e^{z} \\ &\ge -z + (p\tau - q\sigma)e^{-q(\tau-\sigma)}ez \\ &= z[-1 + (p\tau - q\sigma)e^{1-q(\tau-\sigma)}], \end{aligned}$$

which will lead to a contradiction if $p\tau - q\sigma > e^{q(\tau-\sigma)-1}$. □

3. Main Results

It is well-known that the first two conditions (1.2) are necessary for the oscillation of all solutions of (1.1). In this section we first prove a necessary condition which should be useful in estimating how far from the best possible position a sufficient condition actually is.

Theorem 3.1 (A necessary condition for oscillation). *Let $p, q, \tau, \sigma \in R^+$, $p > q$ and $\tau \ge \sigma$. If all solutions of* (1.1) *are oscillatory then $p\tau - q\sigma > 1/e$.*

Proof. We shall prove that otherwise a real root of the characteristic equation must exist indicating a positive solution of the delay equation. Denote the left side of the characteristic equation, i.e. (2.5), by $f(x)$. First assume $p\tau - q\sigma = 1/e$. Since $f(0) = p - q > 0$ and

$$f(-1/\tau) = -1/\tau + pe - qe^{\frac{\sigma}{\tau}} = q[\frac{\mathrm{e}\sigma}{\tau} - e^{\frac{\sigma}{\tau}}] \le 0,$$

hence f has a zero in $[-1/\tau, 0)$.

Next assume $p\tau - q\sigma < 1/e$. If $q\sigma = 0$, it is easy to see that a zero of f will exist in $[-1/\tau, 0)$. Let $q\sigma > 0$ and define $a = q\sigma e$. There exists an $\epsilon > 0$ such that

$$p\tau - q\sigma = \frac{1 - 2\epsilon}{\mathrm{e}} < \frac{1 - \epsilon}{\mathrm{e}}.$$

Without loss of generality we can choose $\epsilon < a$. Thus

$$p\tau < q\sigma + \frac{1 - \epsilon}{\mathrm{e}} = \frac{\mathrm{a} + 1 - \epsilon}{\mathrm{e}}\,. \tag{3.1}$$

We write $f(x) = f_1(x) + f_2(x)$, where

$$f_1(x) = (a + 1 - \epsilon)x + pe^{-\tau x}, f_2(x) = -(a - \epsilon)x - qe^{-\sigma x}.$$

Since an equation of the form $x + ce^{-dx} = 0$, has a real root if and only if $cd \le 1/e$, it follows from (3.1) that $f_1(x) = 0$ has a real root, say x_0, while

$$q\sigma = a/e > \frac{\mathrm{a} - \epsilon}{\mathrm{e}},$$

shows that $f_2(x) = 0$ does not have any real root. Since $f_2(0) < 0$ it follows that $f_2(x_0) < 0$. Now

$$f(x_0) = f_1(x_0) + f_2(x_0) < 0\,.$$

Also $f(0) = p - q > 0$, hence the characteristic equation possesses a root in $(0, x_0)$. This proves the necessity of the condition $p\tau - q\sigma > 1/e$ for the oscillation of all solutions of (1.1). The proof of Theorem 1 is complete. □

Putting the results of Lemmas 2.1-2.3 together, we obtain the following Theorem.

Theorem 3.2 (Sufficient conditions for oscillation)**.** *All solutions of* (1.1) *will be oscillatory if*

$$\begin{gathered} p > q > 0\,, \\ \tau > \sigma \geq 0\,, \\ \ln(\frac{\mathrm{p}}{\mathrm{q}}) > \min\{\frac{(\tau-\sigma)^2}{\tau} q(\frac{\mathrm{p}\tau}{\mathrm{q}\sigma})^{\frac{-\sigma}{\tau-\sigma}},\ \frac{(\tau-\sigma)^2}{\tau} q(\frac{\mathrm{p}}{\mathrm{q}})^{\frac{-\sigma}{\tau-\sigma}} - \frac{(\tau-\sigma)}{\tau}\}, \\ p\tau - q\sigma > \min\{1/b\,, e^{q(\tau-\sigma)-1}\}, \end{gathered}$$

where b is the larger root of the equation

$$x(1 - \ln x) = \frac{\mathrm{q}(\tau - \sigma)}{\mathrm{p}\tau - \mathrm{q}\sigma},$$

and $1 < b \leq e$. If $q = 0$ and/or $\tau = \sigma$ then $b = e$.

A glance at (2.9) indicates that it can be satisfied for arbitrary $q(\tau - \sigma)$ if p is large enough. For example if $q = 3, \tau = 4, \sigma = 2$, the inequality of Lemma 2.2 will hold for $p \geq 6.82$. Also $p\tau - q\sigma = (p - q)\tau + q(\tau - \sigma)$, hence the condition

$$p\tau - q\sigma > e^{q(\tau-\sigma)-1},$$

is equivalent to

$$(p - q)\tau > e^{q(\tau-\sigma)-1} - q(\tau - \sigma). \tag{3.2}$$

Since on $[0, 1]$, $e^x - ex \leq 1 - x$, it is clear that when conditions (1.2) hold, the number on the right side of the fourth condition is larger than its counterpart on the right side of (3.2).

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